Review of some probability concepts: random variables

(A quick tour)

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Random variables

Discrete distributions

Continuous distributions

C.d.f. and quantile functions

Random variables

Random variables

Statistics is about the extraction of information from data that contain an *unpredictable* component.

Random variables (r.v.) are the mathematical device employed to build *models* of this variability.

A r.v. takes a different value at random each time is observed.

Distribution of a r.v.

The main tools used to describe the **distribution** of values taken by a r.v. are:

1. Probability functions
2. Density functions
3. Cumulative distribution functions
4. Quantile functions

Discrete distributions

1. Probability functions

Discrete r.v. take values in a discrete set.

The **probability (mass) function** of a discrete r.v. X is the function $f(x)$ such that

$$f(x) = \Pr(X = x)\,.$$

with $0 \leq f(x) \leq 1$ and $\sum_i f(x_i) = 1$.

The probability function defines the **distribution** of X.

Mean and variance of a discrete r.v.

For many purposes, the first two moments of a distribution provide a useful summary.

The **mean (expected value)** of a discrete r.v. X is

$$E(X) = \sum_i x_i \, f(x_i)\,,$$

and the definition is extended to any function g of X

$$E\{g(X)\} = \sum_i g(x_i)\, f(x_i)\,.$$

The special case $g(X) = (X-\mu)^2$, with $\mu = E(X)$, is the **variance** of X

$$\mathrm{var}(X) = E\{(X-\mu)^2\} = E(X^2) - \mu^2.$$

The **standard deviation** is just given by $\sqrt{\mathrm{var}(X)}$.

Notable discrete random variables

Discrete r.v. often used in applications:

- Binomial (and Bernoulli) distribution
- Poisson distribution
- Negative binomial distribution
- Geometric distribution
- Hypergeometric distribution

Let us give a closer look to some of them.

The binomial distribution

Consider n independent binary trials each with success probability p, $0 < p < 1$. The r.v. X that counts the number of successes has **binomial distribution** with probability function

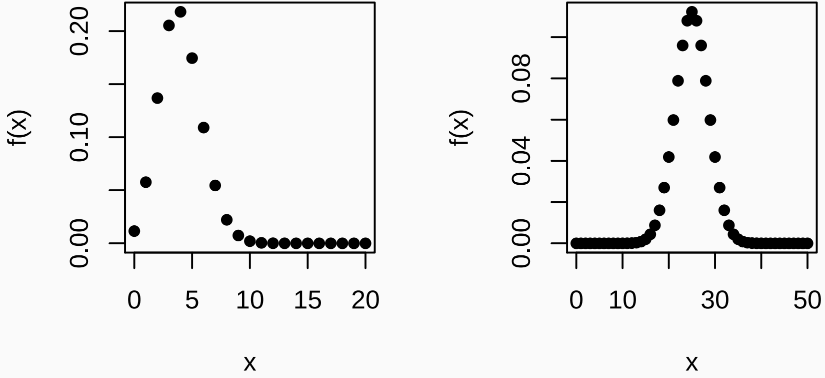
$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, \ldots, n.$$

The notation is $X \sim \mathcal{B}_i(n, p)$, and $E(X) = np$, $\mathrm{var}(X) = np(1-p)$.

The case when $n = 1$ is known as **Bernoulli distribution** and a single binary trial is called **Bernoulli trial**.

R lab: the binomial distribution

```r
par(mfrow=c(1,2), pty="s", pch = 16)
plot(0:20, dbinom(0:20, 20, 0.2), xlab = "x", ylab = "f(x)")
plot(0:50, dbinom(0:50, 50, 0.5), xlab ="x", ylab = "f(x)")
```



The Poisson distribution

The special case of the binomial distribution with $n \to \infty$ and $p \to 0$, while their product is held constant at $\lambda = np$, yields the **Poisson distribution**.

