GENERAL SCHROEDINGER EQUATION

$$H(p,x) = \frac{p^2}{2m} + V(x) = E \qquad \text{Hamiltoniana}$$

operatori

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x,t)\psi(x,t)} \qquad \text{Eq. Schrödinger}$$

ASSUME NOW:

$$V(x,t) \to V(x)$$

potential is only space-dependent

and:

$$\psi(x,t) \to \phi(x)\,\tau(t) \Rightarrow \begin{cases} \dfrac{\partial}{\partial t}\psi = \phi(x)\dfrac{\partial \tau(t)}{\partial t} \\[2ex] \dfrac{\partial^2}{\partial x^2}\psi = \tau(t)\dfrac{\partial^2 \phi(x)}{\partial x^2} \end{cases}$$

wavefunction is separable in x and t

With the assumption that the wavefunction is separable and that the potential is space-dependent only, the Schroedinger equation becomes:

$$\phi = \phi(x)$$

$$\tau = \tau(t)$$

$$i\hbar \frac{\partial \tau}{\partial t} \cdot \phi = -\frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} \cdot \tau + V(x) \cdot \phi \tau$$

so we can separate the variables:

$$\frac{i\hbar}{\tau} \frac{\partial \tau}{\partial t} = -\frac{\hbar^2}{2m} \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} + V(x) = E \quad (\text{costante})$$

$$\Rightarrow \begin{cases} \boxed{\dfrac{i\hbar}{\tau} \dfrac{\partial \tau}{\partial t} = E} \\ \boxed{-\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \phi}{\partial x^2} + V(x) \cdot \phi = E \phi} \end{cases}$$

parte tempo-dipendente

$$\Rightarrow \int \frac{\partial \tau}{\tau} = \int \frac{E}{i\hbar} dt \longrightarrow \boxed{\tau(t) = A e^{-\frac{iEt}{\hbar}}}$$

general form of the time-dependent portion of the wavefuntion

time-independent Schroedinger equation (we need V(x) to solve it) we can write it in a different form, grouping the operators on the function

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y}+V(x)\right)\phi(x)=E\,\phi(x)$$

TIME INDIPENDENT SCHRÖDINGER EQ.

we can define the Hamiltonian operator:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)=\hat{H} \quad\Rightarrow\quad \boxed{\hat{H}\,\phi(x)=E\,\phi(x)}$$

FREE PARTICLE - a first example of solving Schroedinger equation

$V(x) = 0$

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\phi(x) = \phi(x)\cdot E \Rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \phi}{\partial x^2} = E\cdot\phi} \; (*)$$

funzione di prova

$$\boxed{\phi = e^{ikx}} \Rightarrow \frac{\partial \phi}{\partial x} = ik\,e^{ikx} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = -k^2 e^{ikx} = -k^2\phi$$

