28 settembre

<u>Esercizio</u> Sia $p(z) = a_n z^n + \dots + a_0$ un polinomio a coefficienti reali ($a_0, \dots, a_n \in \mathbb{R}$, $a_n \neq 0$)

allora, se $z_0 \in \mathbb{C}$ è una radice anche $\overline{z_0}$ lo è.

<u>Dim</u>

z_0 è una radice di $P(z)$, significa che

$$P(z_0) = 0$$

$$a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0 = 0$$

$$\overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0} = \overline{0} = 0$$

$$\overline{a_n}\, \overline{z_0}^n + \overline{a_{n-1}}\, \overline{z_0}^{n-1} + \quad + \overline{a_1}\, \overline{z_0} + \overline{a_0} = 0$$

$$0 = a_n \overline{z_0}^n + a_{n-1} \overline{z_0}^{n-1} + \cdots + a_1 \overline{z_0} + a_0 = P(\overline{z_0})$$

$$P(\overline{z_0}) = 0$$

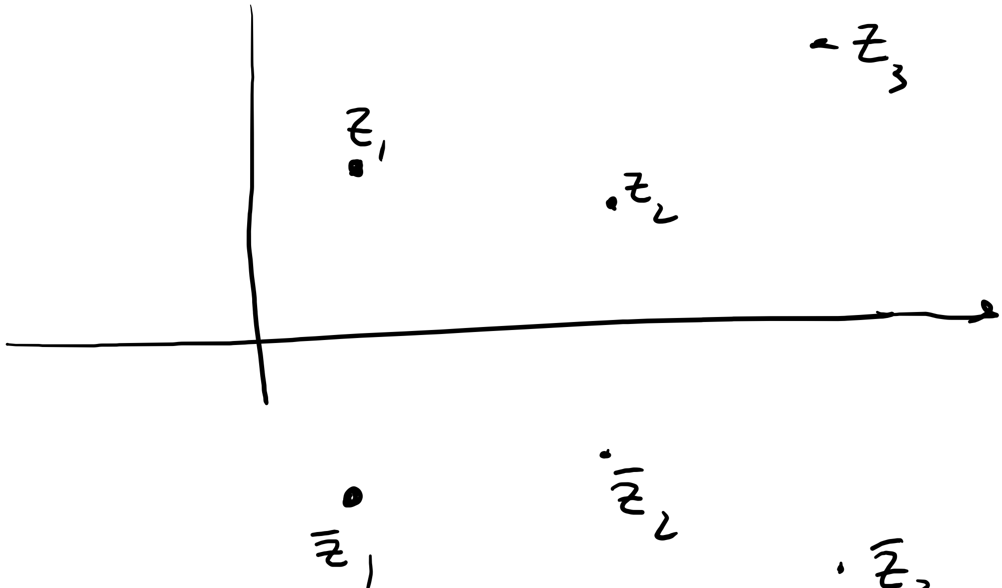
$$p(z) = a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

$$a_5, \dots, a_0 \in \mathbb{R} \qquad a_5 \neq 0$$

Può avere 3 radici non reali, ~~non~~ non complesse coniugate

z_1, z_2, z_3 tutti e tre non in $\mathbb{R}$

a due a due non complessi coniugati l'uno dell'altro e a due a due distinti

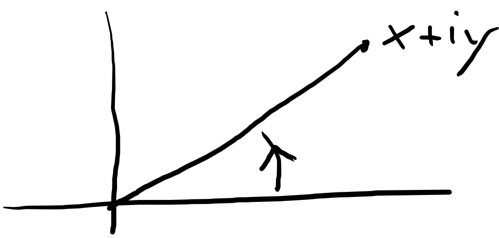


$p(z)$ dovrebbe essere divisibile per $(z-z_1)(z-\bar{z}_1)(z-z_2)(z-\bar{z}_2)(z-z_3)(z-\bar{z}_3)$

$x + iy = z = r\,(\cos\vartheta + i \sin\vartheta)$

$n \in \mathbb{N}$ fissato

$z^n = r^n \left(\cos(n\vartheta) + i \sin(n\vartheta)\right)$



$z^n = 1$ $\qquad z = r\,(\cos\vartheta + i \sin\vartheta)$

$r^n \left(\cos(n\vartheta) + i \sin(n\vartheta)\right) = 1$

$r^n = 1 \implies r = 1$

$$\cos(n\vartheta) + i\sin(n\vartheta) = 1 \iff \begin{cases} \cos(n\vartheta) = 1 \\ \sin(n\vartheta) = 0 \end{cases}$$

