5 ottobre

Robinson Rodrigo Sadoski

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} f(x)\,dx$$

$$\mathcal{F}^* f(x) = \check{f}(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\xi} f(\xi)\,d\xi$$

$$\mathcal{F}[\partial_j f] = -i\xi_j \hat{f}$$

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}} \hat{f}(\xi)\, \hat{g}(\xi)$$

$$C_0^0(\mathbb{R}^d, \mathbb{R}) = \left\{ f \in C^0(\mathbb{R}^d, \mathbb{R}) : \lim_{x\to\infty} f(x) = 0 \right\}$$

$$C_c^0$$

$$e^{-\frac{\varepsilon|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-\frac{|x|^2}{2\varepsilon}}\,dx$$

$$\begin{cases} \partial_t u - \Delta u = 0 & t>0 \qquad \Delta = \sum_{j=1}^{d} \partial_j \partial_j \\ u(0,x) = f \in \mathcal{S}'(\mathbb{R}^d) \end{cases}$$

$\mathcal{F}_x$

$$\begin{cases} \partial_t \hat{u} + |\xi|^2 \hat{u} = 0 \\ \hat{u}(0,\xi) = \hat{f}(\xi) \end{cases} \qquad e^{t|\xi|^2}$$

$$e^{t|\xi|^2} \partial_t \hat{u} + e^{t|\xi|^2} |\xi|^2 \hat{u} = 0$$

$$\partial_t \left(e^{t|\xi|^2} \hat{u} \right) = 0 \,, \quad e^{t|\xi|^2} \hat{u}(t,\xi) = \hat{f}(\xi)$$

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{f}(\xi)$$

$$u(t,x) = \mathcal{F}^* \left(e^{-t|\xi|^2} \hat{f} \right) = (2\pi)^{-\frac{d}{2}} \underline{\mathcal{F}^* e^{-t|\xi|^2} * f}$$

$$\hat{f}\,\hat{g} = (2\pi)^{-\frac{d}{2}} \widehat{f * g}$$

$$\mathcal{F}^* \left(e^{-t|\xi|^2} \right) \overset{(x)}{=} (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$\varepsilon = 2t$

$$e^{-\frac{\varepsilon|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\xi x} e^{-\frac{|x|^2}{2\varepsilon}} dx \quad *$$

$$e^{-t|\xi|^2} = \mathcal{F} \, (2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$$u(t,x) = (4t\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

$$K_t(x) = (4t\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} \qquad \int_{\mathbb{R}^d} K_t(x)\, dx = 1$$

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} f(y)\, dy$$

$$\left(F(\Delta) f\right)^\wedge = F(-|\xi|^2)\, \hat{f}$$

$$|e^{t\Delta} f|_{L^p(\mathbb{R}^d)} \leq |f|_{L^p(\mathbb{R}^d)} \quad \forall\, p \in [1, +\infty]$$

$$|K_t * f|_{L^p(\mathbb{R}^d)} \leq \underbrace{|K_t|_{L^1}}_{1} \, |f|_{L^p}$$

$$|e^{t\Delta} f|_{L^\infty(\mathbb{R}^d)} \leq (4\pi t)^{-\frac{d}{2}} \, |f|_{L^1(\mathbb{R}^d)}$$

$$|e^{t\Delta} f(x)| \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(y)| \, dy$$

$$e^{tA}$$

A

Lemma $\forall\ q \geq p \geq 1$ e $\forall\ j \geq 0$

$\exists\ C_{jpq}$ t.c.

$$\left| \nabla^j e^{t\Delta} f \right|_{L^q(\mathbb{R}^d)} \leq C_{jpq}\, t^{-\frac{j}{2} - \frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} |f|_{L^p(\mathbb{R}^d)}$$

Dim solo $j = 0$

$$\left| e^{t\Delta} f \right|_{L^q} = \left| K_t * f \right|_{L^q} \leq |K_t|_{L^a} |f|_{L^p}$$

$$\frac{1}{q} + 1 = \frac{1}{a} + \frac{1}{p} \qquad \frac{1}{2a} = \frac{1}{2} + \frac{1}{2q} - \frac{1}{2p}$$

$$|K_t|_{L^a} = (4\pi t)^{-\frac{d}{2}} \left| e^{-\frac{|x|^2}{4t}} \right|_{L^a} =$$

$$= (4\pi)^{-\frac{d}{2}}\, t^{-\frac{d}{2}} \left(t^{\frac{1}{2}}\right)^{\frac{d}{a}} \left| e^{-\frac{|x|^2}{4}} \right|_{L^a}$$

$$= C\, t^{d\left(\frac{1}{2a} - \frac{1}{2}\right)} = C\, t^{\frac{d}{2}\left(\frac{1}{q} - \frac{1}{p}\right)}$$

$$= C\, t^{-\frac{d}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}$$

$\rho \in L^1(\mathbb{R}^d)$ $\quad \int \rho \, dx = 1$

$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$ $\quad \varepsilon > 0$

$f \in L^p(\mathbb{R}^d)$ $\quad f * \rho_\varepsilon$

Se $p < +\infty$ si ha

$$\lim_{\varepsilon \to 0^+} f * \rho_\varepsilon = f \quad \text{in } L^p(\mathbb{R}^d) \quad (*)$$

Se $f \in C_0^0(\mathbb{R}^d)$ allora $(*)$ vale per $p = \infty$

$$e^{t\Delta} f = K_t * f \qquad K_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$$

$$= \rho_\varepsilon \quad \varepsilon = t^{\frac{1}{2}}$$

$e^{t\Delta} f \to f$ in $L^p(\mathbb{R}^d)$ $\quad \rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$

$\forall \; p < +\infty$

e in $C_0^0(\mathbb{R}^d) \left(\subset L^\infty(\mathbb{R}^d)\right)$

$$H^k(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \partial_x^\alpha f \in L^2(\mathbb{R}^d) \ \forall\ |\alpha| \le k \}$$

$$f \in H^k(\mathbb{R}^d) \Longleftrightarrow \xi^\alpha \hat{f} \in L^2(\mathbb{R}^d) \quad \forall\ |\alpha| \le k$$

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}$$

$$f \in H^k(\mathbb{R}^d) \Longleftrightarrow \langle \xi \rangle^k \hat{f} \in L^2(\mathbb{R}^d)$$

$\forall\ s \in \mathbb{R}$

$$H^s(\mathbb{R}^d) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \langle \xi \rangle^s \hat{f} \in \underline{L^2(\mathbb{R}^d)} \}$$

$$|f|_{H^s} = |\langle \xi \rangle^s \hat{f}|_{L^2(\mathbb{R}^d)}$$

$s \in \mathbb{R}$, $\dot{H}^s(\mathbb{R}^d)$ è lo spazio delle distribuzioni temperate u in $\mathbb{R}^d$ t.c. $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$

e $\quad |\xi|^s \hat{u} \in L^2(\mathbb{R}^d)$

$$|u|_{\dot{H}^s(\mathbb{R}^d)} = |\,|\xi|^s \hat{u}|_{L^2(\mathbb{R}^d)}$$

Es $\quad u \in \dot{H}^1(\mathbb{R}^d) \Longleftrightarrow \nabla u \in L^2(\mathbb{R}^d)$

e $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$