Funzioni periodiche

$f: E \subseteq \mathbb{R} \to \mathbb{R}$ $\quad T \in \mathbb{R}^+$ $\quad \forall x \in E$ sia $x+T, x-T \in E$; $f(x+T) = f(x)$,

Allora f si dice periodica di periodo T.

Es: la funzione $f(x) = \operatorname{tg}(x)$ è periodica di periodo 2π

$E = \bigcup_{k \in \mathbb{Z}} \left] -\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi \right[$

$x \in E \quad x \pm 2\pi \overset{?}{\in} E$ Vero

$\operatorname{tg}(x + 2\pi) = \operatorname{tg}(x)$ Vero

OSS: Sia f periodica di periodo $T>0$. Allora f è periodica di periodo kT $\forall k \in \mathbb{N}^+$.

Dim induzione: $k=1$ $\quad f(x+T) = f(x) \quad \forall x \in E$ OK

sia vero per $k \Rightarrow$ vero per $k+1$ $\quad \left[\underline{f(z+kT) = f(x)} \quad \forall z \in E \right]$

$$f(x+(k+1)T) = f(\underbrace{(x+kT)}_{z}+T) \overset{1}{=} f(x+kT) = f(x) \quad //$$

Talvolta per "periodo di f" si intende il minimo periodo, se esiste

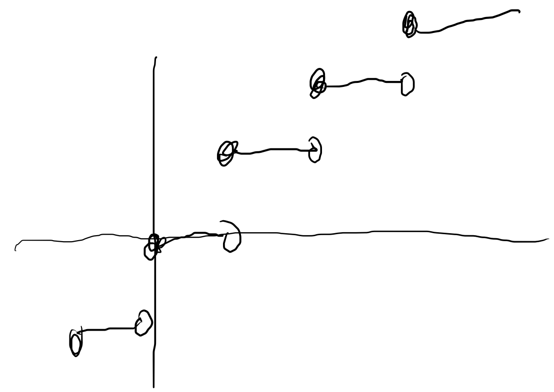
$\min\{T \in \mathbb{R}^+ : T \text{ è periodo}\}$

Es: $f(x) = \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

f è periodica? $\forall T \in \mathbb{Q}^+$ $\quad f(x+T) = f(x) \quad \forall x \in \mathbb{R}$

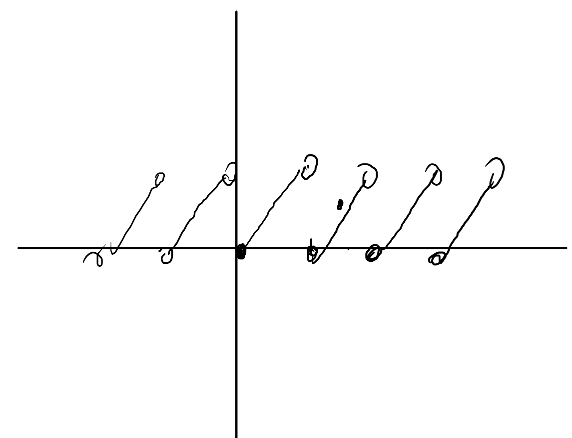
Non esiste periodo minimo!

