5 ottobre

Sia $f: X \longrightarrow Y$ biettiva.

Resta definita la così detta funzione inversa di f, $f^{-1}: Y \longrightarrow X$

Si definisce $f^{-1}(y) = x$ dove x è l'unico elemento di X t.c. $f(x) = y$.

$x = f^{-1}(y)$

$f^{-1}(y) = x$ dove $f(x) = y$

$f(f^{-1}(y)) = f(x) = y$

Quindi, per ogni $y \in Y$

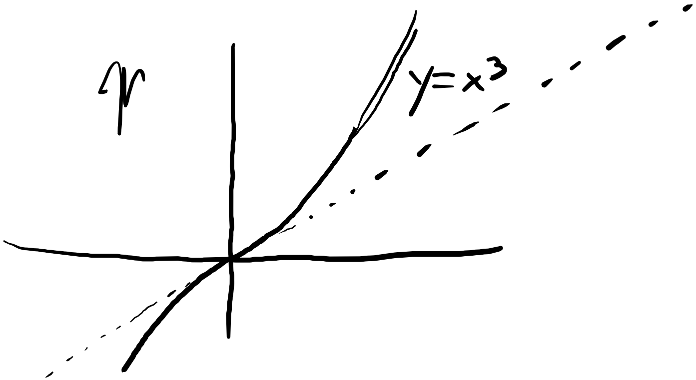
$f(f^{-1}(y)) = y$

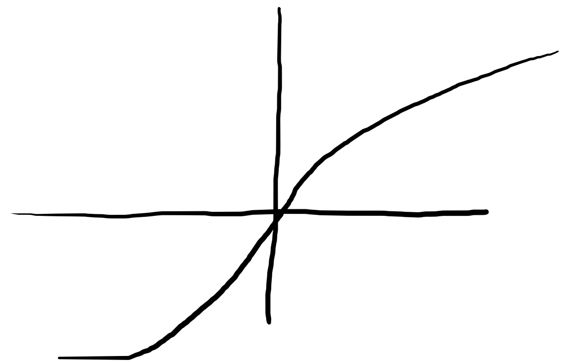
Si ha anche $f^{-1}(f(x)) = x \quad \forall x \in X$

$f(x) = x^3 \qquad g(y) = y^{\frac{1}{3}}$

$$g(f(x)) = g(x^3) = (x^3)^{\frac{1}{3}} = x \qquad \forall x \in \mathbb{R}$$

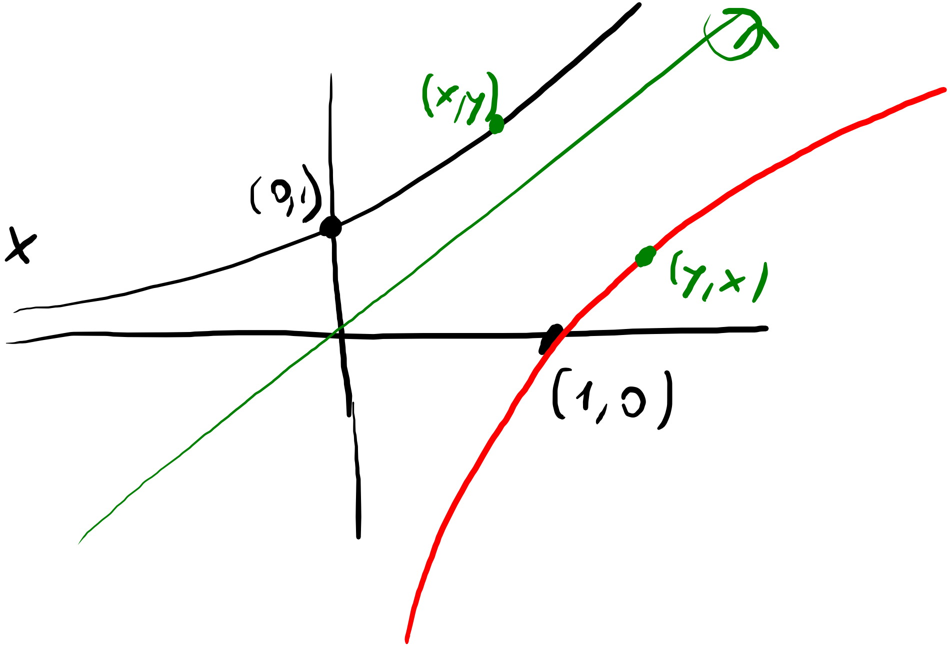
$$f(g(y)) = f(y^{\frac{1}{3}}) = (y^{\frac{1}{3}})^3 = y \qquad \forall y \in \mathbb{R}$$





