4 One-dimensional electron gas

Consider a system of N noninteracting point fermions moving in 1D in [0,L]. Let's study the problem with each of the following boundary conditions:

(a) Born — von Karman (also know as PBC): $\psi(x+L) = \psi(x)$

(b) Hard wall $\psi(L) = \psi(0) = 0$.

In each of cases (a) and (b):

1. Give eigenfunctions and eigenvalues.
2. Calculate the one body density $n(x)$ giving also a qualitative plot and discussing what happens in the thermodynamic limit ($N,\ L \longrightarrow \infty$; $N/L = n$).
3. Calculate the Fermi energy $E_F(N,L)$ and its behavior in the thermodynamic limit.
4. Calculate the total energy $E_{tot}(N,L)$. Discuss whether it is possible to break it in a *volume* term
$$E_V(N,L) = N\mathcal{E}_V\left(\frac{N}{L}\right)$$
and a *surface* term
$$E_S(N,L) = \mathcal{E}_S\left(\frac{N}{L}\right),$$
plus terms that vanish in the thermodynamic limit. Check if there is a relation between the total energy and the Fermi energy.

Suggestion

1. It might be useful to remember:
$$\sum_{n=1}^{m} n^2 = \frac{1}{6}m(m+1)(2m+1)$$
$$\sum_{n=1}^{m} a^n = \frac{a - a^{m+1}}{1-a}$$
2. and the definition of the Bessel function:
$$j_0(x) = \frac{\sin x}{x}.$$