The probability function is

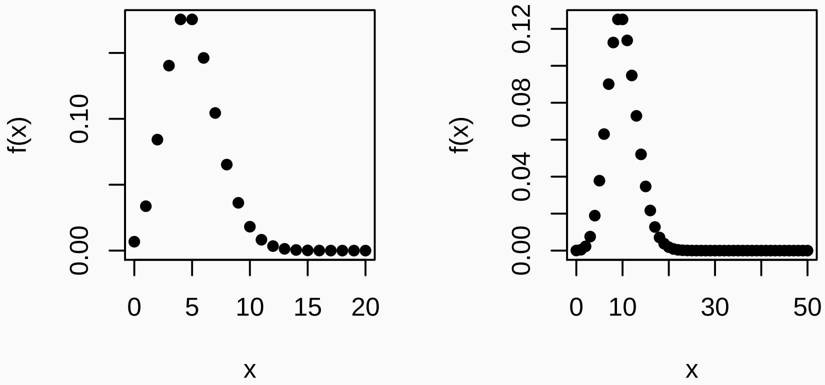
$$\Pr(X = x) = \frac{e^{-\lambda}\,\lambda^x}{x!}\,, \qquad x = 0, 1, 2, \ldots .$$

with $\lambda > 0$.

The notation is $X \sim \mathcal{P}(\lambda)$, and $E(X) = \mathrm{var}(X) = \lambda$.

R lab: the Poisson distribution

```
par(mfrow=c(1,2), pty="s", pch = 16)
plot(0:20, dpois(0:20, 5), xlab = "x", ylab = "f(x)")
plot(0:50, dpois(0:50, 10), xlab ="x", ylab = "f(x)")
```



Negative binomial distribution

Let us consider a sequence of independent Bernoulli trials with success probability p, let X be the count of trials necessary to observe the r-th success. Then X has a **Negative binomial** (or Pascal) distribution with parameters p and r.

The probability function is

$$\Pr(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x = r, r+1, r+2, \ldots.$$

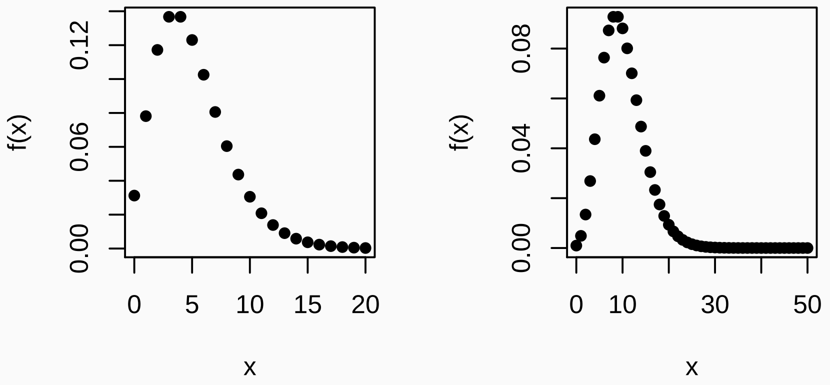
The notation is $X \sim \mathcal{NB}_i(p, r)$, and $E(X) = \frac{r}{p}$, $\mathrm{var}(X) = \frac{r(1-p)}{p^2}$.

It can also be defined with support the Natural numbers by simply considering the variable $Y = X - r$

The case for $r = 1$ is known as the **Geometric** distribution.

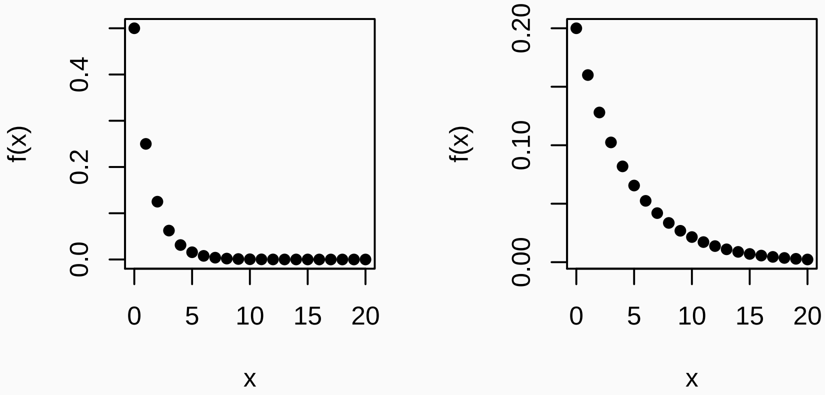
R lab: the Negative Binomial distribution

```
par(mfrow=c(1,2), pty="s", pch = 16)
plot(0:20, dnbinom(0:20, 5, 0.5), xlab = "x", ylab = "f(x)")
plot(0:50, dnbinom(0:50, 10, 0.5), xlab ="x", ylab = "f(x)")
```



