

Esercizi su

variabili elettive

Valore atteso, varianze

Combinatoria lineare di v.a.

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# Somma e differenza di 2 v.a.

Consideriamo 2 v. aleatorie

$$X \text{ t.r.} \quad E(X) = m_x \\ \text{var}(X) = \sigma_x^2$$

$$Y \text{ t.r.} \quad E(Y) = m_y \\ \text{var}(Y) = \sigma_y^2$$

$$Z = X + Y$$

$$E(Z) = ?$$

$$\text{var}(Z) = ?$$

$$W = X - Y$$

$$E(W) = ?$$

$$\text{var}(W) = ?$$

$$Z = X + Y$$

$$E(Z) = E(X + Y) = E(X) + E(Y)$$

ist linear

$$\mu_z = \mu_x + \mu_y$$

$$W = X - Y \quad E(W) = E(X - Y) = E(X) - E(Y)$$

$$\mu_w = \mu_x - \mu_y$$

$$Z = X + Y$$

$$\sigma_z^2 = E[(z - \mu_z)^2]$$

$$= E[z^2 - 2\mu_z z + \mu_z^2]$$

$$\mu_z = \mu_x + \mu_y$$

$$z = X + Y$$

$$= E\left\{ (x+y)^2 - 2(\mu_x + \mu_y)(x+y) + (\mu_x + \mu_y)^2 \right\} = \rightarrow$$

$$\begin{aligned}
\sigma_z^2 &= E[X^2] + E[Y^2] + 2E[XY] + \\
&\quad - 2(\mu_x + \mu_y)(\mu_x + \mu_y) + (\mu_x + \mu_y)^2 \\
&= E[X^2] + E[Y^2] - (\mu_x + \mu_y)^2 + \\
&\quad + 2E[XY] \\
&= \left\{ E[X^2] - \mu_x^2 \right\} + \left\{ E[Y^2] - \mu_y^2 \right\} + \\
&\quad + 2 \left\{ E[XY] - \mu_x \mu_y \right\}
\end{aligned}$$

Ricordando due relazioni:

$$\sigma_x^2 = E[X^2] - \mu_x^2$$

$$\sigma_y^2 = E[Y^2] - \mu_y^2$$

$$\begin{aligned}\sigma_{xy} &\triangleq E[(X - \mu_x)(Y - \mu_y)] = \\ &= E[XY - \mu_y X - \mu_x Y + \mu_x \mu_y]\end{aligned}$$

$$\sigma_{xy} \stackrel{\Delta}{=} E[(X - \mu_x)(Y - \mu_y)]$$
$$= E(XY) - \mu_x \mu_y$$

In definition:

$$\sigma_x^2 = \left\{ E[X^2] - \mu_x^2 \right\} + \left\{ E[Y^2] - \mu_y^2 \right\} + 2 \left\{ E[XY] - \mu_x \mu_y \right\}$$

$\sigma_{xy}$  ←

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2\sigma_{xy}$$

Se  $X, Y$  sono correlate  $\Rightarrow \sigma_{xy} \neq 0$

e quindi

$$\sigma_z^2 > \sigma_x^2 + \sigma_y^2$$



$$W = X - Y$$

$$\sigma_W^2 = E \left[ (W - \mu_W)^2 \right]$$

$$= E \left\{ W^2 - 2\mu_W W + \mu_W^2 \right\} \quad \mu_W = \mu_X - \mu_Y$$

$$= E \left\{ X^2 + Y^2 - 2XY - 2(\mu_X - \mu_Y)(X - Y) + \mu_X^2 + \mu_Y^2 - 2\mu_X\mu_Y \right\} \Rightarrow$$

$$\begin{aligned}
 \sigma_w^2 &= E\{X^2\} + E\{Y^2\} - 2E(XY) - 2\mu_x^2 + \\
 &\quad + 2\mu_x\mu_y + 2\mu_x\mu_y - 2\mu_y^2 + \mu_x^2 + \mu_y^2 - 2\mu_x\mu_y \\
 &= \underbrace{\left\{ E[X^2] - \mu_x^2 \right\}}_{\sigma_x^2} + \underbrace{\left\{ E[Y^2] - \mu_y^2 \right\}}_{\sigma_y^2} + \\
 &\quad - 2 \underbrace{\left\{ E(XY) - \mu_x\mu_y \right\}}_{\sigma_{xy}}
 \end{aligned}$$