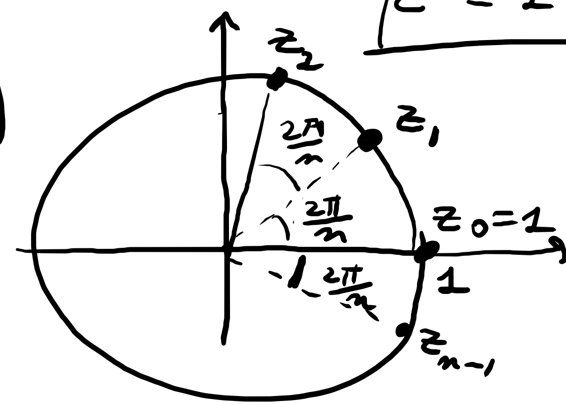
$$\iff \quad n\vartheta = 2k\pi \qquad \text{per } k \in \mathbb{Z}$$

$$\iff \quad \vartheta_k = \frac{2k\pi}{n} \qquad \text{per } k \in \mathbb{Z}$$

$$\boxed{z^n = 1}$$

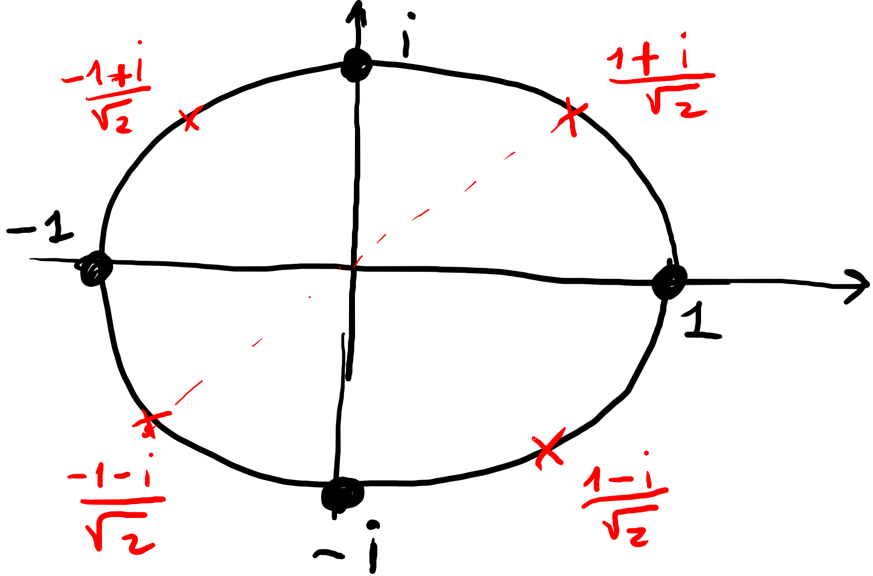
$$\implies \quad z_k = \cos(\vartheta_k) + i\sin(\vartheta_k)$$

$$\boxed{\begin{array}{c} k = 0, 1, \dots, n-1 \\ z_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \end{array}}$$



$z^4 = 1$

$z = \pm 1, \pm i$



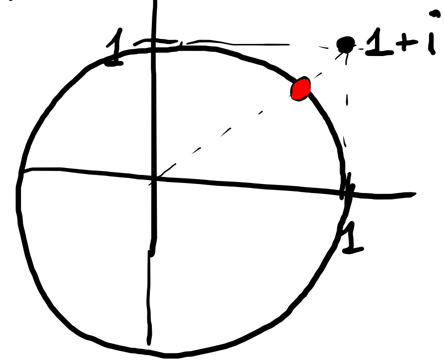
$z^8 = 1$

$z^{50} - 1 = 0$

$$(1+i)^{\frac{1}{8}} = ?$$

$$z^8 = 1+i$$

$$z^8 = 1+i = \sqrt{2}\ \frac{1+i}{\sqrt{2}} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right)$$



$$z^8 = 2^{\frac{1}{2}}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$z = r\left(\cos(\vartheta) + i\sin(\vartheta)\right)$$

$$r^8\left(\cos(8\vartheta) + i\sin(8\vartheta)\right) = 2^{\frac{1}{2}}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$

$$r^8\left(\cos(8\vartheta)+i\sin(8\vartheta)\right) = 2^{\frac{1}{2}}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$$

