5 ottobre

$u \in \dot{H}^s(\mathbb{R}^d)$

$u \in \mathcal{S}'(\mathbb{R}^d)$, $\hat{u} \in L^1_{loc}(\mathbb{R}^d)$

$|\xi|^s \hat{u} \in L^2(\mathbb{R}^d)$

$\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$ solo se

$s > -\frac{d}{2}$

se infatti $u \in \mathcal{S}(\mathbb{R}^d)$ è t.c.

$\hat{u}(0) \neq 0 \implies |\xi|^{-\frac{d}{2}} \hat{u} \notin L^2(\mathbb{R}^d)$

Lemma. Per $s > -\frac{d}{2}$ risulta $C_c^\infty(\mathbb{R}^d)$ è denso in $\dot{H}^s(\mathbb{R}^d)$

- $\dot{H}^s(\mathbb{R}^d)$

$$\langle f, g \rangle_{\dot{H}^s} = \langle |\xi|^s \hat{f}, |\xi|^s \hat{g} \rangle_{L^2}$$

Prop Per $s < \frac{d}{2}$ $\dot{H}^s(\mathbb{R}^d)$ è uno spazio di Hilbert. In particolare

$$\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi)$$

è un isomorfismo.

Per $s \geq \frac{d}{2}$, $\dot{H}^s(\mathbb{R}^d)$ non è completo

Dim Segue da

Lemma $s < \frac{d}{2}$

- $L^2(\mathbb{R}^d \setminus \{0\}, |\xi|^{2s} d\xi) \subseteq L^1_{loc}(\mathbb{R}^d)$
- $\subseteq \mathcal{S}'(\mathbb{R}^d)$
- $\mathcal{F}(\dot{H}^s(\mathbb{R}^d)) = L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$

Dim $g \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \Rightarrow$

$\Rightarrow g \in L^2_{loc}(\mathbb{R}^d \setminus \{0\}, d\xi)$. Verifichiamo che $g \in L^1(B)$. $\int_B |g(\xi)| d\xi =$

$$= \int_B |\xi|^{-s} |\xi|^{s} |g(\xi)| d\xi \leq$$

$$\leq \boxed{\left(\int_B |\xi|^{-2s} d\xi\right)^{\frac{1}{2}}}_{(C_s \; 2s<d)} \underbrace{\left(\int_{\mathbb{R}^d} |\xi|^{2s} |g(\xi)|^2 d\xi\right)^{\frac{1}{2}}}_{<+\infty}$$

Verifichiamo che $g \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ è $g \in \mathcal{S}'(\mathbb{R}^d)$ se $s < \frac{d}{2}$.

$$g = \underbrace{\chi_B g}_{\in L^1(\mathbb{R}^d)} + (1-\chi_B) g$$

$$(1-\chi_B) g \in \overbrace{L^2(\mathbb{R}^d, \langle\xi\rangle^{2s} d\xi)}^{f \in} \subseteq \mathcal{S}'(\mathbb{R}^d)$$

$$\left|\int f(\xi)\varphi(\xi)\, d\xi\right| \leq \left(\int_{\mathbb{R}^d} \langle\xi\rangle^{2s} |f|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \langle\xi\rangle^{-2s} |\varphi|^2 d\xi\right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\mathbb{R}^d} \langle\xi\rangle^{-2s-2m} d\xi\right)^{\frac{1}{2}} \left|\langle\xi\rangle^m \varphi\right|_{L^\infty(\mathbb{R}^d)}$$

$$2(s+m) > d$$

$s \in (0,1)$ $\quad u \in L^1_{loc}(\mathbb{R}^d)$

$$|u|^2_{\dot{H}^s} = C_{ds} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} dx\, dy$$

$$u : \mathbb{R}^3 \to \mathbb{R}^3$$

$$u \in \mathcal{S}'(\mathbb{R}^d, \mathbb{R}^d)$$

$$\operatorname{div} u = \nabla \cdot u = \partial_j u^j = \sum_{j=1}^{d} \partial_j u^j = 0$$

$$\widehat{\operatorname{div} u} = -i \xi_j \hat{u}^j = 0$$

$$H(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d, \mathbb{R}^d) \ : \ \nabla \cdot u = 0 \}$$

$$V(\mathbb{R}^d) = \{ u \in H^1(\mathbb{R}^d, \mathbb{R}^d) \ : \ \nabla \cdot u = 0 \}$$

$$C^\infty_{\sigma c}(\mathbb{R}^d, \mathbb{R}^d) = \{ u \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^d) \ : \ \nabla \cdot u = 0 \}$$

Vogliamo dimostrare che $C^\infty_{\sigma c}$ è denso sia in H che in V

Lemma $\forall \ u \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ ho

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u) \qquad (1)$$

<u>Dim</u> $\Delta u = \Delta\, u_i \vec{e}_i = \Delta u_i \, \vec{e}_i = \partial_j \partial_j u_i \vec{e}_i$

$$= \underbrace{\cancel{\partial_i \partial_j u_j \vec{e}_i}} - \underbrace{\left(\cancel{\partial_i \partial_j u_j} \vec{e}_i - \partial_j \partial_j u_i \vec{e}_i\right)}$$

$$\partial_i \partial_j u_j \vec{e}_i = \nabla(\nabla \cdot u) = \partial_i (\nabla \cdot u) \vec{e}_i \doteq \partial_i \partial_j u_j \vec{e}_i$$

$$\nabla \times (\nabla \times u) = \epsilon_{ijk} \partial_j (\nabla \times u)_k \vec{e}_i$$