$$f(x) = e^{x}$$

$$g(x) = \lg x = \ln x$$



Sia $f: X \to Y$ biettiva, sia $g: Y \to X$ l'inversa. Allora $f(x) = y \iff x = g(y)$

$$G_f = \{ (x,y) \in X \times Y \, : \, y = f(x) \} =$$

$$= \{ (x,y) \in X \times Y \, : \, x = g(y) \}$$

$$G_g = \{ (y,x) \in Y \times X \, : \, x = g(y) \}$$

La mappa $Y \times X \to X \times Y$ definita da $(x,y) \to (y,x)$

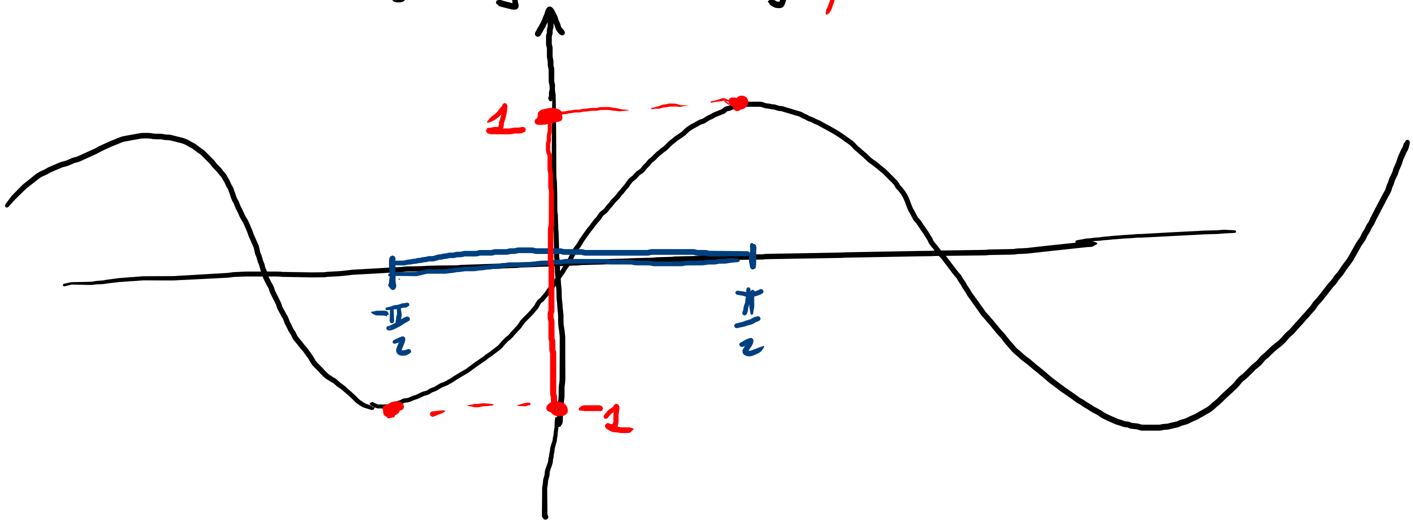
quindi $(y,x) \longrightarrow (x,y)$, ristretta a $G_g \subseteq Y \times X$

