8 ottobre

Teor (Unicità del limite) Sia $X \subseteq \mathbb{R}$ con $\sup X = +\infty$ e sia $f : X \to \mathbb{R}$. Se L_1 ed $L_2 \in \mathbb{R}$ sono limiti di f per $x \to +\infty$, allora $L_1 = L_2$

Dim $\lim_{x \to +\infty} f(x) = L_1$:

$$\boxed{\forall \varepsilon > 0 \quad \exists\, M_1(\varepsilon) \in \mathbb{R} \ \text{t.c.}\ x > M_1(\varepsilon) \text{ e } x \in X \Rightarrow |f(x) - L_1| < \varepsilon .}$$

$$\lim_{x \to +\infty} f(x) = L_2$$

$$\boxed{\forall \varepsilon > 0 \quad \exists M_2(\varepsilon) \in \mathbb{R} \text{ t.c. } x > M_2(\varepsilon) \text{ e } x \in X \Rightarrow |f(x) - L_2| < \varepsilon}$$

Osserviamo che $\forall x \in X$

$$|L_1 - L_2| = |(L_1 - f(x)) + (f(x) - L_2)| \leq$$

$$\leq |L_1 - f(x)| + |f(x) - L_2|$$

Sia $\quad M_3(\varepsilon) = \max\{M_1(\varepsilon), M_2(\varepsilon)\}$

Per $x > M_3(\varepsilon)$ ed $x \in X$ si ha

$$|L_1 - f(x)| < \varepsilon \quad e \quad |L_2 - f(x)| < \varepsilon .$$

Per $x > M_3(\varepsilon)$ con $x \in X$ si ha

$$|L_1 - L_2| \le \underbrace{|f(x) - L_1|}_{< \varepsilon} + \underbrace{|f(x) - L_2|}_{< \varepsilon} < 2\varepsilon \quad \forall \varepsilon > 0$$

Quindi concludiamo che

$$\boxed{|L_1 - L_2| < 2\varepsilon \quad \forall \varepsilon > 0} \iff \boxed{|L_1 - L_2| < \varepsilon \quad \forall \varepsilon > 0}$$

$$\Downarrow \; L_1 = L_2$$

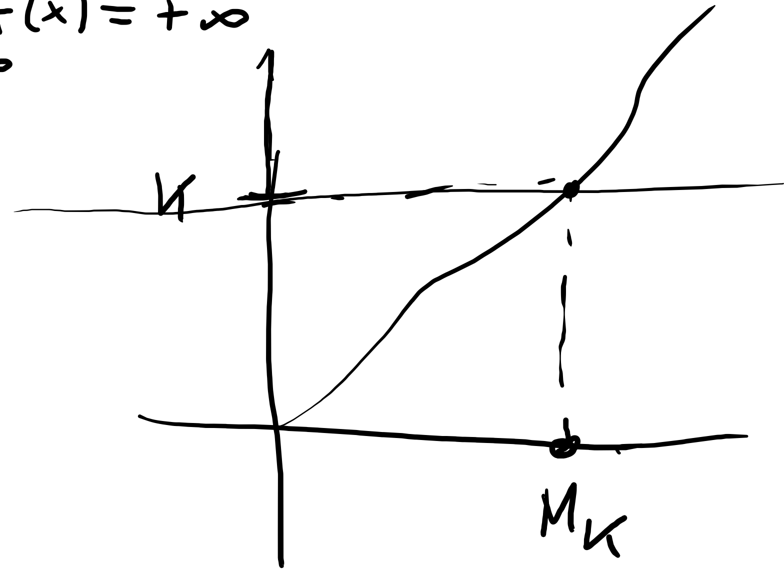
<u>Def</u> Sia $X \subseteq \mathbb{R}$, $\sup X = +\infty$ ed $f : X \to \mathbb{R}$

Scriviamo che $\lim_{x \to +\infty} f(x) = +\infty$

$\forall K \in \mathbb{R} \ \exists M_K \in \mathbb{R}$ t.c.

$x > M_K$ e $x \in X$

$\Rightarrow f(x) > K$.