$\epsilon_{123} = 1$

$\epsilon_{ijk} = \pm 1$ se ijk è una permutazione di $1,2,3$

$$= \epsilon_{ijk} \, \epsilon_{ki'j'} \partial_j \partial_{i'} u_{j'} \vec{e}_i$$

$$= \epsilon_{ijk} \, \epsilon_{i'j'k} \partial_j \partial_{i'} u_{j'} \vec{e}_i$$

$$= (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}) \partial_j \partial_{i'} u_{j'} \vec{e}_i$$

$$= \underbrace{\partial_j \partial_i u_j \vec{e}_i - \partial_j \partial_j u_i \vec{e}_i}$$

P roiettore di Leray. $\mathbb{P}$

$$\mathcal{F}(\mathbb{P}u)^j = \hat{u}^j - \frac{1}{|\xi|^2}\xi_j \xi_k \hat{u}^k$$

Notare che se $\operatorname{div} u = 0$, $\mathbb{P}u = u$

$$\widehat{\operatorname{div} \mathbb{P}u} = 0$$

$$= -i\xi_j \widehat{\mathbb{P}u}^j = -i\left(\xi_j \hat{u}^j - \frac{1}{\cancel{|\xi|^2}}\cancel{\xi_j}\xi_j \xi_k \hat{u}^k\right)$$

$$= 0$$

$$\mathbb{P} : \dot{H}^1 \to \dot{H}^1$$

$$\dot{H}^1(\mathbb{R}^d, \mathbb{R}^d) = \text{Range } \mathbb{P} \oplus \ker \mathbb{P}$$

Se $\mathbb{P}u = 0$

$$\hat{u}^j = \frac{1}{|\xi|^2}\xi_j \xi_k \hat{u}^k =$$

$$= -i\xi_j \frac{i\xi_k \hat{u}_k}{|\xi|^2} = -i\xi_j \hat{V}(\xi) = \widehat{\nabla V}$$

$$\hat{V}(\xi) = \frac{i\xi_k \hat{u}_k}{|\xi|^2}$$

Lemma $C^\infty_{c\sigma}(\mathbb{R}^3,\mathbb{R}^3)$ è denso in $\underline{H}(\mathbb{R}^3)$

Dim

Preliminarmente si dimostra che

$$\boxed{\mathbb{P}u = -\Delta^{-1}\nabla\times(\nabla\times u)} =$$

$$\widehat{\mathbb{P}u} = -\frac{1}{|\xi|^2}\, -i\xi\times(-i\xi\times\hat{u}) =$$

$$= \frac{1}{|\xi|^2}\,\xi\times(\xi\times\hat{u})$$

$$= \left(\hat{u} - \frac{1}{|\xi|^2}\xi_j\xi_k\hat{u}^k\vec{e}_j\right)$$

$$\Delta u = \nabla(\nabla\cdot u) - \nabla\times(\nabla\times u)$$

$$-|\xi|^2\hat{u} = -i\xi(-i\xi_j\hat{u}_j) - \xi_j\xi_k\hat{u}^k\vec{e}_j$$

$$\mathbb{P}u = \nabla\times(-\Delta^{-1}\nabla\times u)$$

$$u\in H\subseteq L^2(\mathbb{R}^3,\mathbb{R}^3)$$

$$u = \mathbb{P}u = \nabla\times(\overbrace{-\Delta^{-1}\nabla\times u}^{A})$$

dove $A\in\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$ $\qquad 1<\frac{3}{2}$

$C^\infty_c(\mathbb{R}^3,\mathbb{R}^3)\subset\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$ è una immersione densa. $\Rightarrow\ \exists\, A_n\to A$ in $\dot{H}^1(\mathbb{R}^3,\mathbb{R}^3)$

$A_n\in C^\infty_c(\mathbb{R}^3,\mathbb{R}^3)$

$$u_n = \underbrace{\nabla\times A_n}_{C^\infty_c(\mathbb{R}^3,\mathbb{R}^3)} \longrightarrow \nabla\times A \quad \text{in } L^2(\mathbb{R}^3,\mathbb{R}^3)$$

Lemma $C^\infty_{c\sigma}(\mathbb{R}^3, \mathbb{R}^3)$ è denso in $V(\mathbb{R}^3) = H(\mathbb{R}^3) \cap H^1(\mathbb{R}^3, \mathbb{R}^3)$.

in $L^2(\mathbb{R}^3)$ $\lambda > 0$

$$\widehat{P_\lambda u} = \chi_{|x| \le \lambda} \hat{u}$$

$$P_\lambda = \chi_{[0,\lambda]}(\sqrt{-\Delta})$$

$\cancel{f(\sqrt{-\Delta}}$

$f(\Delta)$

$1 < p < +\infty$
$s \in \mathbb{R}$

$$W^{s,p}(\mathbb{R}^d)$$

$$= \{ u \in \mathcal{S}'(\mathbb{R}^d) : \left(\langle \xi \rangle^s \hat{u} \right)^\vee \in L^p(\mathbb{R}^d) \}$$

$$W^{k,p}(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : \partial_x^\alpha u \in L^p(\mathbb{R}^d) \text{ per } |\alpha| \le k \}$$

Theor se $1 < p < +\infty$

$$\mathcal{W}^{k,p}(\mathbb{R}^d) = W^{k,p}(\mathbb{R}^d)$$

$\forall \; k \in \{0, 1, 2, \dots\}$