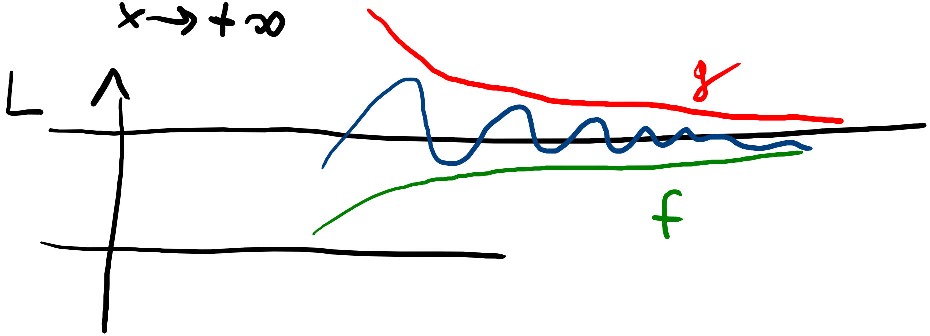
11 ottobre $\qquad X \subseteq \mathbb{R}$

Teor Se $f, g, h : X \to \mathbb{R}$, $\sup X = +\infty$,

$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} g(x) = L \in \overline{\mathbb{R}}$, e se $f(x) \leq h(x) \leq g(x)$,

allora $\lim_{x \to +\infty} h(x) = L$



Dim (con $L \in \mathbb{R}$)

$\lim_{x \to +\infty} f(x) = L \in \mathbb{R}$ significa che

$\forall \varepsilon > 0 \quad \exists M_1(\varepsilon) \in \mathbb{R}$ t.c. $x > M_1(\varepsilon)$ e $x \in X$

$\Rightarrow \quad |f(x) - L| < \varepsilon \iff L - \varepsilon < f(x) < L + \varepsilon$

$\lim_{x \to +\infty} g(x) = L$

$\forall \varepsilon > 0 \quad \exists M_2(\varepsilon) \in \mathbb{R}$ t.c. $x > M_2(\varepsilon)$ e $x \in X$

$\Rightarrow \quad |g(x) - L| < \varepsilon \iff L - \varepsilon < g(x) < L + \varepsilon$

$M_3(\varepsilon) = \max\{M_1(\varepsilon), M_2(\varepsilon)\}$

Allora per $x \in X$ e $x > M_3(\varepsilon)$ si hanno entrambe

$$L-\varepsilon < f(x) < L+\varepsilon$$

$$L-\varepsilon < g(x) < L+\varepsilon$$

Ricordando $f(x) \leq h(x) \leq g(x) \quad \forall x \in X$, per $x > M_3(\varepsilon)$ e $x \in X$ risulta

$$\boxed{L-\varepsilon < f(x) \leq h(x) \leq g(x) < L+\varepsilon}.$$

Concludiamo che $\forall \varepsilon > 0$, se $x > M_3(\varepsilon)$ e $x \in X$

allora $L - \varepsilon < h(x) < L + \varepsilon \iff |h(x) - L| < \varepsilon$

Cioè abbiamo dimostrato che

$$\forall \varepsilon > 0 \ \exists M(\varepsilon) \text{ t.c. } x > M(\varepsilon) \text{ e } x \in X$$

$$\implies |h(x) - L| < \varepsilon$$

$$M(\varepsilon) = M_3(\varepsilon).$$

Esempio Sia $b>1$. Allora $\displaystyle\lim_{n\to+\infty} \frac{b^n}{n} = +\infty$ $\frac{\infty}{\infty}$

Noi abbiamo già dimostrato che se $b>1$ allora

$$\lim_{n\to+\infty} b^n = +\infty$$

Naturalmente abbiamo anche $\displaystyle\lim_{n\to+\infty} n = +\infty$

L'esempio ci dice che se $n>>1$ allora

$$b^n >> n$$

$$\frac{b^n}{n}$$

$$b > 1 \Rightarrow \sqrt{b} > 1$$

$$\sqrt{b} = 1 + a \quad \text{con} \quad a > 0$$

$$= \frac{(\sqrt{b})^{2n}}{n} = \frac{(1+a)^{2n}}{n} = \frac{\left((1+a)^n\right)^2}{n} \geq$$

$$\boxed{\begin{gathered}(1+a)^n \geq 1 + na \\ \left((1+a)^n\right)^2 \geq (1+na)^2\end{gathered}}$$

$$\geq \frac{(1+na)^2}{n} =$$

$$= \frac{1 + 2na + n^2a^2}{n}$$

$$= \frac{1}{n} + 2a + na^2$$

$$+\infty > \frac{b^n}{n} \geq \frac{1}{n} + 2a + n a^2 \qquad \forall\, n \in \mathbb{N}$$

$$\lim_{n \to +\infty} \left(\frac{1}{n} + 2a + n a^2 \right) = \frac{1}{+\infty} + 2a + a^2 (+\infty) = 0 + 2a + \infty$$
$$= +\infty$$

$$\Longrightarrow \qquad \lim_{n \to +\infty} \frac{b^n}{n} = +\infty$$

Quindi per $n >> 1$ abbiamo $b^n >> n$.

