## ES. 1

3. $\bar{q} \to \bar{\varphi}(\alpha, \bar{q})$ $\qquad P = \sum_{h=1}^{n} p_h \frac{\partial \varphi_h}{\partial \alpha}$

$q_1$ è coord. ciclica $\Rightarrow \varphi_h(\alpha, \bar{q}) = q_h + \alpha \delta_{h1}$ è simm. di L

$$P = \sum_{h=1}^{n} p_h \delta_{h1} = p_1$$ mom. coniugato alla coord. ciclica.

4. se $P$ è ind. da $t$

$P$ è cost. del moto $\iff \{P, H\} = 0$

5. coord. ciclica non comp. espl. in $H$, cioè $\frac{\partial H}{\partial q_1} = 0 \iff$

$\iff q_1$ è ciclica

$$\{p_1, H\} = \sum_{h=1}^{n} \Big( \underbrace{\frac{\partial p_1}{\partial q_h}}_{=0} \frac{\partial H}{\partial p_h} - \underbrace{\frac{\partial p_1}{\partial p_h}}_{=\delta_{1h}} \frac{\partial H}{\partial q_h} \Big) = - \frac{\partial H}{\partial q_1} = 0$$

6. Teor. di Arnold, $\quad M_z, \bar{M}^2, H$ sono 3 cost. del moto in involuzione.

ES. 2

$$\bar{A} = \frac{B}{2}\begin{pmatrix} -q_2 \\ q_1 \\ 0 \end{pmatrix} \qquad \bar{B} = \nabla \times \bar{A} = B\bar{e}_z$$

$$L = \frac{1}{2}m\dot{\bar{q}}^2 + e\dot{\bar{q}}\cdot\bar{A} = \frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2) + \frac{eB}{2}(q_1\dot{q}_2 - q_2\dot{q}_1)$$

1. $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \frac{d}{dt}\left(m\dot{q}_1 - \frac{eB}{2}q_2\right) = m\ddot{q}_1 - \frac{eB}{2}\dot{q}_2$

$\frac{\partial L}{\partial q_1} = \frac{eB}{2}\dot{q}_2 \qquad \Big\} \to \quad m\ddot{q}_1 - eB\dot{q}_2 = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = \frac{d}{dt}\left(m\dot{q}_2 + \frac{eB}{2}q_1\right) = m\ddot{q}_2 + \frac{eB}{2}\dot{q}_1$

$\frac{\partial L}{\partial q_2} = -\frac{eB}{2}\dot{q}_1 \qquad \Big\} \to \quad m\ddot{q}_2 + eB\dot{q}_1 = 0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_3} = \frac{d}{dt}(m\dot{q}_3) = m\ddot{q}_3$

$\frac{\partial L}{\partial q_3} = 0 \qquad \to \quad \ddot{q}_3 = 0$

2) $q_1 = r\cos\varphi \qquad \dot{q}_1 = \dot{r}\cos\varphi - r\dot{\varphi}\operatorname{sen}\varphi$

$q_2 = r\operatorname{sen}\varphi \qquad \dot{q}_2 = \dot{r}\operatorname{sen}\varphi + r\dot{\varphi}\cos\varphi$

$q_3 = z \qquad \dot{q}_3 = \dot{z}$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2) + \frac{eB}{2}r^2\dot{\varphi}$$

$\frac{eB}{2}(q_1\dot{q}_2 - q_2\dot{q}_1) = \frac{eB}{2}\big(r\cos\varphi(\dot{r}\cancel{\operatorname{sen}\varphi} + r\dot{\varphi}\cos\varphi)$
$- r\operatorname{sen}\varphi(\dot{r}\cancel{\cos\varphi} - r\dot{\varphi}\operatorname{sen}\varphi)\big) = r^2\dot{\varphi}\frac{eB}{2}$

$p_r = m\dot{r}$

$L = L_{\text{di particella}} + V(r)$

3. coord. cicliche: $\varphi$, $z$

$M_z \to p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} + \frac{eB}{2}r^2 \leftarrow$ rotaz. attorno asse $z$

$p_z = m\dot z \quad \leftarrow$ traslaz. lungo asse $z$

$p_z$ impulso

4. Sist. indip. da $t$, cioè $\frac{\partial L}{\partial t} = 0 \rightarrow$ energia è cost. del moto

$$E = \sum_{h=1}^{n} \dot q_h \underbrace{\frac{\partial L}{\partial \dot q_h}}_{p_h} - L = \dot r\, m\dot r + \dot\varphi\left(mr^2\dot\varphi + \cancel{\frac{eB}{2}r^2}\right) + \dot z(m\dot z)$$

$$-\frac{1}{2}(m\dot r^2 + mr^2\dot\varphi^2 + m\dot z^2) - \cancel{\frac{eB}{2}r^2\dot\varphi} + V(r)$$

$$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\varphi^2 + \frac{1}{2}m\dot z^2 + V(r)$$

5. $\dot z = \frac{p_z}{m} \qquad p_z$ cost.