Es Sia $b > 1$. Allora $\lim_{n \to +\infty} b^n = +\infty$

$b = 1 + a$ dove $a = b - 1 > 0$

$b^n = (1+a)^n \geq 1 + na$ per Bernoulli.

Vogliamo dimostrare che

$$* \quad \boxed{\forall K \in \mathbb{R} \ \exists M_K \in \mathbb{R} \text{ t.c. } n > M_K \Rightarrow b^n > K}$$

Considero $K \in \mathbb{R}$. Risolvo

$$b^n \geq 1 + na > K \qquad n > \frac{K-1}{a} =: M_K$$

$$n > M_K = \frac{K-1}{a} \iff 1 + na > K$$

In particolare $n > M_K \implies b^n \geq 1 + na > K$

$$\implies b^n > K$$

Abbiamo dimostrato ~~che~~ cioè che $\forall K \in \mathbb{R} \; \exists M_K \in \mathbb{R}$ t.c.

$$n > M_K \implies b^n > K$$

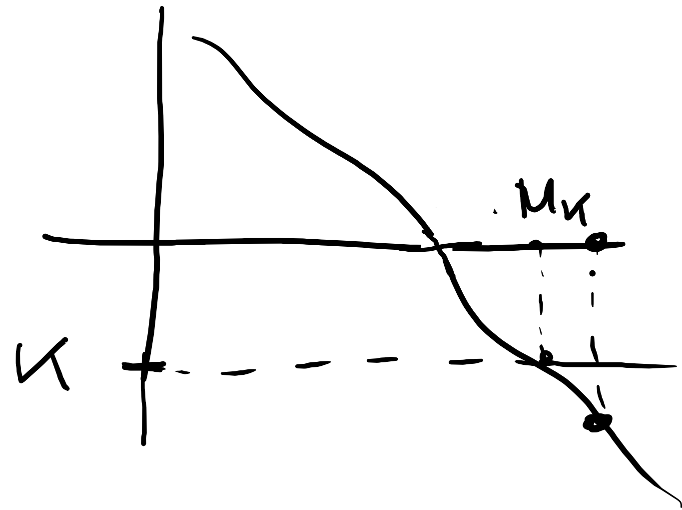
Def Sia $X \subseteq \mathbb{R}$ con $\sup X = +\infty$ ed $f: X \to \mathbb{R}$

Scriviamo che $\lim_{x \to +\infty} f(x) = -\infty$ se

$\forall K \in \mathbb{R}\ \exists M_K \in \mathbb{R}$ t.c.

$x > M_K$ e $x \in X$

$\Rightarrow f(x) < K$.



$\lim_{x \to +\infty} -x = -\infty$

In modi analoghi si definiscono

$$\lim_{x \to -\infty} f(x)$$

quando $f: X \to \mathbb{R}$, $\inf X = -\infty$,

"Regole" dei limiti

$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$

$a + (+\infty) = +\infty$ se $a > -\infty$

$a + (-\infty) = -\infty$ se $a < +\infty$

$a\,(+\infty) = +\infty$ se $a > 0$

$a\,(+\infty) = -\infty$ se $a < 0$

$a\,(-\infty) = -\infty$ se $a > 0$

$a\,(-\infty) = +\infty$ se $a < 0$

$\frac{1}{\pm\infty} = 0$

$+\infty + (-\infty)$ esclus[o]

$0\,(+\infty)$ escluso

$0\,(-\infty)$ escluso

Teor (Regole dei limiti) Siano $f, g : X \to \mathbb{R}$, $\sup X = +\infty$ e supponiamo che $\lim_{x\to+\infty} f(x) = a$ e $\lim_{x\to+\infty} g(x) = b$

$a, b \in \overline{\mathbb{R}}$. Vale quanto segue

1) $\lim_{x\to+\infty} \left(f(x)+g(x)\right) = a+b$ salvo che $(a,b) = \begin{cases} (+\infty,-\infty) \\ (-\infty,+\infty) \end{cases}$

2) $\lim_{x\to+\infty} f(x)\,g(x) = a\,b$ salvo che per $(a,b) = \begin{cases} (0,\pm\infty) \\ (\pm\infty,0) \end{cases}$