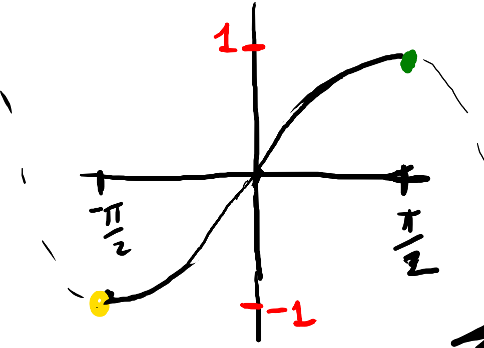
$$Y \times X$$

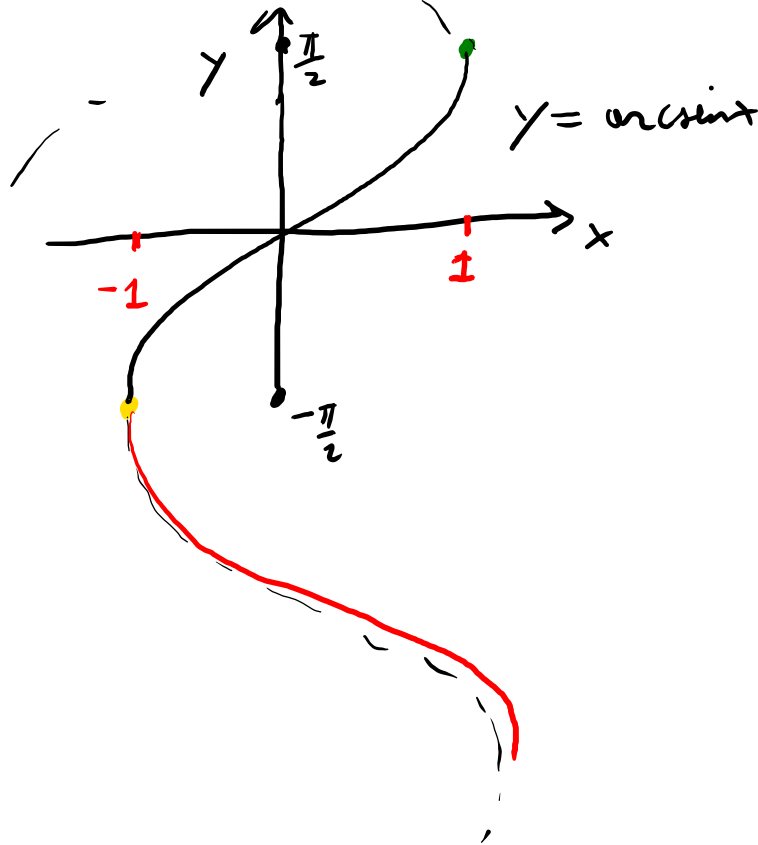
ci dà una funzione biettiva $G_g \longrightarrow G_f$

$\sin x : \mathbb{R} \to [-1, 1]$ non è biettiva.

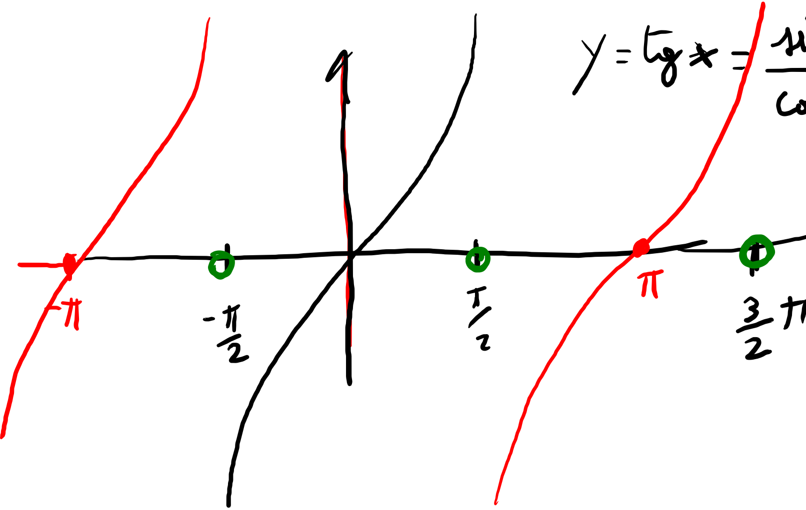
però $\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, arcsin : $[-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$