Es: funzione parte intera $[x]$



funzione mantissa $f(x) = x - [x]$ è periodica

$f(x) = (x)$



$\left(\frac{3}{2}\right) = \frac{1}{2}$ $\qquad T = 1$

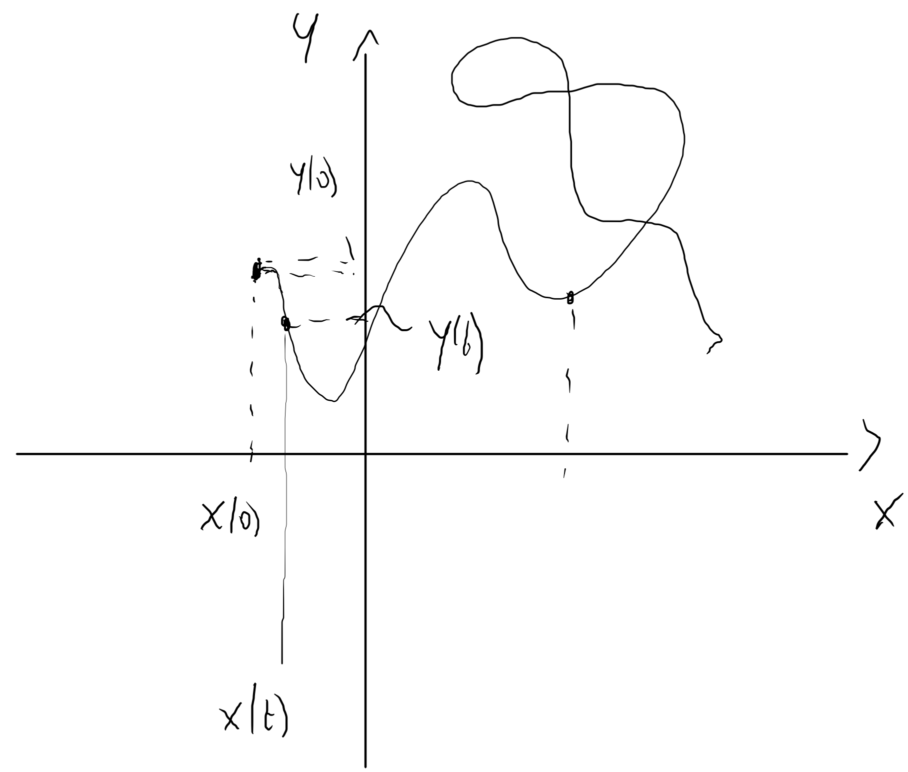
Curve parametriche nel piano

$$\gamma : [a, b] \to \mathbb{R}^2$$

$$\boxed{\gamma(t) = \big(x(t), y(t)\big)}$$

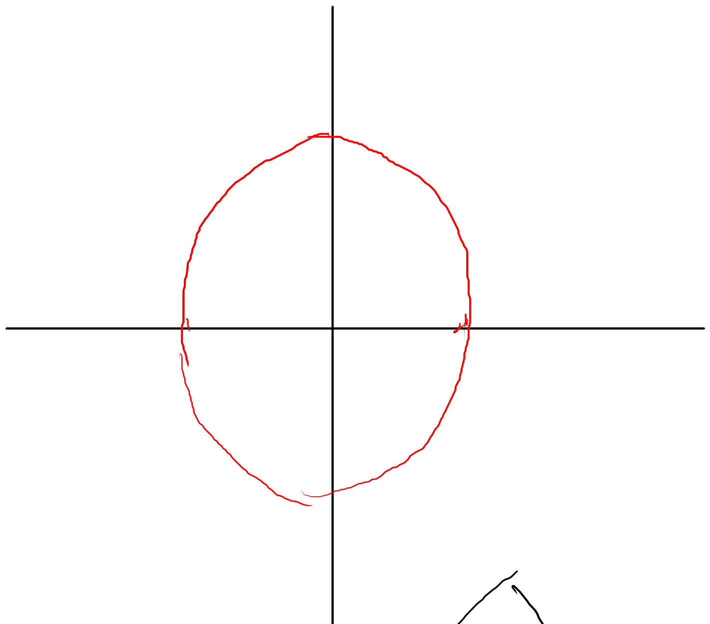
$$x : [a,b] \to \mathbb{R}$$

$$y : [a,b] \to \mathbb{R}$$



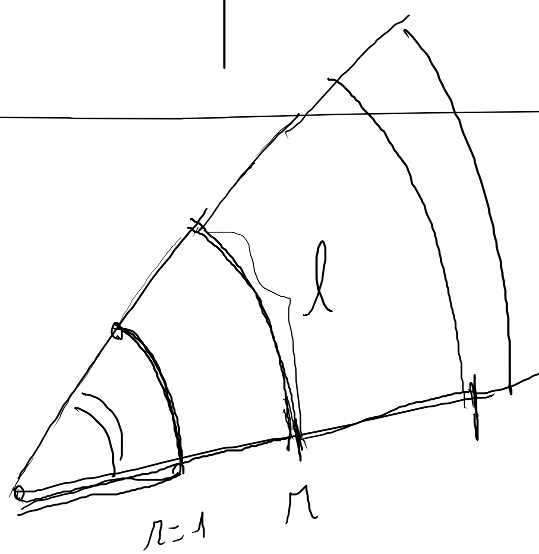
NB: si può considerare $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^2$ I qualunque intervallo

se $x(t) = t$ $\quad \gamma(t) = (t, y(t))$ è proprio il grafico della funzione $y : I \to \mathbb{R}$