R lab: the Geometric distribution

```
par(mfrow=c(1,2), pty="s", pch = 16)
plot(0:20, dnbinom(0:20, 1, 0.5), xlab = "x", ylab = "f(x)")
plot(0:20, dnbinom(0:20, 1, 0.2), xlab ="x", ylab = "f(x)")
```



Continuous distributions

2. Density functions

Continuous r.v. take values from intervals on the real line.

The **(probability) density function** (p.d.f.) of a continuous r.v. X is the function $f(x)$ such that, for any constants $a \leq b$

$$\Pr(a \leq X \leq b) = \int_a^b f(x)dx \, .$$

Note that $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

The probability density function defines the **distribution** of X.

Mean and variance of a continuous r.v.

The definitions given in the discrete case are readily extended.

The **mean (expected value)** of a continuous r.v. X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) dx\,,$$

and the definition is extended to any function g of X

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx\,.$$

This includes the **variance** as a special case.

Two results, quite useful for continuous r.v., apply to a *linear transformation* $a + b\,X$, with a, b constants:

$$\begin{aligned} E(a + b\,X) &= a + b\,E(X) \\ \mathrm{var}(a + b\,X) &= b^2\,\mathrm{var}(X)\,. \end{aligned}$$

Notable continuous random variables

Important continuous distributions include:

- Normal distribution
- Gamma, exponential and χ^2 distribution
- F distribution
- t and Cauchy distributions
- Beta distribution

The normal distribution has a major role in statistics. The χ^2, t and F distributions are *relative* of the normal distribution.

The normal distribution

A r.v. X has a normal (or *Gaussian*) distribution if it has p.d.f.

$$f(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\,\exp\left\{-\frac{1}{2\,\sigma^2}(x-\mu)^2\right\}\,, \qquad -\infty < x < \infty\,.$$

The notation is $X \sim \mathcal{N}(\mu, \sigma^2)$, and $E(X) = \mu$ and $\mathrm{var}(X) = \sigma^2$, $\sigma^2 > 0$, $\mu \in \mathbb{R}$.

An important property is that for any constants a, b

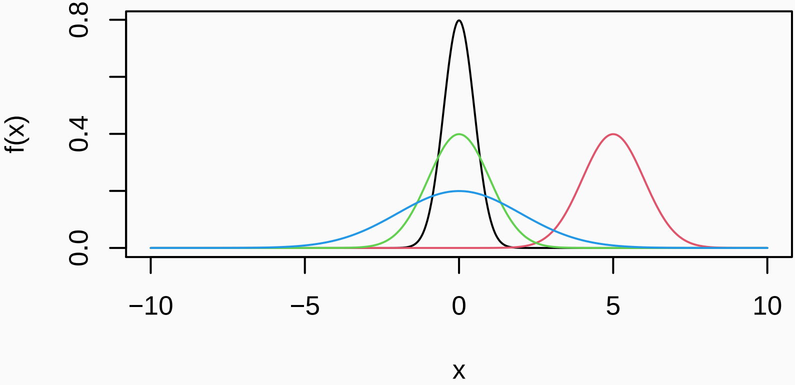
$$a + b\,X \sim \mathcal{N}(a + b\,\mu, b^2\,\sigma^2)\,,$$

so that $Z = (X - \mu)/\sigma \sim \mathcal{N}(0, 1)$, the **standard normal distribution**.

Finally, $Y = e^X$ has a **lognormal distribution**, useful for asymmetric variables with occasional right-tail outliers.