$$\sigma_w^2 = \sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}$$

If  $X, Y$  uncorrelated  $\rightarrow \sigma_{xy} = 0$

$$\sigma_w^2 = \sigma_x^2 + \sigma_y^2$$

Valore atteso e varianza di una  
v.a.  $\sim \mathcal{U}[0, 1]$

v.a. con distribuzione uniforme sull'intervallo  
 $[0, 1]$

$$v \sim \mathcal{U}[0, 1] \Leftrightarrow f_v(q) = \begin{cases} 1 & q \in [0, 1] \\ 0 & q \in \mathbb{R} \setminus [0, 1] \end{cases}$$

$$E(v) = ? \quad \Delta_v^2 = ?$$

Data  $f_v(q)$ , quale è  $F_v(q)$ ?

$$F_v(q) = \int_{-\infty}^q f_v(q) dq = \begin{cases} \text{almeno a distanza} \\ \text{i casi:} \end{cases}$$

$$q < 0$$

$$0 \leq q \leq 1$$

$$q > 1$$

$$q < 0$$

$$F_v(q) = \int_{-\infty}^q f_v(q) dq = 0$$

$$f_v \equiv 0$$

$$0 \leq q \leq 1 \quad f_v(q) = 1 \quad \text{for } q \in [0, 1]$$

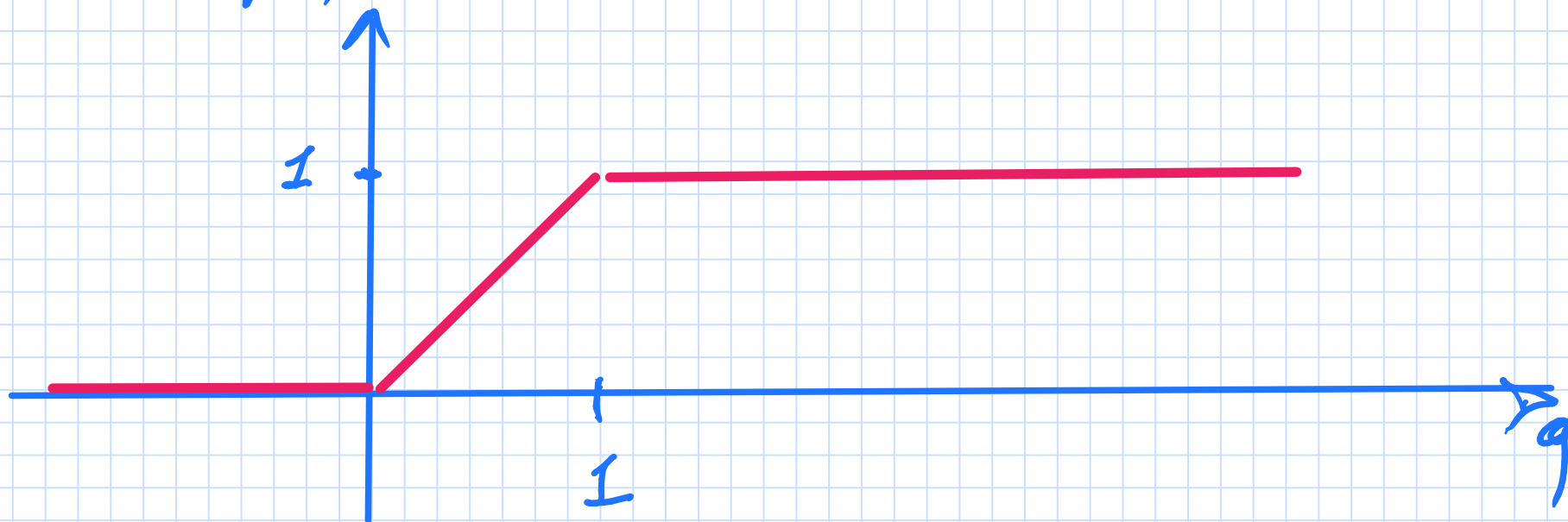
$$F_v(q) = \int_{-\infty}^q f_v(q) dq = \int_0^q 1 dq = q$$