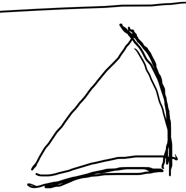


$x^2+y^2=1$

Radianti



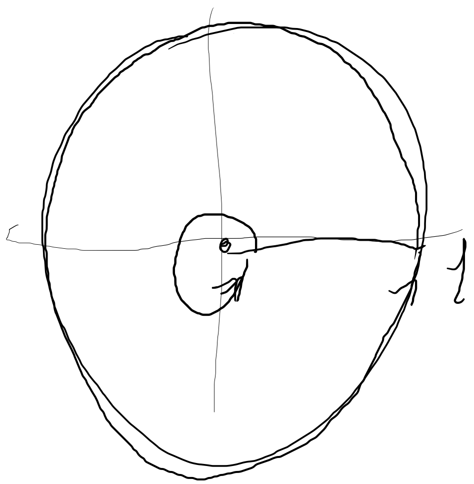
angolo

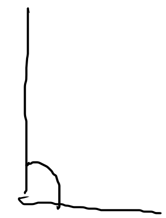


$$\frac{\ell}{r} = \text{costante che dipende solo dall'ampiezza dell'angolo}$$

Diremo che un angolo misura 1 radiante se il rapporto tra la lunghezza dell'arco di cerchio di raggio r e il raggio r è 1.

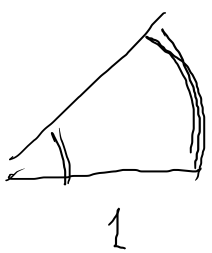
2π ampiezza dell'angolo giro



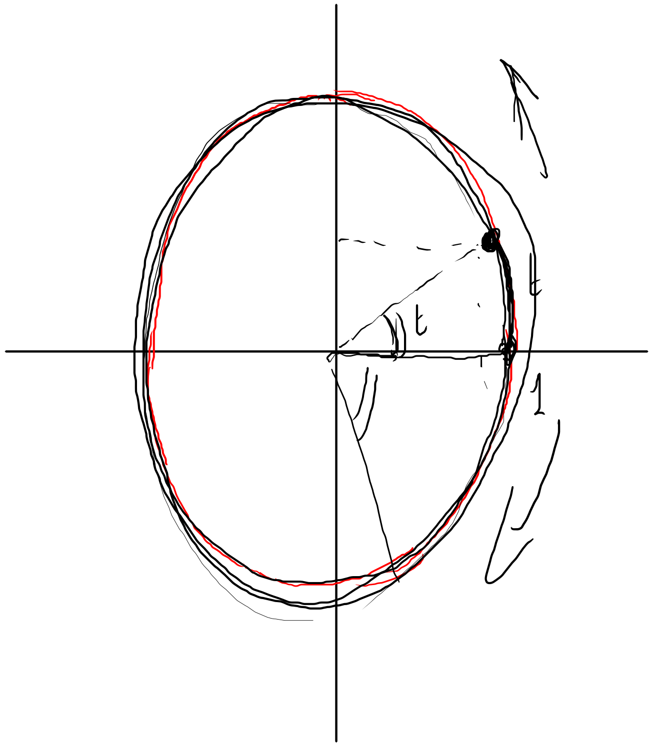


$\frac{\pi}{2}$

π



nella circonferenza di raggio 1 possiamo confondere la lunghezza d'arco e misura dell'angolo.