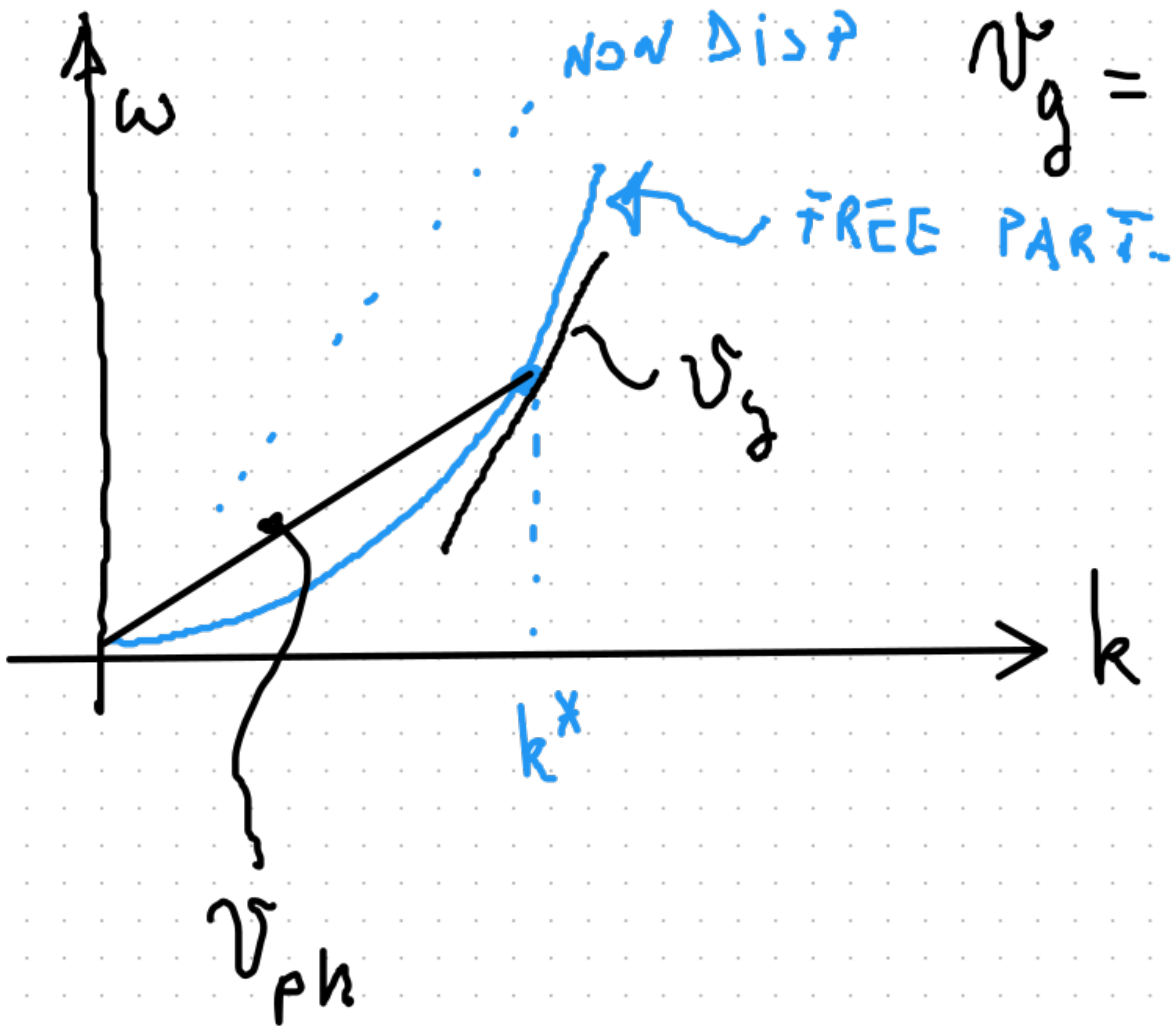
$$(*) \quad \frac{\hbar^2}{2m}k^2\phi = E\phi \Rightarrow \boxed{E = \frac{\hbar^2 k^2}{2m}} \qquad E = h\nu = \hbar\omega$$

$$\Rightarrow \boxed{\omega = \frac{\hbar}{2m}k^2}$$ relazione di dispersione

Alcune considerazioni sulla relazione di dispersione per una particella libera

$$\omega = \frac{\hbar}{2m} k^2 \longrightarrow v_{ph} = \frac{\omega}{k} = \frac{\hbar k}{2m}$$

relazione quadratica e quindi dispersiva!