<u>Esempio</u> $\forall\ N \in \mathbb{N}$ (con $b>1$) si ha $\lim_{n \to +\infty} \frac{b^n}{n^N} = +\infty$

cioè $\forall\ N \in \mathbb{N}$, se $n >> 1$ allora

$b^n >> n^N$.

$b > 1 \implies b^{\frac{1}{N+1}} > 1$. Si ha

$b^{\frac{1}{N+1}} = 1 + a \quad$ con $\quad a > 0$

$$b^n = \left(\left(b^{\frac{1}{N+1}} \right)^n \right)^{N+1} = \left((1+a)^n \right)^{N+1}$$

$$\geq \left(1 + na \right)^{N+1} = \sum_{k=0}^{N+1} \binom{N+1}{k} n^k a^k$$

$$\frac{b^n}{n^N} \geq \frac{\sum_{k=0}^{N+1} \binom{N+1}{k} n^k a^k}{n^N} =$$

$$= \sum_{k=0}^{N+1} \binom{N+1}{k} n^{k-N} a^k$$

$$+\infty > \frac{b^n}{n^N} \geq \sum_{k=0}^{N+1} \binom{N+1}{k} n^{k-N} a^k =$$

$$= \underbrace{\sum_{k=0}^{N-1} \binom{N+1}{k} n^{\overbrace{k-N}^{<0}} a^k}_{\substack{\downarrow n\to+\infty \\ 0}} + \underbrace{\overbrace{\binom{N+1}{N}}^{(N+1)} a^N}_{\substack{\downarrow \\ (N+1)a^N}} + \underbrace{n\, a^{N+1}}_{\substack{\downarrow \\ +\infty}} \xrightarrow{n\to+\infty} +\infty$$

$$\Rightarrow \lim_{n\to+\infty} \frac{b^n}{n^N} = +\infty ,$$

Esempio

Siano $R(x) = \dfrac{P(x)}{Q(x)}$

con $P(x) = a_N x^N + a_{N-1} x^{N-1} + \dots + a_0 \qquad a_N \neq 0$

$Q(x) = b_m x^m + b_{m-1} x^{m-1} + \dots + b_0 \qquad b_m \neq 0$

Allora

$$\lim_{n \to +\infty} R(n) = \lim_{n \to +\infty} \frac{a_N n^N}{b_m n^m}$$

$$\lim_{n \to +\infty} \frac{a_N n^N + \cancel{a_{N-1} n^{N-1} + \dots + a_0}}{b_m n^m + \cancel{b_{m-1} n^{m-1} + \dots + b_0}}$$

$$\frac{P(n)}{Q(n)} = \frac{a_N n^N + a_{N-1} n^{N-1} + \cdots + a_0}{b_m n^m + b_{m-1} n^{m-1} + \cdots + b_0} =$$

$$= \frac{a_N n^N \left(1 + \underbrace{\frac{a_{N-1}}{a_N}\frac{n^{N-1}}{n^N} + \cdots + \frac{a_0}{a_N}\frac{1}{n^N}}_{\to 0}\right)}{b_m n^m \left(1 + \underbrace{\frac{b_{m-1}}{b_m}\frac{n^{m-1}}{n^m} + \cdots + \frac{b_0}{b_m}\frac{1}{n^m}}_{\to 0}\right)}$$

$$\frac{P(n)}{Q(n)} = \frac{a_N n^N}{b_m n^m} \frac{(1 + o(1))}{(1 + o(1))}$$

dove denotiamo con $o(1)$ una successione t.c. $\lim_{n\to\infty} o(1) = 0$

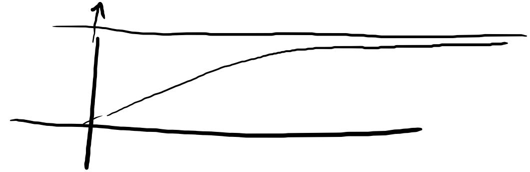
$$\lim_{n \to +\infty} \frac{P(n)}{Q(n)} = \lim_{n \to +\infty} \frac{a_N n^N}{b_m n^m}$$

Es $$\lim_{n \to +\infty} \frac{2n^3 + n^2 + n + 1}{-n^2 + n + 2} = \lim_{n \to +\infty} \frac{2n^3}{-n^2} =$$

$$= \lim_{n \to +\infty} (-2n) = -\infty$$

Teor (Limiti di funzioni monotone)

Sia $X \subseteq \mathbb{R}$ t.c. $\sup X = +\infty$ e sia $f : X \to \mathbb{R}$ monotona



1) Se f è crescente

$$\lim_{x \to +\infty} f(x) = \sup f(X) = \sup\{f(x) : x \in X\}$$

2) Se f è decrescente

$$\lim_{x \to +\infty} f(x) = \inf f(X) = \inf\{f(x) : x \in X\}$$

Dim Dimostriamo solo 1) e solo nel caso

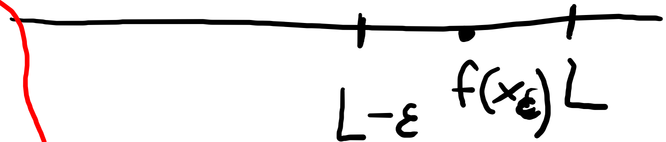
$\sup f(X) < +\infty$. Poniamo $L = \sup f(X) \in \mathbb{R}$