$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

Circonferenza di raggio 1

Questo insieme si può rappresentare in modo parametrico, con parametro la lunghezza d'arco (o equivalentemente l'angolo)

$t \in [0, 2\pi]$

$\gamma(t) = (x(t), y(t))$

$x(t) = \cos t$

$y(t) = \operatorname{sen} t$

← componenti della parametrizzazione della circonferenza

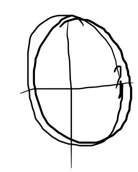
Ha significato considerare $t \in \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \cos x$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \operatorname{sen} x$

$\forall x \in \mathbb{R} \quad [\cos(x)]^2 + [\operatorname{sen}(x)]^2 = 1$

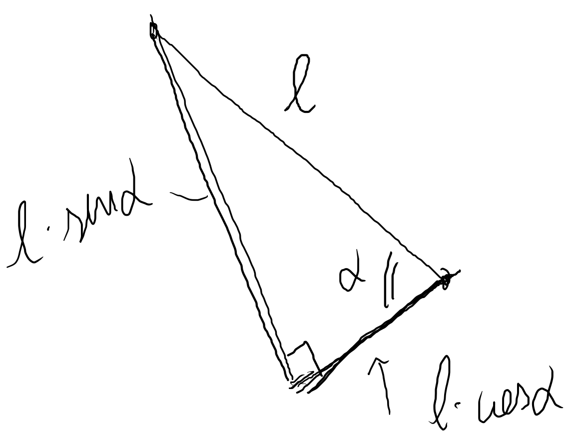
$$\boxed{\cos^2 x + \operatorname{sen}^2 x = 1}$$



Oss

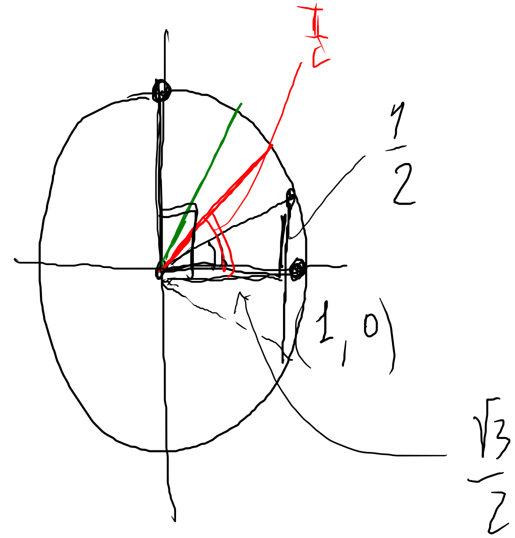
Sono funzioni periodiche di periodo 2π

e 2π è periodo minimo



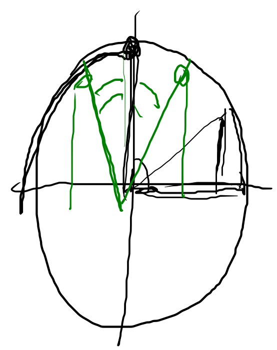
Angoli significativi

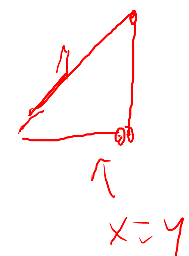
α	$\cos\alpha$	$\sin\alpha$
0	1	0
$\pi/6$	$\sqrt{3}/2$	$1/2$
$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\pi/3$	$1/2$	$\sqrt{3}/2$
$\pi/2$	0	1