$$\sin(\arcsin x) = x$$



$$y = \operatorname{tg} x = \frac{\sin x}{\cos x}$$

tg

tang

$$\tan(x + k\pi) = \tan(x) \quad \forall\, k \in \mathbb{Z}$$

$$\operatorname{tg} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

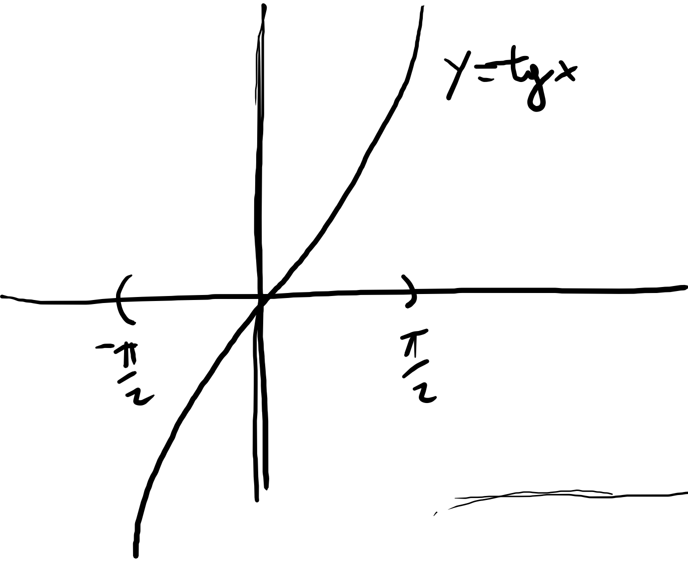
è biettiva.

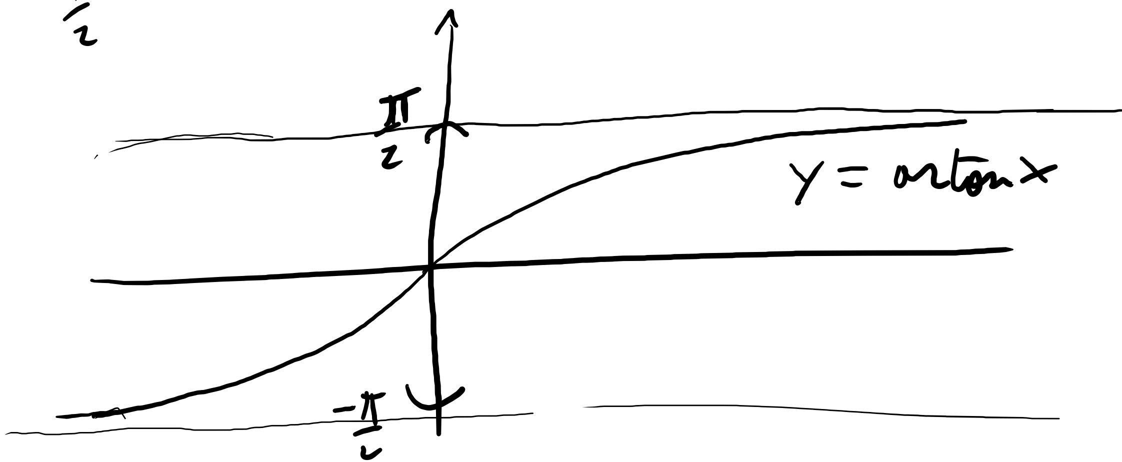
$$\operatorname{tg} : \mathbb{R} \setminus \left(\frac{\pi}{2} + \pi\mathbb{Z}\right) \to \mathbb{R}$$

$$\frac{\pi}{2} + \pi\mathbb{Z} = \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$$