$$q > 1 \quad f_v(q) = 0$$

$$F_v(q) = \int_{-\infty}^q f_v(q) dq = \int_0^1 1 dq = 1$$

In definitiva:

$$F_2(q) = \begin{cases} 0 & q < 0 \\ q & 0 \leq q \leq 1 \\ 1 & q > 1 \end{cases}$$



$$v \sim \mathcal{U}(0,1) \quad E(v) = ?$$

$$E(v) = \int_{-\infty}^{+\infty} g f(g) dg = \int_0^1 g dg = \frac{1}{2} g^2 \Big|_0^1 = \frac{1}{2}$$

$$D_v^2 = \int_{-\infty}^{+\infty} [g - E(v)]^2 f(g) dg \longrightarrow$$



$$\Delta^2 = E \left\{ [v - E(v)]^2 \right\}$$

$$= E[v^2] - [E(v)]^2$$

$$E(v) = \frac{1}{2}$$

$$E(v^2) = \int_{-\infty}^{+\infty} q^2 f_v(q) dq = \int_0^1 q^2 dq = \frac{1}{3} q^3 \Big|_0^1 = \frac{1}{3}$$

$$\Delta_v^2 = E[v^2] - [E(v)]^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

# Trasformazione lineare applicata a v.a.

Dato v.a.  $X$ , con  $\mu_x = E(X)$ ,  $\sigma_x^2 = \text{var}(X)$

Sia

$$Y = aX + b \quad \text{con } a, b \in \mathbb{R}$$

$$E(Y) ? \quad \text{var}(Y) = ?$$

Per lineare:

$$E(Y) = E[aX + b] = aE(X) + b$$

quindi

$$\mu_y = a\mu_x + b$$

Per la varianza:

$$\sigma_y^2 = E[(Y - \mu_y)^2]$$

sostituendo  
 $Y = aX + b$   
 $\mu_y = a\mu_x + b$

$$\sigma_y^2 = E \left[ (aX + b) - (a\mu_x + b) \right]^2 =$$

$$= E \left[ a(X - \mu_x) + \cancel{b} - \cancel{b} \right]^2$$

$$= E \left[ a^2 (X - \mu_x)^2 \right] = a^2 E \left[ (X - \mu_x)^2 \right]$$

$$\Delta_y^2 = \sigma^2 \mathbb{E} \left[ (X - \mu_x)^2 \right]$$
$$= \sigma^2 \Delta_x^2$$

Exemple d'usage :

$$v \sim \mathcal{U}[0, 1] \implies z = a + (b-a)v$$

$$z \sim \mathcal{U}[a, b]$$

$$E(z) = a + (b-a)u_{\sigma} = a + \frac{b-a}{2} = \frac{b+a}{2}$$

$$\sigma_z^2 = (b-a)^2 \sigma_{\sigma}^2 = \frac{(b-a)^2}{12}$$

In Matlab: v. aleatoare cu  
distributie uniforme in  $[0,1]$   
`rand()`

Altro esempio d'uso delle transf. lineari

$$v \sim \mathcal{N}(0, 1)$$

v.a. gaussiana

$$E(v) = 0$$

$$\Delta_v^2 = \mathbf{I}$$

$$X = \mu + \sigma v$$

X è v.a. gaussiana

con  $\mu, \sigma \in \mathbb{R}$

$$E(X) = \mu$$

$$\Delta_X^2 = \sigma^2 \cdot \mathbf{I}$$



In Matlab:

v. a. gaussiana con media attesa  $\phi$   
e varianza 1;

randn ( )

```
N = 1000;
a = -2; b = +2;

v_unif = a + (b-a) * rand(N,2);
% N bidimensional random variables, with unif.✓
distribution in [-1 +1]

figure;
plot(v_unif(:,1), v_unif(:,2),...
     'd', 'MarkerEdgeColor', 'b',...
     'MarkerFaceColor', 'b',...
     'MarkerSize',10);
grid on
hold on;

v_gauss = randn(N,2);
% gaussian random variable, with expected value 0 and✓
variance 1
plot(v_gauss(:,1), v_gauss(:,2),...
     'o', 'MarkerEdgeColor', 'r',...
     'MarkerFaceColor', 'r',...
     'MarkerSize',10);
axis square
```