$$\left(\frac{1}{2}\right)^2 + \cos^2\alpha = 1$$

$$1 - \frac{1}{4} = \frac{3}{4}$$





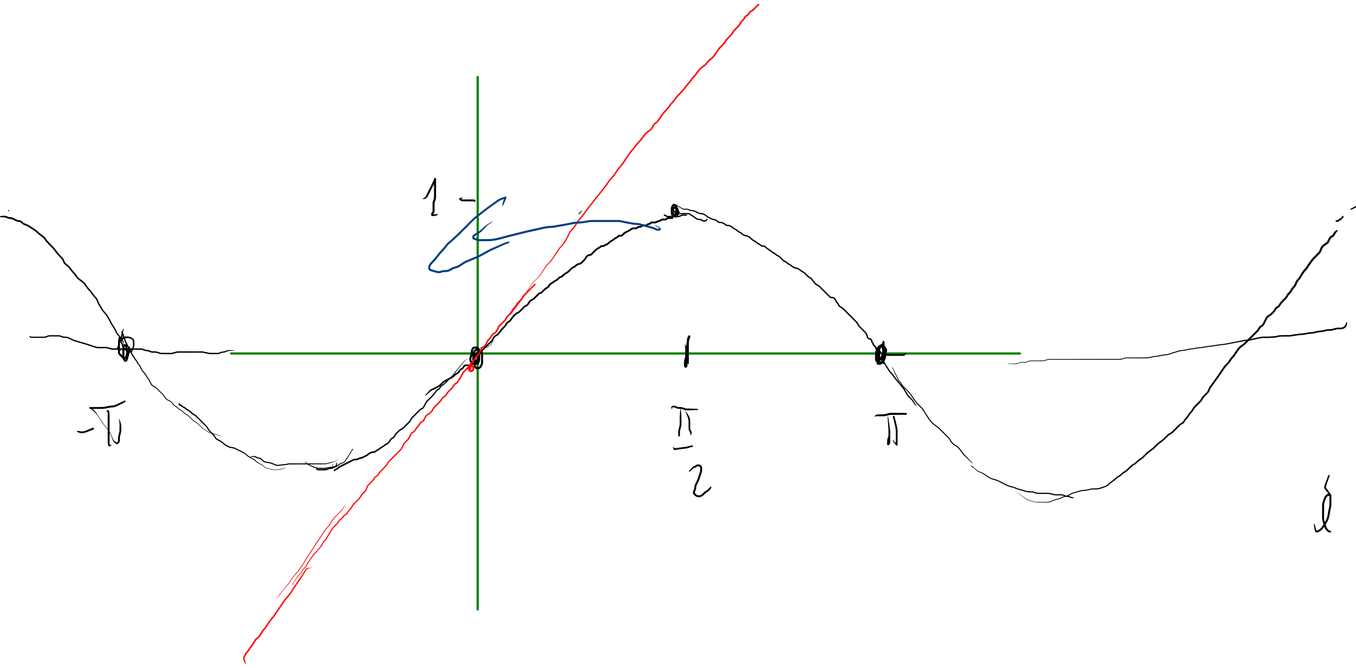
$x^2 + y^2 = 1$

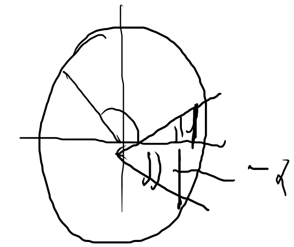
$x^2 = \frac{1}{2} \qquad x = \frac{1}{\sqrt{2}}$

$f : [0, \frac{\pi}{2}] \to \mathbb{R} : f(x) = \sin x$ è strettamente crescente

$f(0) = 0 \quad f(\frac{\pi}{2}) = 1$

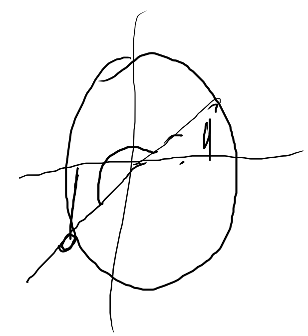
OSS: $\sin x$ è simmetrico rispetto $\frac{\pi}{2}$





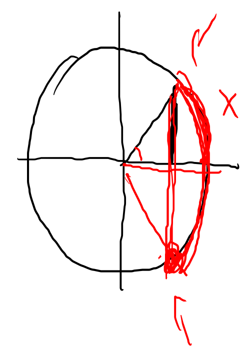
$\operatorname{sen}(-x) = -\operatorname{sen}(x)$