Vogliamo dimostrare che $\lim_{x \to +\infty} f(x) = L$

Dobbiamo dimostrare che

$$* \quad \boxed{\forall \varepsilon > 0 \ \exists M_\varepsilon \text{ t.c. } x > M_\varepsilon \text{ e } x \in X \Rightarrow |f(x) - L| < \varepsilon \Longleftrightarrow L - \varepsilon < f(x) < L + \varepsilon}$$

$$\forall \varepsilon > 0 \quad \exists\, x_\varepsilon \in X \quad \text{t.c.} \quad L-\varepsilon < f(x_\varepsilon) \le L$$



Infatti se questa ultima proposizione fosse falsa

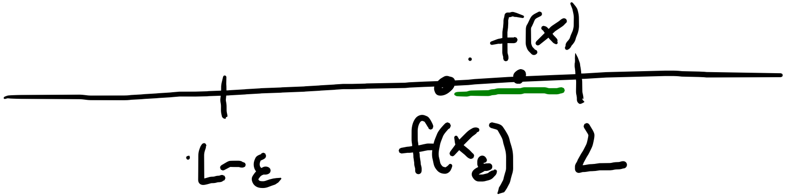
$$\exists\, \varepsilon_0 > 0 \quad \text{t.c.} \quad f(x) \le L-\varepsilon_0 \quad \forall x \in X$$

$$\overset{\text{2 prop sup}}{\Longrightarrow} \quad L-\varepsilon_0 \ge \sup\{f(x) : x \in X\} = L$$

$$L-\varepsilon_0 \ge L \iff -\varepsilon_0 \ge 0 \iff 0 < \varepsilon_0 \le 0 \quad \text{Assurdo}$$

Quindi ci siamo convinti che

$\forall \varepsilon > 0 \ \exists \, x_\varepsilon \in X$ t.c. $\quad L-\varepsilon < f(x_\varepsilon) \le L$



Per $\quad x > x_\varepsilon \quad$ e $x \in X \quad$ si ha

$$L-\varepsilon < f(x_\varepsilon) \le f(x) \le L$$

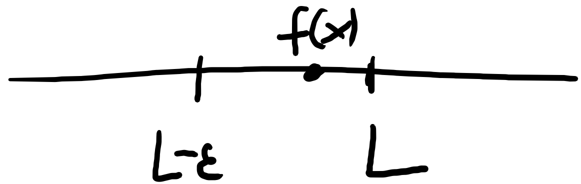
Concludiamo che per $\quad x > x_\varepsilon \quad$ e $\quad x \in X \quad$ ho

$L-\varepsilon < f(x) \le L$

Quindi abbiamo dimostrato che

$$\boxed{\forall \varepsilon > 0 \quad \exists M_\varepsilon \quad t.c. \quad x > M_\varepsilon \text{ e } x \in X \quad \Longrightarrow \quad |f(x) - L| < \varepsilon}$$

$M_\varepsilon = x_\varepsilon$



$$\Rightarrow \lim_{x \to +\infty} f(x) = L.$$

Teor (Numero di Neper e)

1) La successione $\left\{ \left(1+\frac{1}{n}\right)^n \right\}$ è strettamente crescente

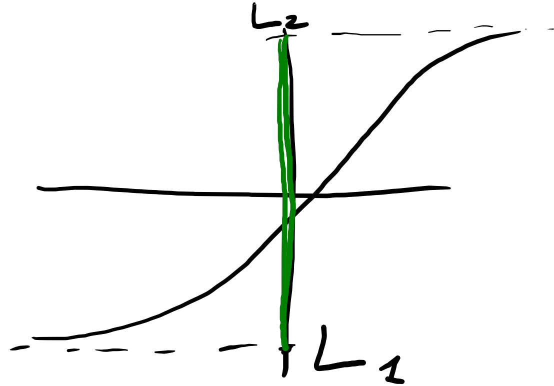
2) La successione $\left\{ \left(1+\frac{1}{n}\right)^{n+1} \right\}$ è strettamente decrescente

Abbiamo $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1} \doteq e$$

$2 < e < 3$

<u>Teor</u> Sia $f : X \to \mathbb{R}$ dove $X \subseteq \mathbb{R}$ con $\inf X = -\infty$

f monotona



1) Se f è crescente

$$\lim_{x \to -\infty} f(x) = \inf f(X)$$

2) Se f è decrescente

$$\lim_{x \to -\infty} f(x) = \sup f(X)$$

Esempio Sia $b > 1$. Allora $\lim_{x \to +\infty} b^x = +\infty$

Qui b^x è crescente in x

$$\lim_{x \to +\infty} b^x = \sup\{ b^x : x \in \mathbb{R} \} =$$

$$\geq \sup\{ b^n : n \in \mathbb{N} \} = \lim_{n \to +\infty} b^n = +\infty$$

$$\lim_{x \to +\infty} b^x = +\infty$$