R lab: the normal distribution

```
xx <- seq(-10, 10, l=1000)
plot(xx, dnorm(xx, 0, 0.5), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  dnorm(xx, 5, 1), col = 2)
lines(xx,  dnorm(xx, 0, 1), col = 3)
lines(xx,  dnorm(xx, 0, 2), col = 4)
```



The Gamma and the exponential distributions

A r.v. X has a Gamma distribution if it has the following pdf

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} \quad e^{-\lambda x}}{\Gamma(\alpha)}, \qquad\qquad x \geq 0$$

where $\lambda, \alpha > 0$ and $\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1} dx$.

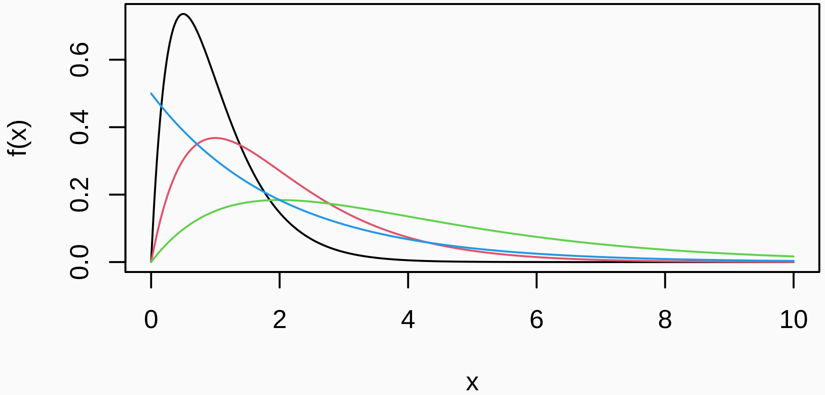
The notation is $X \sim Ga(\alpha, \lambda)$, $E(X) = \frac{\alpha}{\lambda}$ and $\mathrm{var}(X) = \frac{\alpha}{\lambda^2}$.

When α is an integer it is also called **Erlang** distribution.

When $\alpha = 1$ it is called **exponential** distribution. The exponential distribution is related to the Poisson r.v. since X represents the waiting times between two arrivals in a Poisson process (The process which generates the Poisson rv)

Rlab: The Gamma and the exponential distributions

```
xx <- seq(0, 10, l=1000)
plot(xx, dgamma(xx, 2, 2), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  dgamma(xx, 2, 1), col = 2)
lines(xx,  dgamma(xx, 2, .5), col = 3)
lines(xx,  dgamma(xx, 1, .5), col = 4) # exponential distributio
```



The Beta (and the uniform) distribution

A r.v. X has a Beta distribution if it has the following pdf

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1$$

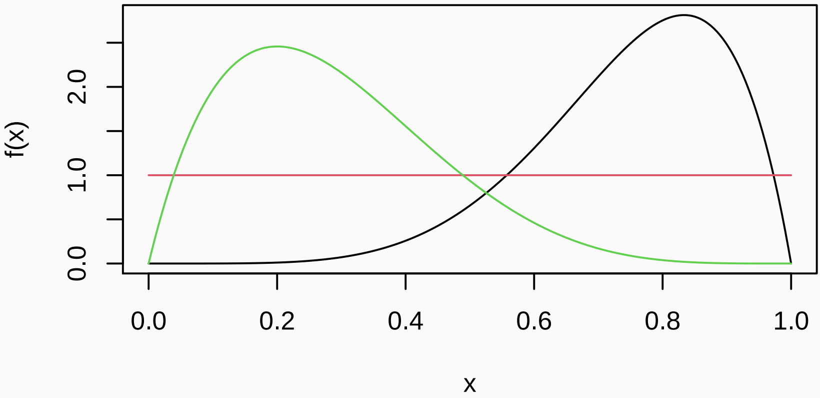
$\alpha, \beta > 0$

The notation is $X \sim Be(\alpha, \beta)$, $E(X) = \frac{\alpha}{\alpha+\beta}$ and $\mathrm{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

The **Uniform** distribution on $[0, 1]$ is a special case when $\alpha = 1$ and $\beta = 1$.

R lab: the Beta distribution

```
xx <- seq(0, 1, l=1000)
plot(xx, dbeta(xx, 6,2), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  dbeta(xx, 1,1), col = 2)
lines(xx,  dbeta(xx, 2, 5), col = 3)
```



The χ^2 distribution

Let $Z_1, \ldots, Z_k$ be a set of independent $\mathcal{N}(0,1)$ r.v., then $X = \sum_{i=1}^{k} Z_i^2$ is a r.v. with a χ^2 **distribution with** k **degrees of freedom**.