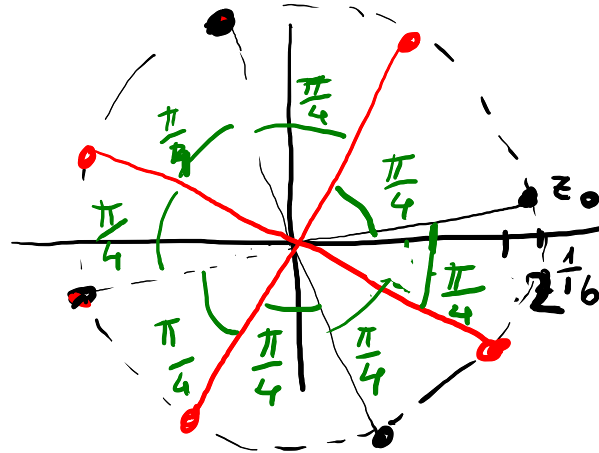
$$r^8 = 2^{\frac{1}{2}} \iff r = 2^{\frac{1}{16}}$$

$$\cos(8\vartheta)+i\sin(8\vartheta) = \cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)$$

$$\cos(8\vartheta) = \cos\left(\frac{\pi}{4}\right), \quad \sin(8\vartheta) = \sin\left(\frac{\pi}{4}\right)$$

$$8\vartheta = \frac{\pi}{4} + 2\pi k \qquad k \in \mathbb{Z}$$

$$\vartheta_k = \frac{\pi}{32} + \frac{2\pi k}{8}$$



$$k = 0, 1, 2, 3, 4, 5, 6, 7$$

$$z_k = 2^{\frac{1}{16}}\left(\cos\vartheta_k + i\sin\vartheta_k\right)$$

$$z^6 - |z|^4 + |z|^2 = 1 \qquad z = 1 \qquad z = r(\cos\vartheta + i\sin\vartheta)$$

$$z^6 = r^6\left(\cos(6\vartheta) + i\sin(6\vartheta)\right)$$

$$|z|^2 = r^2 \quad , \quad |z|^4 = r^4$$

$$r^6\left(\cos(6\vartheta) + i\sin(6\vartheta)\right) - r^4 + r^2 = 1$$

$$\begin{cases} r^6\cos(6\vartheta) - r^4 + r^2 = 1 \\ r^6\sin(6\vartheta) = 0 \end{cases}$$

$$r^6\sin(6\vartheta) = 0$$

$\cancel{r = 0}$ oppure

$$\sin(6\vartheta) = 0$$

$$r^6 \cos(6\vartheta) - r^4 + r^2 = 1 \qquad \sin(6\vartheta) = 0$$

$$\sin(6\vartheta) \Rightarrow \cos(6\vartheta) = \begin{cases} 1 \\ -1 \end{cases}$$

$$\cos(6\vartheta) = -1$$

$$-r^6 - r^4 + r^2 = 1$$

$$r^6 + r^4 - r^2 = -1$$

$$\boxed{r^6 + r^4 + 1 = r^2}$$ non ha soluzioni $r \geq 0$.

$r^6 + r^4 + 1 = r^2$ non ha soluzioni

Ci sono due casi

1) $0 \le r < 1 \implies r^2 < 1 \implies r^2 < 1 + r^4 + r^6$

2) $r \ge 1 \implies$

$$r^6 \ge r^2 \qquad \implies \boxed{r^6 + r^4 + 1 > r^2}$$

$$r^4 \ge r^2 \qquad r^6 + r^4 + 1 > 2r^2 > r^2$$

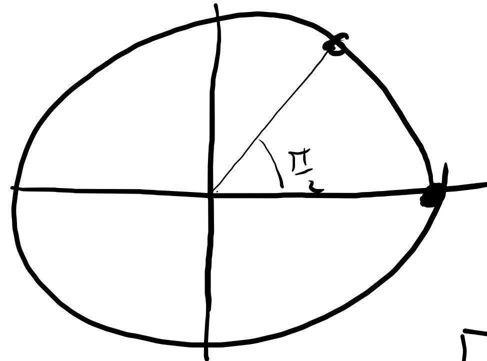
$\cos(6\vartheta) = 1$ $\qquad$ $r^6 \cos(6\vartheta) - r^4 + r^2 = 1$

$$r^6 - r^4 + r^2 - 1 = 0$$

$$r^4(r^2-1) + r^2 - 1 = 0$$

$$(r^2-1)(r^4+1) = 0$$

$$(r-1)(r+1)(r^4+1) = 0 \iff r - 1 = 0 \iff \boxed{r = 1}$$



$$\begin{cases} \sin(6\vartheta) = 0 \\ \cos(6\vartheta) = 1 \end{cases}$$

$$6\vartheta = 2\pi k \qquad k \in \mathbb{Z}$$

$$\vartheta = \frac{2\pi k}{6} \qquad k = 0, \dots ; 5$$

$$z^4 + 2|z|^2 = 1$$