$tg : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$

$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$





Def Date $X \xrightarrow{f} Y$ e $Y \xrightarrow{g} Z$

Resta definito

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & Z \end{tikzcd}$$

la composizione è la funzione

$$\begin{tikzcd} x \in X \arrow[r] \arrow[rr, bend right] & f(x) \in Y \arrow[r] & g(f(x)) \end{tikzcd}$$

$$g(f(x)) = g \circ f(x)$$

$$g \circ f$$

$$f(x) = x + 1 \qquad g(y) = e^{y}$$

$$g \circ f(x) = g(f(x)) = e^{f(x)} = e^{x+1}$$

$$f \circ g(x) = f(g(x)) = g(x) + 1 = e^{x} + 1$$

$$h(z) = z^3 + z^2$$

$$h \circ g \circ f(x) = h(g \circ f(x)) = (g \circ f(x))^3 + (g \circ f(x))^2$$

$$= (e^{x+1})^3 + (e^{x+1})^2 = e^{3x+3} + e^{2x+2}$$

Def Sia $X \subseteq \mathbb{R}$ e sia $f : X \to \mathbb{R}$. Allora

1) f si dice crescente se $x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$

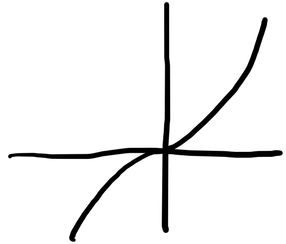
(strettamente crescente) $\left(x_1 < x_2 \Longrightarrow f(x_1) < f(x_2) \right)$

2) f decrescente $x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2)$

(strettamente decrescente) $\left(x_1 < x_2 \Longrightarrow f(x_1) > f(x_2) \right)$

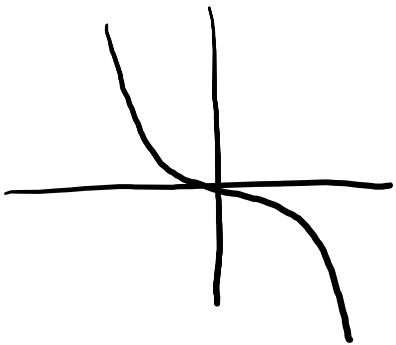
Esempi

$f(x) = x^3$



e' strettamente crescente

$f(x) = -x^3$

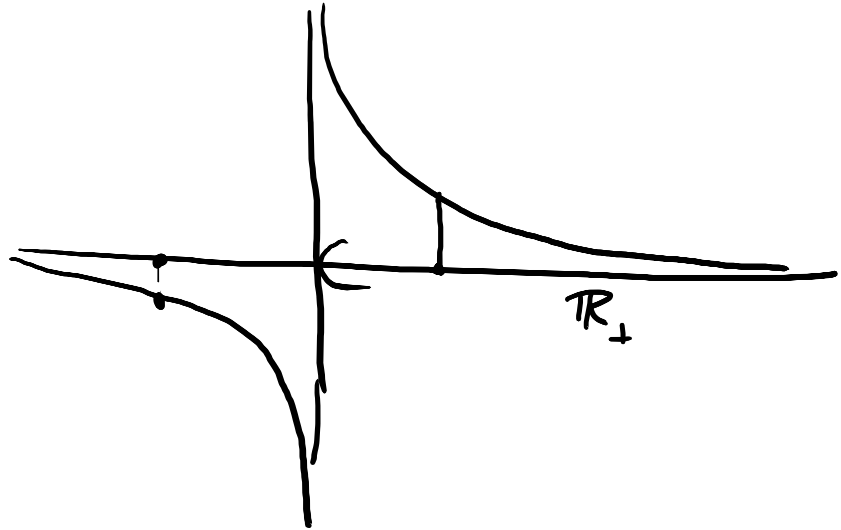


e' strettamente decrescente

$f(x) = \frac{1}{x}$

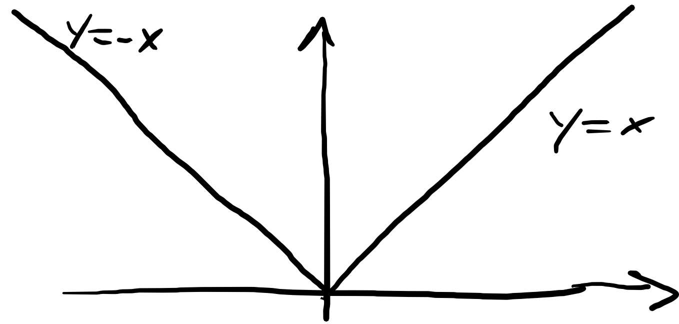
$f : \mathbb{R}_+ \longrightarrow \mathbb{R}$