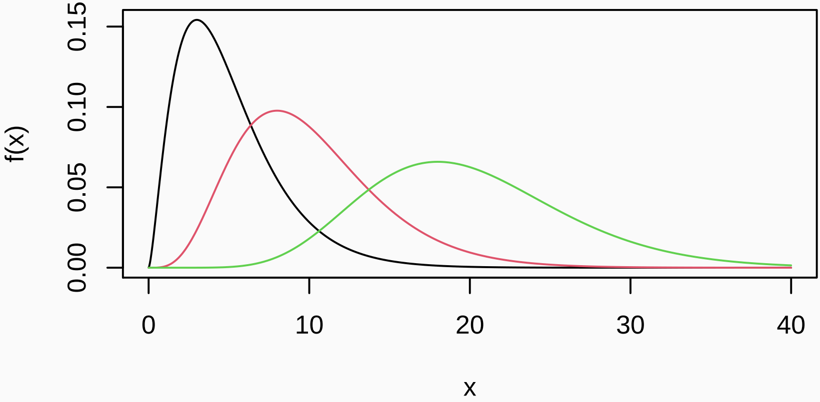
The notation is $X \sim \chi^2_k$, $E(X) = k$ and $\mathrm{var}(X) = 2k$.

It is a special case of the Gamma distribution. In fact a χ^2 distribution with k degrees of freedom is a Gamma distribution with parameters $\alpha = k/2$ and $\lambda = 1/2$.

It plays an important role in the theory of hypothesis testing in statistics.

R lab: the χ^2 distribution

```
xx <- seq(0, 40, l=1000)
plot(xx, dchisq(xx, 5), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  dchisq(xx, 10), col = 2)
lines(xx,  dchisq(xx, 20), col = 3)
```



The F distribution

Let $X \sim \chi^2_n$ and $Y \sim \chi^2_m$, independent, then the r.v.

$$F = \frac{X/n}{Y/m}$$

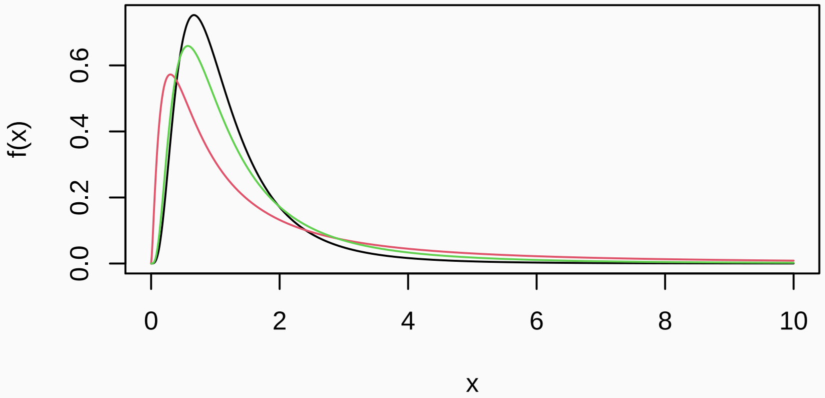
has an F **distribution with** n **and** m **degrees of freedom**.

The notation is $F \sim \mathcal{F}_{n,m}$, and $E(F) = m/(m-2)$ provided that $m > 2$.

The distribution is almost never used as a model for observed data, but it has a central role in hypothesis testing involving linear models.

R lab: the F distribution

```
xx <- seq(0, 10, l=1000)
plot(xx, df(xx, 10, 10), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  df(xx, 5, 2), col = 2)
lines(xx,  df(xx, 10, 5), col = 3)
```



The t and Cauchy distributions

Let $Z \sim \mathcal{N}(0, 1)$ and $X \sim \chi^2_n$, independent, then the r.v.

$$T = \frac{Z}{\sqrt{\frac{X}{n}}}$$

has an t **distribution with** n **degrees of freedom**.

The notation is $T \sim t_n$, and $E(T) = 0$ provided that $n > 1$, whereas $\mathrm{var}(T) = n/(n-2)$ provided that $n > 2$.

