12 ottobre.

Disuguaglianza di Young per conv.

$$\|f*g\|_{L^r} \leq |f|_{L^p} \, |g|_{L^q}$$

$$\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}$$

Funzione di distribuzione.

$g: \mathbb{R}^d \to \mathbb{R}$ per $\alpha>0$

$$d_g(\alpha)=\left|\{x\in\mathbb{R}^d : |g(x)|>\alpha\}\right|$$

$d_g$ è decrescente

Per $g\in L^p(\mathbb{R}^d)$ $1\leq p<+\infty$

$$\boxed{\int_{\mathbb{R}^d}|g(x)|^p\,dx} = \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\,\alpha^{p-1}\,d\alpha$$

$$= \int_0^{+\infty} d\alpha \; p\,\alpha^{p-1} \int_{\{x:\,|g(x)|>\alpha\}} dx =$$

$$= \boxed{\int_0^{+\infty} p\,\alpha^{p-1}\, d_g(\alpha)\; d\alpha}$$

$$F(x,\alpha)=|\alpha|^{p-1}\,\chi_{\mathbb{R}_+}(|g(x)|-\alpha)\,\chi_{\mathbb{R}_+}(\alpha)$$

**Def** $\quad L^{p,\infty}(\mathbb{R}^d) = \{ f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) :$ $\quad 1 \le p < +\infty$

**Lorentz** $\quad \|f\|_{L^{p,\infty}} = \sup_{\alpha > 0} \alpha \, d_f^{\frac{1}{p}}(\alpha) < +\infty \}$

Se $f \in L^p(\mathbb{R}^d) \Rightarrow |f|_{L^{p,\infty}} \le |f|_{L^p}$

$$d_f(\alpha) = |\{x : |f(x)| > \alpha\}| = |\{x : |f(x)|^p > \alpha^p\}|$$

$$\le \frac{|f|_{L^p}^p}{\alpha^p}$$

$$\sup_{\alpha > 0} \alpha \, d_f^{\frac{1}{p}}(\alpha) \le |f|_{L^p}$$

$$|f|_{L^p}^p = \int_{\mathbb{R}^d} |f(x)|^p dx \ge \int_{\{x : |f(x)|^p > \alpha^p\}} |f(x)|^p dx$$

$$\ge \alpha^p \, d_f(\alpha)$$

Es 1) $\alpha \in (0,1)$ $\quad t^{-\alpha}\chi_{\mathbb{R}_+} \in L^{p,\infty}(\mathbb{R}) \Leftrightarrow \alpha p = 1$

2) $T > 0$ $\quad t^{-\alpha}\chi_{[0,T]} \in L^{p,\infty}(\mathbb{R}) \Leftrightarrow \alpha p \leq 1$.

per $\alpha p = 1$ caso 2 è una conseguenza di caso 1, mentre per $\alpha p < 1$

$t^{-\alpha}\chi_{[0,T]} \in L^p(\mathbb{R})$ e questo implica il resto di 2)

Esempio Se $a \in (0,d)$ allora $|x|^{-a} \in L^{p,\infty}(\mathbb{R}^d)$

$\Leftrightarrow \quad ap = d$.

$$\alpha \, |\{x \in \mathbb{R}^d : |x|^{-a} > \alpha\}|^{\frac{1}{p}}$$ limitato?

$$|x|^{-a} > \alpha \qquad \frac{1}{\alpha} > |x|^a$$

$$|x| < \frac{1}{\alpha}^{\frac{1}{a}}$$

$$= \alpha \left(c_d \, \alpha^{-\frac{d}{a}}\right)^{\frac{1}{p}} = c_d^{\frac{1}{p}} \, \alpha^{1 - \frac{d}{ap}}$$

è limitato in $\mathbb{R}_+ \Leftrightarrow 1 - \frac{d}{ap} = 0$ ✓

**Teor** Siano $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ con

$1<p,q,r<+\infty$. Allora $\exists\, C$ t.c.

