Lock-in amplifier

Let us take a signal V(t) that is modulated at the frequency ω that we know with a phase ϕ that we know. We can write it as a Fourier series

$$V(t) = V_0 + \sum_{n=1}^{\infty} V_n \sin(n\omega t + \phi_n)$$

There will always be some additional noise N(t) that we can write in terms of its Fourier transform N(ω)

$$N(t) = \int_{-\infty}^{+\infty} N(\omega')\sin(\omega' t + \phi_{\omega'})d\omega'$$

What we measure is $V_{tot}(t)$, the sum of the signal that we want to study and of the noise

$V_{tot}(t) = V(t) + N(t)$

Even if the noise is much larger than the signal, we can measure the signal in the following way. First we multiply the measured $V_{tot}(t)$ by a reference signal $R(t) = R\sin(\omega t + \phi')$. The reference signal must have the same frequency of the first harmonic of V(t) and the difference $\phi' - \phi_1$ must be constant.

$$V_{tot}(t)\, R(t) = V_0 R\sin(\omega t + \phi') +$$
$$+\frac{R}{2}\sum_{n=1}^{\infty} V_n(\cos((n-1)\omega t + (\phi_n - \varphi')) \; - \cos((n+1)\omega t + \phi_n + \phi')) +$$
$$+\frac{R}{2}\int_{-\infty}^{+\infty} N(\omega')(\cos((\omega' - \omega)t + \phi_{\omega'} - \phi') - (\cos((\omega' + \omega)t + \phi_{\omega'} + \phi'))d\omega'$$

This expression contains a constant term in the summation, i.e. $\frac{R}{2} V_1\cos(\phi_1 - \phi')$.

Let's calculate the time average of $V_{tot}(t)$ R(t), that we call U(t), i.e. integrate this product in time from t=0 to t=T and divide by T

$$U(T) = \frac{1}{T}\int_0^T V_{tot}(t)R(t)dt \; = \frac{R}{2} V_1\cos(\phi_1 - \phi') + RF(T)$$

F(T) contains a term that in modulus is $\leq 2\frac{V_0}{T\omega}$, a summation of terms that in modulus are $\leq \frac{V_n}{T(n \pm 1)\omega}$, and an integral of terms that in modulus are $\leq \frac{N_{\omega'}}{T(\omega \pm \omega')}$.

In the limit T -> ∞ F(T) -> 0.

Therefore if we multiply the noisy measured signal by a sinusoidal signal with the same frequency and a constant phase shift with respect of the modulation, and we average the product for a time long enough, we measure the first harmonic of the modulated signal and all the rest – noise + higher harmonics- is filtered out. Only the noise at the same frequency of the modulation can survive, but only if its phase remains constant.

In principle, from the mathematics, a signal can be measured in this way, no matter how large the noise is. In the real world you need an analog or digital electronic device to generate the reference signal, multiply it by the measured signal and

integrate the product. Any electronic circuit has some non-linearity, offset, saturation… and these unavoidable defects limit the accuracy of the treatment of the signal. Even the digital systems introduce errors, truncation errors, limited number of bits generated by the analog to digital converter that reads the measured signal….
The device that perform the operations described above is called lock-in amplifier.
If it uses analog circuits it can measure signals that are 10^4 times smaller than the noise, if it uses digital circuits this limit reaches 10^6. This number is called the dynamical reserve of the lock-in. These numbers are valid for noises with appreciable Fourier components not to close to the modulation frequency ω. But, for instance, if $\omega = 1000\ s^{-1}$, you are integrating for 1 s, and the noise as $\omega' = 999\ s^{-1}$, the noise in not killed because $\frac{1}{T(\omega - \omega')}$ is 1. In this case you have to integrate for much longer times (T>>1 s) to kill this noise. For a noise frequency very close to the modulation frequency ω the dynamical reserve is much less than 10^4 or 10^6 and it depends on T.

Let us take I(U(t)) and put $U(t) = U_0 + \Delta U \sin(\omega t)$ with $|\Delta U/U_0| \ll 1$ (we are modulating U at the frequency ω). Therefore

$$I(t) = I(U_0) + \frac{dI}{dU} \Delta U \sin(\omega t) + \frac{1}{2}\frac{d^2 I}{dU^2}(\Delta U \sin(\omega t))^2 + \ldots\ldots$$

The third term contains the second harmonic, the fourth the third harmonic, and so on.
If we use Rsin(ωt) as a reference signal for the lock-in we can measure $\frac{dI}{dU}$, if we use Rsin(2ωt) as a reference we can measure $\frac{d^2 I}{dU^2}$, and so on.

A lock-in amplifier contains filters to cut the 50 Hz noise (60 Hz in USA) and its second harmonic, filters to cut frequencies far from the reference frequency, amplifiers to amplify the signal, an oscillator to generate the reference signal - the frequency and the phase can be set by the user, and the reference signal send out to modulate the system to be measured, or can be given by an external oscillator that modulate the system to be measured -, a circuit that multiply the signal by the reference, and a circuit that integrate the result of the multiplication.

The choice of the amplification, of the phase, of the amount of the modulation, of the integration time, of the kind of dynamical reserve to use, etc. depends on the experimental conditions and has to be optimized each time.