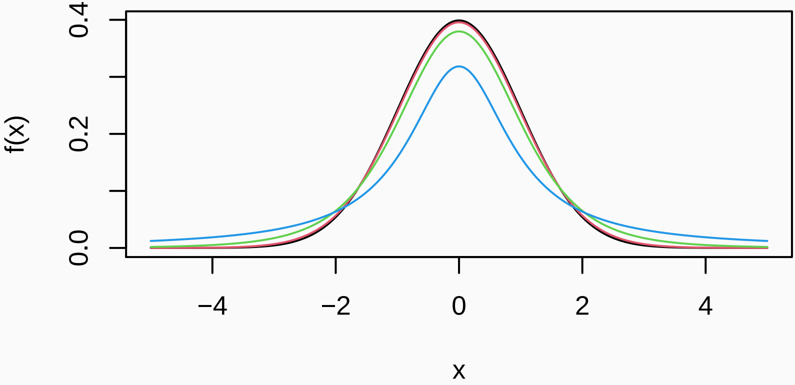
t_∞ is $\mathcal{N}(0, 1)$, while for n finite the distribution has heavier tails than the standard normal distribution.

The case t_1 is the **Cauchy distribution**.

The distribution has a central role in statistical inference; at times it is used for modelling phenomena presenting *outliers*.

R lab: the t and Cauchy distributions

```
xx <- seq(-5, 5, l=1000)
plot(xx, dnorm(xx, 0, 1), xlab ="x", ylab ="f(x)", type ="l")
lines(xx,  dt(xx, 30), col = 2)
lines(xx,  dt(xx, 5), col = 3)
lines(xx,  dt(xx, 1), col = 4)
```



C.d.f. and quantile functions

3. Cumulative distribution functions

The **cumulative distribution function** (c.d.f.) of a r.v. X is the function $F(x)$ such that

$$F(x) = \mathrm{Pr}(X \leq x)\,,$$

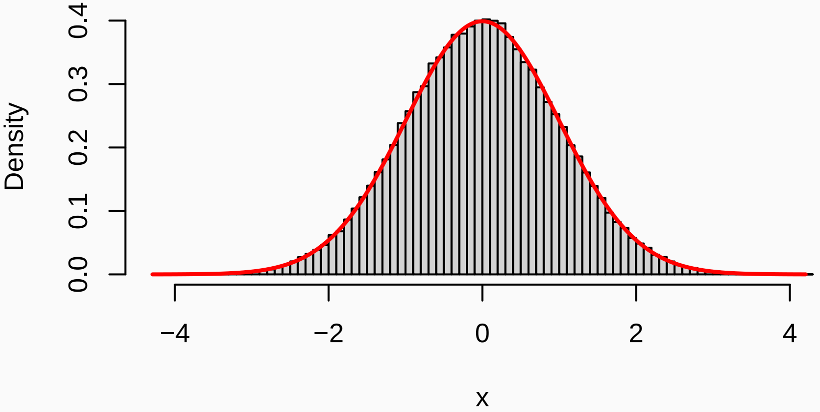
and it can be obtained from the probability function or the density function: the c.d.f. *identifies* the distribution.

From the definition of F it follows that $F(-\infty) = 0$, $F(\infty) = 1$, $F(x)$ is monotonic.

A useful property is that if F is a continuous function then $U = F(X)$ has a uniform distribution.

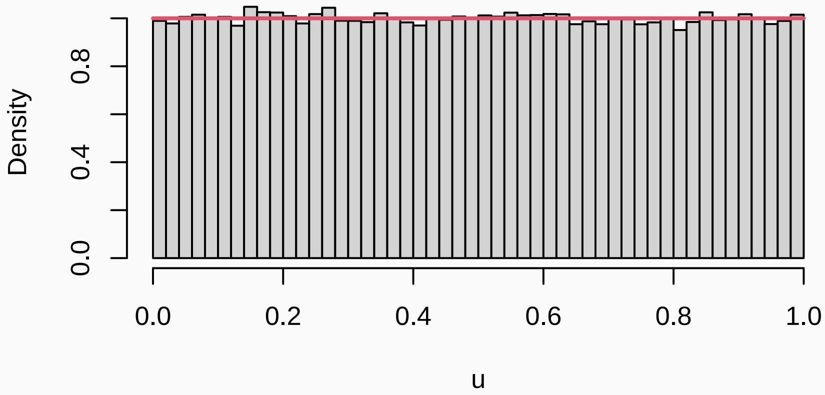
R lab: uniform transformation

```
x <- rnorm(10^5)    ### simulate values from N(0,1)
xx <- seq(min(x), max(x), l = 1000)
hist.scott(x, main = "") ### from MASS package
lines(xx, dnorm(xx), col = "red", lwd = 2)
```