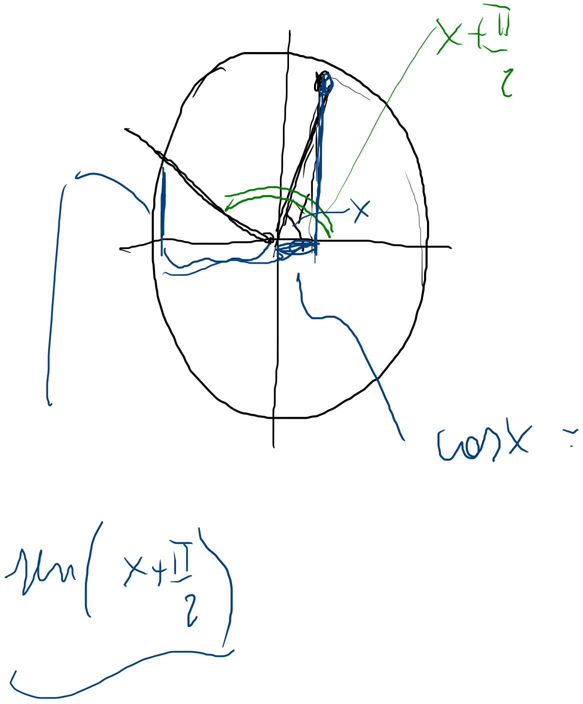
è una funzione dispari



Oss) $\operatorname{sen}(x+\pi) = -\operatorname{sen} x$

⚠ se $x \in \left]0, \frac{\pi}{2}\right[$ $\boxed{\operatorname{sen} x < x}$

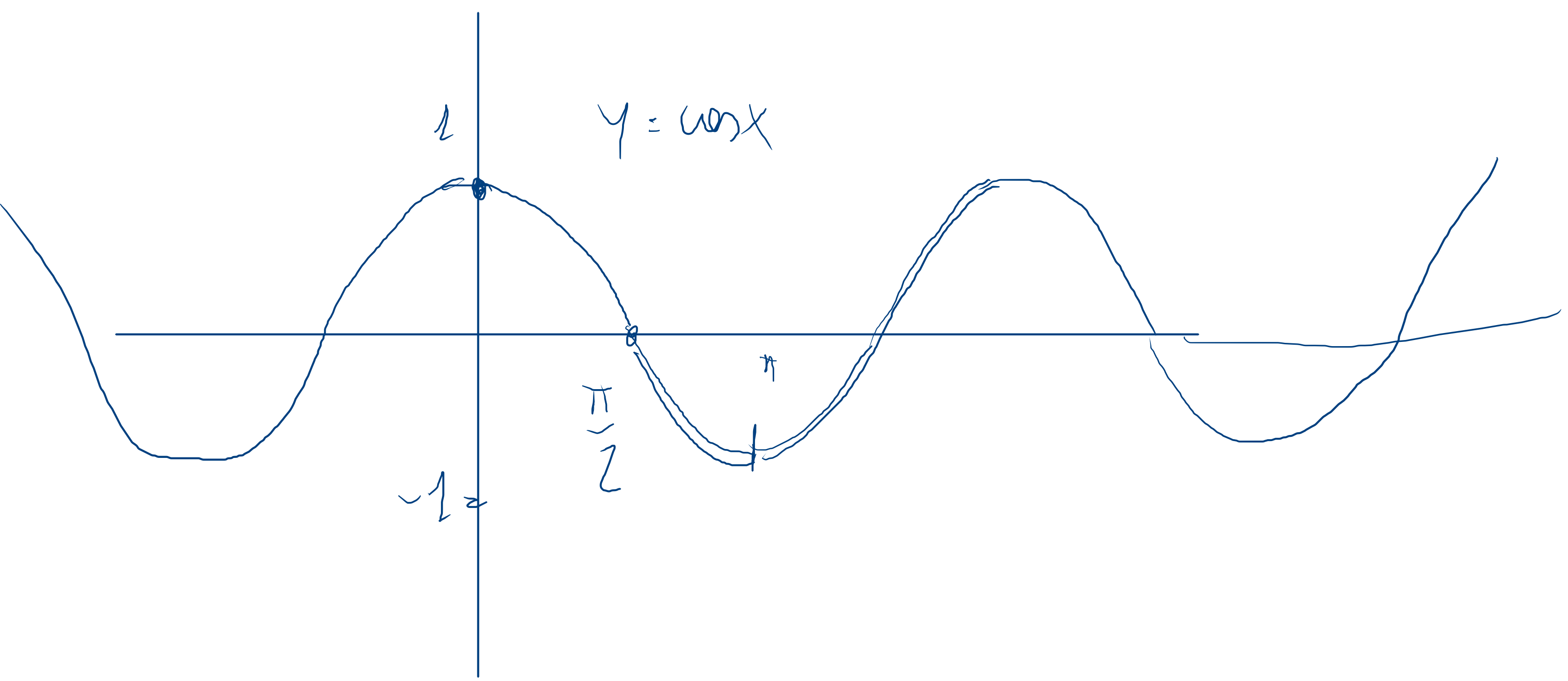




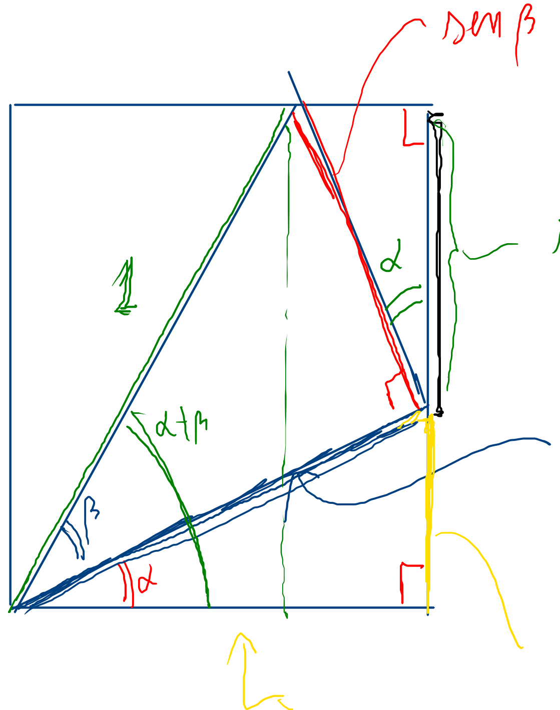
$$\cos(x) = \operatorname{sen}\left(x+\frac{\pi}{2}\right)$$

il grafico di $\cos x$ si ottiene traslando a sinistra il grafico di $\operatorname{sen} x$ di $\frac{\pi}{2}$

$\cos x$ è pari $\quad \cos(-x) = \cos(x)$



Formule di addizione



sen β · cos α

cos β · sen α

cos β · sen α

$$\boxed{\operatorname{sen}\beta \cdot \cos\alpha + \cos\beta \cdot \operatorname{sen}\alpha = \operatorname{sen}(\alpha+\beta)}$$

Esercizio: ricavare la formula $\cos(\alpha+\beta) = \cos(\alpha)\cdot\cos(\beta) - \operatorname{sen}(\alpha)\cdot\operatorname{sen}(\beta)$

$$\operatorname{sen}(2x) = \operatorname{sen}(x+x) = \operatorname{sen}(x)\cos(x) + \cos(x)\operatorname{sen}(x) = 2\operatorname{sen}(x)\cos(x)$$

$$\operatorname{sen}(x-y) = \operatorname{sen}(x)\cos(-y) + \cos(x)\operatorname{sen}(-y) = \operatorname{sen}(x)\cos(y) - \cos(x)\operatorname{sen}(y)$$