3) $\lim_{x\to+\infty} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$ per $b \neq 0$ $(a,b) \neq (\pm\infty,\pm\infty)$

$$\lim_{x \to +\infty} (x^3 - x^2 + x + 1) \neq +\infty + (-\infty) + (+\infty) + 1$$

$$\lim_{x \to +\infty} x = +\infty \qquad \lim_{x \to +\infty} 1 = 1$$

$$\lim_{x \to +\infty} x^n = (+\infty)^n = +\infty \qquad n = 2, 3$$

$$\lim_{x \to +\infty} (x^3 - x^2 + x + 1) = \lim_{x \to +\infty} x^3 \left(1 - \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right) = +\infty \cdot 1 = +\infty$$

$$x^3 - x^2 + x + 1 = x^3 \left(1 - \frac{x^2}{x^3} + \frac{x}{x^3} + \frac{1}{x^3}\right) =$$

$$= x^3 \left(1 - \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right)$$

$$\lim_{x \to +\infty} \left(1 - \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right) = 1 - \frac{1}{+\infty} + \frac{1}{+\infty} + \frac{1}{+\infty}$$

$$= 1$$

$$\lim_{x \to +\infty} \frac{1-x}{x^2} \qquad \frac{-\infty}{+\infty}$$

$$\frac{1-x}{x^2} = \frac{1}{x^2} - \frac{1}{x} \xrightarrow{x \to +\infty} 0$$

$$\lim_{x \to +\infty} \left(x - \sqrt{x^2+1}\right) \qquad (+\infty) - (+\infty)$$

$$\lim_{x \to +\infty} \sqrt{x^2+1} = \lim_{x \to +\infty} \sqrt{x^2\left(1+\frac{1}{x^2}\right)} = \lim_{x \to +\infty} x \sqrt{1+\frac{1}{x^2}}$$

$$= +\infty \cdot 1 = +\infty$$

$$x-\sqrt{x^2+1} = \left(x-\sqrt{x^2+1}\right)\frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}} =$$

$$= \frac{x^2-\left(\sqrt{x^2+1}\right)^2}{x+\sqrt{x^2+1}}$$

$$(a-b)(a+b)=a^2-b^2$$

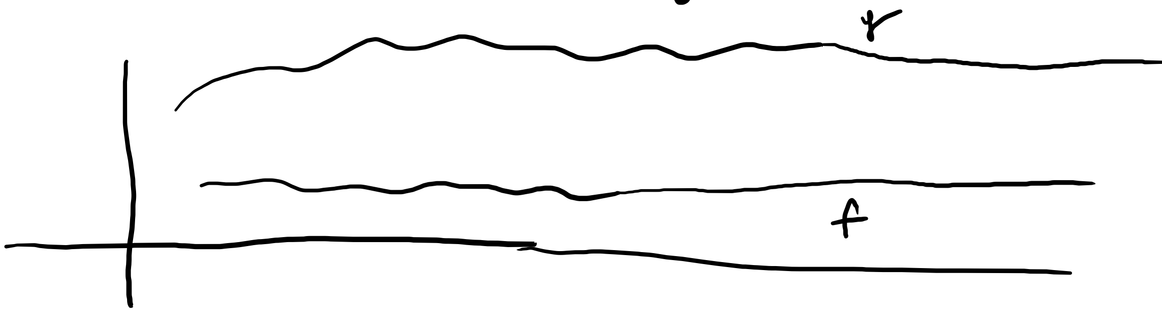
$$= \frac{\cancel{x^2}-(\cancel{x^2}+1)}{x+\sqrt{x^2+1}} = \frac{-1}{x+\sqrt{x^2+1}} \xrightarrow{x\to+\infty} -$$

$$\frac{-1}{(+\infty)+(+\infty)} = \frac{-1}{+\infty} = 0$$

Teoremi del confronto

<u>Teor</u> 1 Siano $f, g : X \to \overline{\mathbb{R}}$, $\sup X = +\infty$ e supponiamo che $\lim_{x \to +\infty} f(x) = a \in \overline{\mathbb{R}}$ e $\lim_{x \to +\infty} g(x) = b \in \overline{\mathbb{R}}$

Allora, se $f(x) \leq g(x)$ $\forall x \in X$, segue che $a \leq b$.