$$\dot\varphi = \frac{\left(p_\varphi - \frac{eB}{2}r^2\right)}{mr^2} \qquad p_\varphi \text{ cost.}$$

$$L^* = \left(L - p_z\dot z - p_\varphi\dot\varphi\right)\Big|_{\substack{\dot z = \dots \\ \dot\varphi = \dots}}$$

$$V(r) = \frac{\alpha}{m}\ln\left(\frac{r}{a}\right)$$

$$= \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\frac{\left(p_\varphi - \frac{eB}{2}r^2\right)^2}{(mr^2)^2} + \frac{1}{2}m\frac{p_z^2}{m^2} + \frac{eB}{2}r^2\frac{\left(p_\varphi - \frac{eB}{2}r^2\right)}{mr^2}$$

$$-\frac{p_z^2}{m} - p_\varphi\frac{\left(p_\varphi - \frac{eB}{2}r^2\right)}{mr^2} - V(r)$$

$$= \frac{m}{2}\dot r^2 + \frac{\left(p_\varphi - \frac{eB}{2}r^2\right)^2}{2mr^2} + \frac{p_z^2}{2m} - V(r) - \frac{\left(p_\varphi - \frac{eB}{2}r^2\right)\left(p_\varphi - \frac{eB}{2}r^2\right)}{mr^2}$$

$$= \frac{m\dot r^2}{2} - V(r) - \frac{1}{2mr^2}\left(p_\varphi - \frac{eB}{2}r^2\right)^2 + \cancel{\frac{p_z^2}{2m}}$$

trascuro pezzi costanti

6. $$V_{eff} = \frac{\alpha}{m}\ln\left(\frac{r}{a}\right) + \frac{1}{2mr^2}\left(p_\varphi - \frac{eB}{2}r^2\right)^2$$

$$= \frac{\alpha}{m}\ln\left(\frac{r}{a}\right) + \frac{1}{2mr^2}\left(p_\varphi^2 - eBp_\varphi r^2 + \frac{(eB)^2}{4}r^4\right)$$

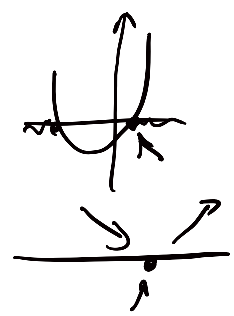
$$= \frac{\alpha}{m}\ln\left(\frac{r}{a}\right) + \frac{p_\varphi^2}{2mr^2} + \frac{e^2B^2}{8m}r^2 - \cancel{\frac{eBp_\varphi}{2m}}$$

trascuro perché cost.

$$V_{eff}'(r) = \frac{\alpha}{mr} - \frac{p_\varphi^2}{mr^3} + \frac{e^2B^2 r}{4m} = 0$$

$$= \frac{1}{mr^3}\left(\alpha r^2 - p_\varphi^2 + \frac{e^2B^2}{4}r^4\right) = 0$$

$$= \frac{e^2B^2}{4mr^3}\left(\underbrace{r^4 + \frac{4\alpha}{e^2B^2}r^2 - \frac{4p_\varphi^2}{e^2B^2}}_{=0}\right) = 0$$




$$(r^2)_{1/2} = -\frac{2\alpha}{(eB)^2} \pm \sqrt{\left(\frac{2\alpha}{(eB)^2}\right)^2 + \frac{4p_\varphi^2}{(eB)^2}}$$

$r^2 > 0 \Rightarrow$ quindi posso accettare solo la soluzione col $+$

$$r_0 = \sqrt{\frac{-2\alpha}{(eB)^2} + \sqrt{\frac{4\alpha^2}{(eB)^4} + \frac{4p_\varphi^2}{(eB)^2}}}$$

$V(r) = \frac{\alpha}{mr} - \frac{p_\varphi^2}{mr^3} + \frac{e^2 B^2 r}{4m}$

7. $V''_{eff}(r) = -\frac{\alpha}{mr^2} + \frac{3p_\varphi^2}{mr^4} + \frac{e^2B^2}{4m}$

$$V''_{eff}(r_0) = -\frac{\alpha}{mr_0^2} + \frac{3p_\varphi^2}{mr_0^4} + \frac{e^2B^2}{4m}$$

$$\hat{L}^* = \frac{1}{2}m\dot{r}^2 - \frac{1}{2}V''_{eff}(r_0)(r-r_0)^2$$

$$\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$$

$$\omega^2 = \frac{V''_{eff}(r_0)}{m}$$

8.

$$L = \frac{1}{2}m\dot{\vec{q}}^2 + e\dot{\vec{q}}\cdot\vec{A} - V(r) = \frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2) + \frac{eB}{2}(q_1\dot{q}_2 - q_2\dot{q}_1) - V(r)$$

Sist. è inv. per traslazioni in $\mathbb{R}^3$

$\hookrightarrow$ 3 parametri $\to$ 3 cost. del moto

$$L(\dot{q}_1, \dot{q}_2, \dot{q}_3, q_1+\alpha_1, q_2+\alpha_2, q_3) =$$

$$= \frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2) + \frac{eB}{2}(q_1\dot{q}_2 - q_2\dot{q}_1) +$$

$$+ \frac{eB}{2}(\alpha_1\dot{q}_2 - \alpha_2\dot{q}_1) =$$

$$L + \underbrace{\frac{eB}{2}(\alpha_1\dot{q}_2 - \alpha_2\dot{q}_1)}_{\text{se valutato su un moto}} = \frac{d}{dt}\left(\frac{eB}{2}(\alpha_1 q_2 - \alpha_2 q_1)\right)$$