$$\cos(2x) = \cos^2(x) - \operatorname{sen}^2(x) = 2\cos^2(x) - 1$$

$$\boxed{\cos^2(x) = \frac{1+\cos(2x)}{2}}$$

$$\operatorname{sen}^2(x) = 1 - \cos^2(x)$$

Formule di prostoferesi e di Werner

$$\cos(x+y) = \cos(x)\cos(y) - \operatorname{sen}(x)\operatorname{sen}(y) \qquad \cos(x-y) = \cos(x)\cos(y) + \operatorname{sen}(x)\operatorname{sen}(y)$$

$$\operatorname{sen}(x+y) = \operatorname{sen}(x)\cos(y) + \cos(x)\operatorname{sen}(y) \qquad \operatorname{sen}(x-y) = \operatorname{sen}(x)\cos(y) - \cos(x)\operatorname{sen}(y)$$

$$\cos(x+y) + \cos(x-y) = 2\cos(x)\cos(y)$$

$$\rightarrow \cos(x+y) - \cos(x-y) = -2\operatorname{sen}(x)\operatorname{sen}(y)$$

$$\operatorname{sen}(x+y) + \operatorname{sen}(x-y) = 2\operatorname{sen}(x)\cos(y)$$

$$\operatorname{sen}(x+y) - \operatorname{sen}(x-y) = 2\cos(x)\operatorname{sen}(y)$$

$$\begin{cases} x+y = \alpha \\ x-y = \beta \end{cases} \qquad x = \frac{\alpha+\beta}{2} \qquad y = \frac{\alpha-\beta}{2}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$\vdots$

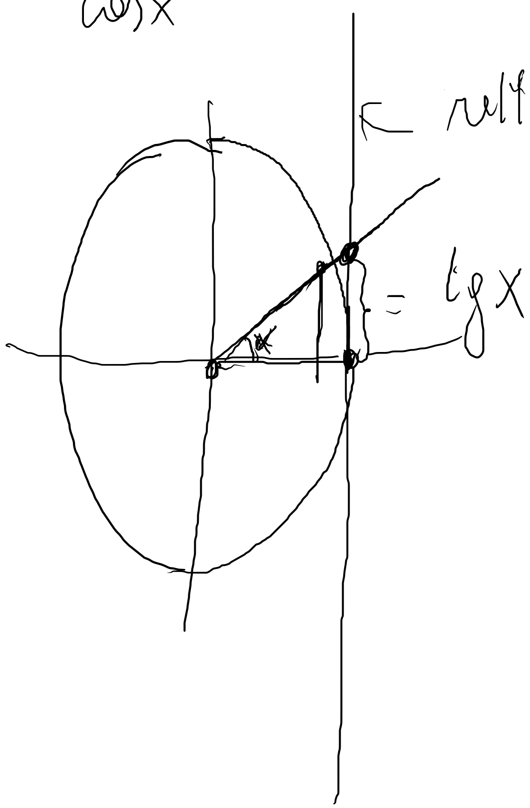
$$\cos(\alpha) - \cos(\beta) = -2\operatorname{sen}\left(\frac{\alpha+\beta}{2}\right)\operatorname{sen}\left(\frac{\alpha-\beta}{2}\right)$$

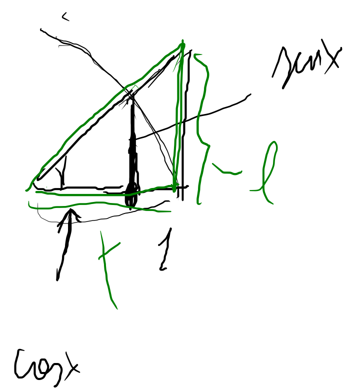
Cosmetico Medusense

funzione tangente

$$f(x) = \frac{\operatorname{sen} x}{\cos x} \qquad f : \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \right\} \qquad \operatorname{tg} x = \tan x$$

← retta tangente la circonferenza nel punto $(1,0)$





$$\frac{\operatorname{sen} x}{\cos x} = \frac{l}{1}$$