R lab: uniform transformation (cont'd.)

```
u <- pnorm(x)    ### that's the uniform transformation
hist.scott(u, prob = TRUE, main="")
segments(0, 1, 1, 1, col = 2, lwd = 2)
```



The quantile function

The inverse of the c.d.f. is defined as

$$F^{-}(p) = \min\left(x | F(x) \geq p\right) , \qquad 0 \leq p \leq 1 \, .$$

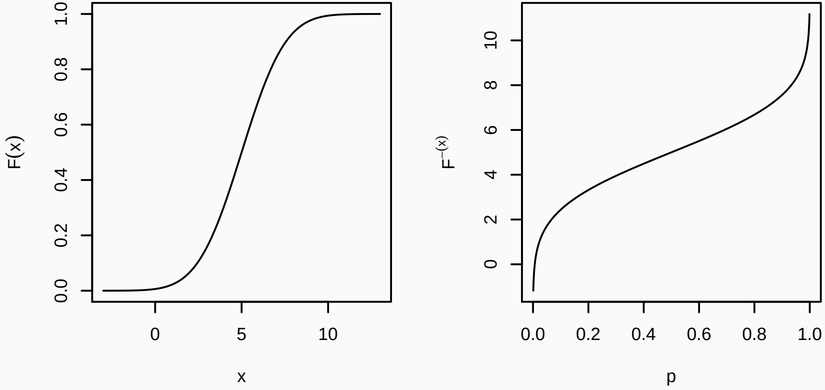
This is the usual inverse function of F when F is continuous.

Another useful property is that if $U \sim \mathcal{U}(0,1)$, namely it has a *uniform distribution* in [0,1], then the r.v. $X = F^{-}(U)$ has c.d.f. F.

This provides a simple method to generate random numbers from a distribution with known quantile function: it is the **inversion sampling method**, that only requires the ability to simulate from a uniform distribution.

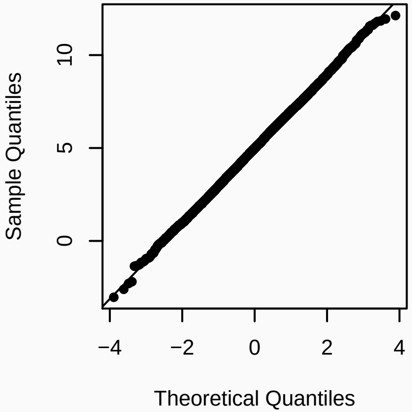
Example: normal cdf and quantile functions

Let us consider the case of $X \sim \mathcal{N}(5, 2^2)$, with c.d.f. and quantile functions given by `pnorm` and `qnorm`



R lab: inversion sampling

```
u <- runif(10^4); y <- qnorm(u, m = 5, s = 2)
par(pty = "s",  cex = 0.8)
qqnorm(y, pch = 16, main = "")
qqline(y)
```



Side note: quantile-quantile plot

The previous slide demonstrated the usage of the quantile function to build a tool for **model goodness-of-fit**.

The *quantile-quantile plot* visualizes the plausibility of a theoretical distribution for a set of observations $y = (y_1, \ldots, y_n)$.

This is done by comparing the quantile function of the assumed model with the sample quantiles, which are the points that lie on the inverse of the **empirical distribution function**

$$\widehat{F}_n(t) = \frac{\text{number of elements of } y \leq t}{n}.$$

If the agreement between the data and the theoretical distribution is good, the points on the plot would approximately lie on a line.