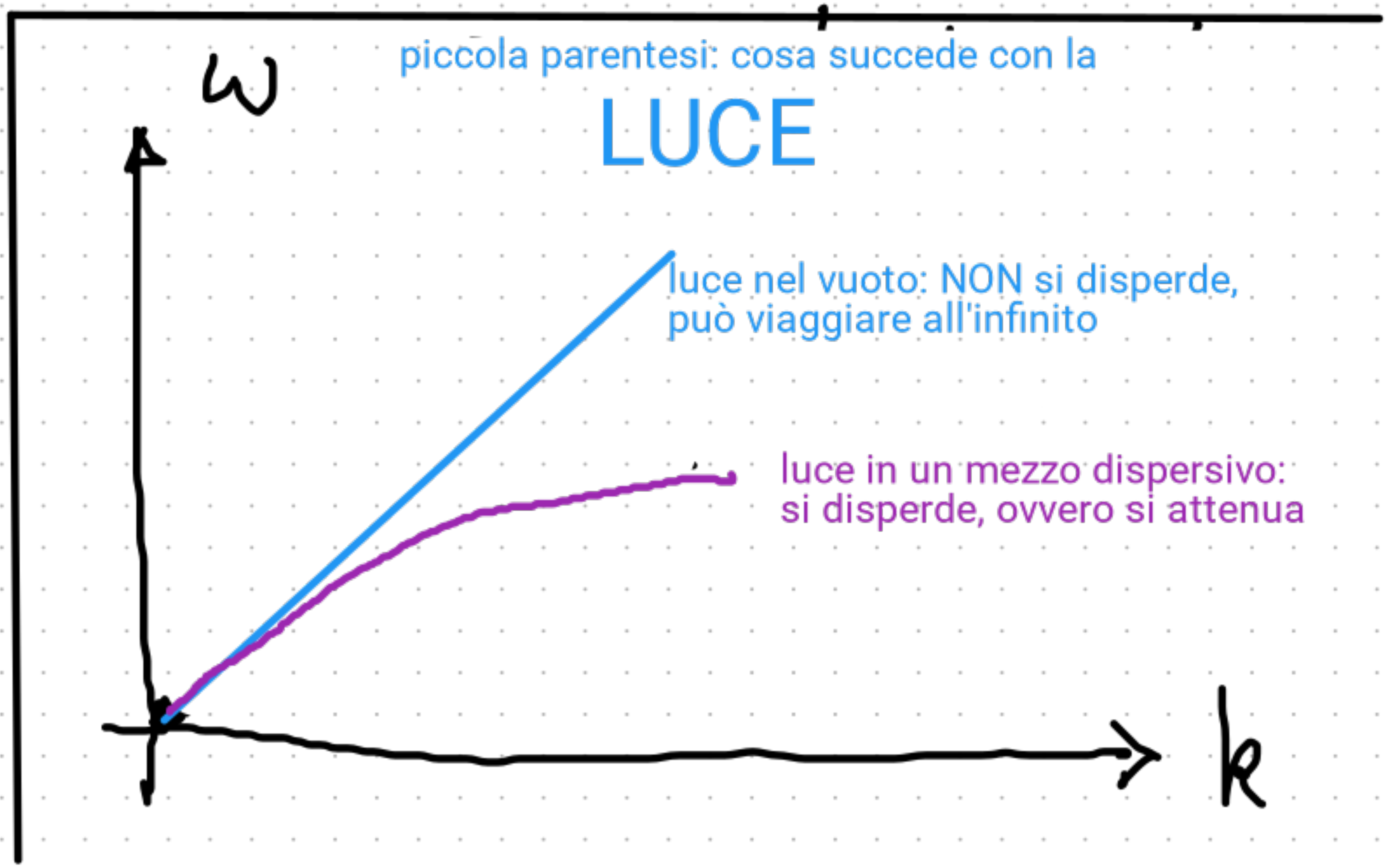


$$v_g = \frac{\partial \omega}{\partial k} = \frac{\hbar k}{m}$$

$$v_g = 2 v_{ph}$$

piccola parentesi: cosa succede con la

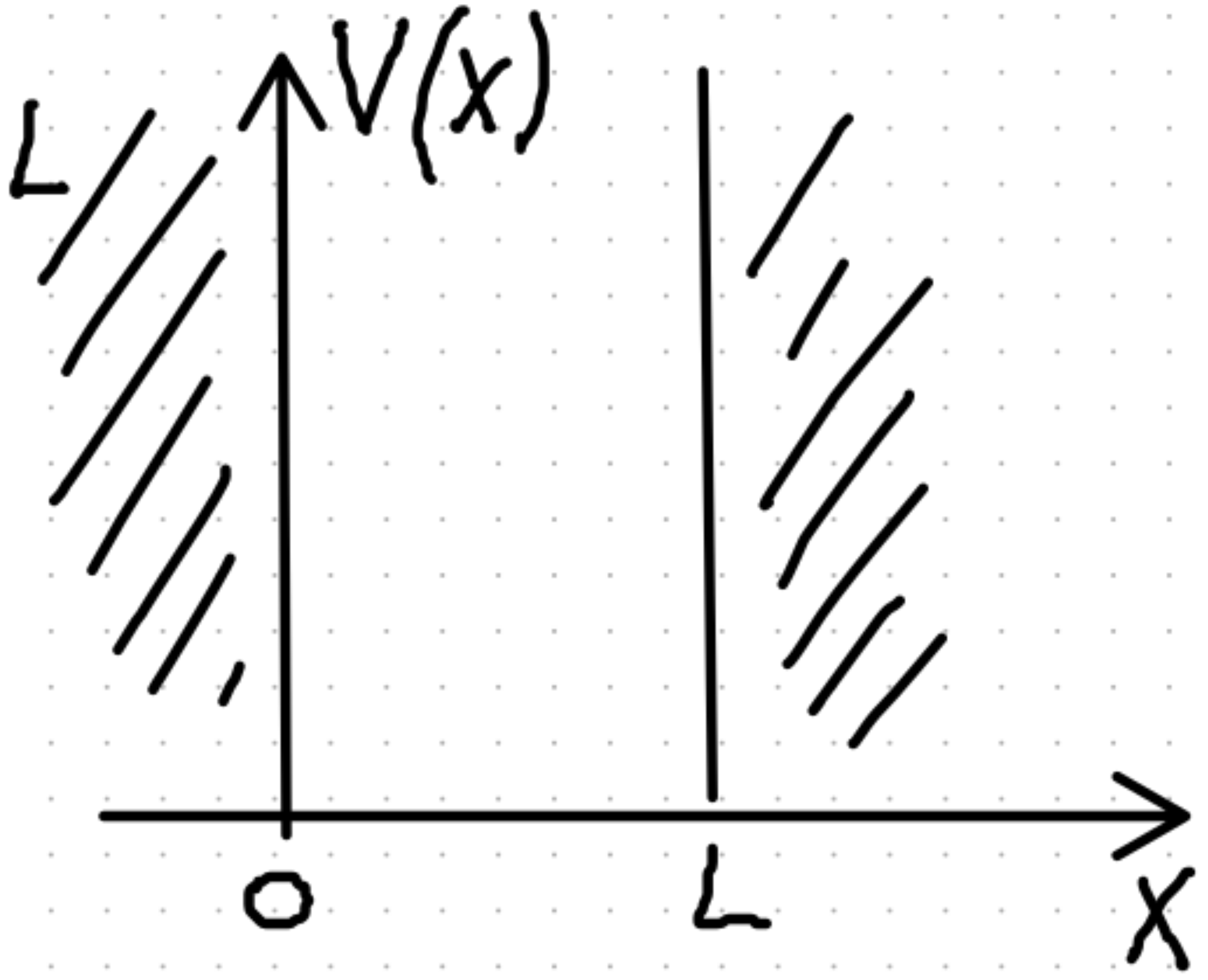
LUCE



another important example:

PARTICLE IN A BOX

$$\rightarrow \quad V(x) = \begin{cases} V = 0 & 0 < x < L \\ V = \infty & x \leq 0 \\ & x \geq L \end{cases}$$



Per $0 < x < L$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi = E\phi$$

uso come funzione di prova:

$$\phi(x) = A \sin kx + B \cos kx \Rightarrow$$

condizioni al contorno:

$$\begin{cases} \phi(0) = 0 \Rightarrow B = 0 \\ \phi(L) = 0 \Rightarrow A \sin kL = 0 \quad k = \frac{n\pi}{L} \end{cases}$$

condizione di normalizzazione:

$$\int_{-\infty}^{\infty} |\phi|^2 dx = 1 \quad \therefore \quad \int_{-\infty}^{\infty} A^2 \sin^2 kx\, dx = \int_0^L A^2 \left(\frac{1 - \cos 2kx}{2}\right) dx = 1$$

...continua...

...segue dalla lavagna precedente...

$$\Rightarrow \frac{A^2}{2}\left[x - \frac{\sin 2kx}{2k}\right]_0^L = \frac{A^2}{2}\left[L - \cancel{\frac{\sin 2kL}{2k}} - \cancel{0} + \cancel{\frac{\sin 2k\,0}{2k}}\right] = 1$$

$$\Rightarrow \frac{A^2}{2}L = 1 \qquad A = \sqrt{\frac{2}{L}}$$

$$\Rightarrow \boxed{\phi = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x}$$

Forma generale delle funzioni d'onda che descrivono una particella in una scatola

Sostituisco in $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\phi = E\phi$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = -k^2\phi \Rightarrow E = \frac{\hbar^2 k^2}{2m}$$

$$\boxed{E = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{L^2}} \Rightarrow \boxed{\omega = \frac{\hbar}{2m}\frac{n^2\pi^2}{L^2}}$$

valori di energia permessi (sono discreti!!!)

relazione di dispersione