è decrescente.



in $\mathbb{R} \setminus \{0\}$ non è monotona

$$|x| = \begin{cases} x & \text{se } x \geq 0 \\ 0 & x = 0 \\ -x & x \leq 0 \end{cases}$$



<u>Lemma</u> Sia $a \geq 0$. Allora

$$|x| \leq a \iff -a \leq x \leq a$$

$$|x| < a \iff -a < x < a$$

1) $|x \; y| = |x| \; |y|$

$|x| \le a \iff -a \le x \le a$

$a = |x| + |y|$

2) $|x+y| \le |x| + |y| \quad \forall \; x, y \in \mathbb{R}.$

$x \rightsquigarrow x+y$

$|x| \le |x| \iff -|x| \le x \le |x|$

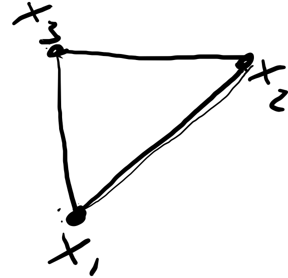
$|y| \le |y| \iff -|y| \le y \le |y|$

$-|x| - |y| \le x+y \le |x| + |y|$

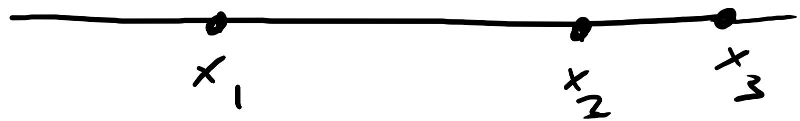
$-(|x| + |y|) \le x+y \le |x| + |y| \iff |x+y| \le |x| + |y|$

Esercizio

$\forall n \in \mathbb{N}$ $$\left| \sum_{j=1}^{n} x_j \right| \le \sum_{j=1}^{n} |x_j|$$



Def Dati $x_1, x_2 \in \mathbb{R}$ la loro distanza in $\mathbb{R}$ è data $|x_1 - x_2|$



$$|x_1 - x_2| \le |x_1 - x_3| + |x_3 - x_2| \qquad \forall\, x_1, x_2, x_3 \in \mathbb{R}.$$

$$|x_1 - x_2| = |(x_1 - x_3) + (x_3 - x_2)| \le |x_1 - x_3| + |x_3 - x_2|$$

$$\forall\, x_1, x_2, x_3 \in \mathbb{R}$$

Esercizio Dimostrare che dati $L_1, L_2 \in \mathbb{R}$ la seguente proposizione è vera

$$|L_1 - L_2| < \varepsilon \quad \forall \varepsilon > 0 \implies L_1 = L_2$$

__Esercizio__ Se $X \subseteq \mathbb{R}$ è finito

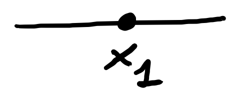
$\Rightarrow \max X$ e $\min X$ esistono.

Dim per induzione sul numero degli elementi di X

Sia $n = \# X$ $\quad n = \operatorname{card} X$

$n = 1$. Sia X t.c. $\operatorname{card} X = 1$

Allora $\exists\ x_1 \in \mathbb{R}$ t.c. $X = \{x_1\}$



$\min X = \max X = x_1$

$n-1 \Rightarrow n$

Sia $X \subseteq \mathbb{R}$ t.c. $\operatorname{card} X = n$.

sia $X = \{x_1, \dots, x_n\}$

sia $Y = \{x_1, \dots, x_{n-1}\}$

$\operatorname{card} Y = n-1 \Rightarrow \exists \max Y \in Y \subseteq X$

Se $x_n > \max Y \Rightarrow x_n = \max X$

Se $x_n < \max Y \Rightarrow \max Y = \max X$