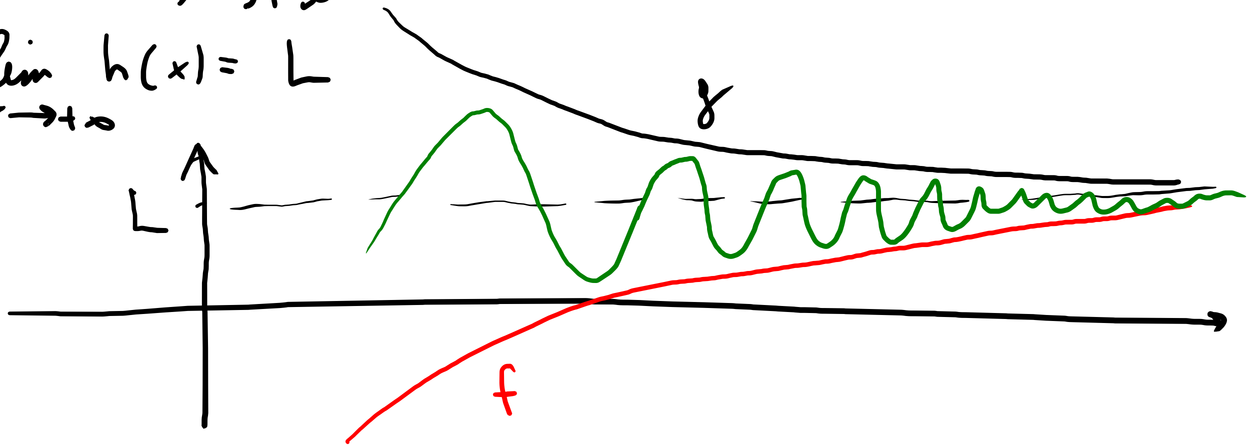


Teor (Carabinieri) Siano $f, g, h : X \to \mathbb{R}$, $\sup X = +\infty$

Supponiamo $f(x) \le h(x) \le g(x)$ $\forall x \in X$ e

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = L \in \overline{\mathbb{R}}$. Allora

$\lim_{x \to +\infty} h(x) = L$



Sia $b \in \mathbb{R}_+$. Allora $\lim_{n \to +\infty} b^{\frac{1}{n}} = 1$

3 casi

$0 < b < 1$

$b = 1$ $\quad 1^{\frac{1}{n}} = 1 \quad \lim_{n \to +\infty} 1^{\frac{1}{n}} = 1$

$b > 1$

$b > 1$. $\qquad b^{\frac{1}{n}} > 1 \quad \forall\, n \in \mathbb{N}$

Scriviamo $\qquad b^{\frac{1}{n}} = 1 + a_n \qquad$ (dimostreremo $\lim_{n\to+\infty} a_n = 0$) dove $a_n > 0$.

$$b = \left(b^{\frac{1}{n}}\right)^n = (1+a_n)^n \geq 1 + n\,a_n$$

$$b \geq 1 + n\,a_n \iff \boxed{0 < a_n \leq \frac{b-1}{n}}$$

$$\lim_{n\to+\infty} 0 = 0$$

$$\lim_{n\to+\infty} \frac{b-1}{n} = \frac{b-1}{+\infty} = 0 \implies \lim_{n\to+\infty} a_n = 0$$

$$\implies \lim_{n\to+\infty} b^{\frac{1}{n}} = 1$$

Sia $0 < b < 1$. Come dimostriamo $\lim_{n \to +\infty} b^{\frac{1}{n}} = 1$

$\frac{1}{b} > 1$

$$b^{\frac{1}{n}} = \left(\frac{1}{\frac{1}{b}} \right)^{\frac{1}{n}} = \frac{1}{\left(\frac{1}{b}\right)^{\frac{1}{n}}} \xrightarrow{n \to +\infty} \frac{1}{1} = 1$$