$$\| g * f \|_{L^r} \le C \, \| g \|_{L^{q,\infty}} \, \| f \|_{L^p} .$$

**Teor** (Hardy - Littlewood - Sobolev)

Per ogni

$\gamma \in (0,d)$ e $1<p<q<\infty$ con

$\frac{1}{p}=\frac{1}{q}+\frac{d-\gamma}{d}$ $\exists\, C$ t.c.

$$\left\| \int_{\mathbb{R}^d} f(x-y)\, |y|^{-\gamma}\, dy \right\|_{L^q(\mathbb{R}^d)} \le C \, \| f \|_{L^p(\mathbb{R}^d)}$$

**Dim** $|x|^{-\gamma} \in L^{\frac{d}{\gamma},\infty}(\mathbb{R}^d)$ si ha

$$\| f * |x|^{-\gamma} \|_{L^q(\mathbb{R}^d)} \le C \, \| |x|^{-\gamma} \|_{L^{\frac{d}{\gamma},\infty}} \, \| f \|_{L^p}$$

se $\frac{1}{q}+1=\frac{d}{\gamma}+\frac{1}{p}$ grazie al precedente teorema.

<u>Lemma</u> $\forall\ \gamma \in (0,d)$ $\exists\ c_\gamma > 0$ t.c.

$$\mathcal{F}\left(|x|^{-\gamma}\right)(\xi) = c_\gamma\, |\xi|^{-(d-\gamma)}$$

$$\int_{\mathbb{R}^d} |x|^{-\gamma} \phi(x)\, dx = c_\gamma \int_{\mathbb{R}^d} |\xi|^{\gamma-d}\, \hat{\phi}(\xi)\, d\xi$$

$$\int_{\mathbb{R}^d} \varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \phi(x) = \int_{\mathbb{R}^d} e^{-\varepsilon\frac{|\xi|^2}{2}}\, \hat{\phi}(\xi)\, d\xi$$

$$e^{-\varepsilon\frac{|\xi|^2}{2}} = \mathcal{F}\left(\varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}}\right)$$

$$\int_0^\infty \frac{d\varepsilon}{\varepsilon}\, \varepsilon^{\frac{\cancel{d}-\gamma}{2}} \int_{\mathbb{R}^d} \varepsilon^{-\frac{\cancel{d}}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \phi(x)\, dx$$

$$= \int dx\, \phi(x) \int_0^{+\infty} \left(\frac{\varepsilon}{|x|^2}\right)^{-\frac{\gamma}{2}} e^{-\frac{|x|^2}{2\varepsilon}}\, \frac{d\varepsilon}{\varepsilon} \qquad \frac{\varepsilon}{|x|^2}$$

$$= \int_{\mathbb{R}^d} dx\, \phi(x)\, |x|^{-\gamma} \underbrace{\int_0^{+\infty} \varepsilon^{-\frac{\gamma}{2}} e^{-\frac{1}{2\varepsilon}}\, \frac{d\varepsilon}{\varepsilon}}_{a_\gamma}$$

$$\int_0^{+\infty} \frac{d\varepsilon}{\varepsilon}\, \varepsilon^{\frac{d-\gamma}{2}} \int_{\mathbb{R}^d} e^{-\varepsilon\frac{|\xi|^2}{2}}\, \hat{\phi}(\xi)\, d\xi =$$

$$= \int_{\mathbb{R}^d} d\xi\, \hat{\phi}(\xi) \int_{\mathbb{R}^d} \left(|\xi|^2\varepsilon\right)^{\frac{d-\gamma}{2}} e^{-\varepsilon\frac{|\xi|^2}{2}}\, \frac{d\varepsilon}{\varepsilon} \qquad |\xi|^{d-\gamma}$$

$$= \int_{\mathbb{R}^d} d\xi\, \hat{\phi}(\xi)\, |\xi|^{\gamma-d}\, b_\gamma \qquad \Bigg|\ c_\gamma = \frac{2^{\frac{d-\gamma}{2}}\, \Gamma\left(\frac{d-\gamma}{2}\right)}{2^{\frac{\gamma}{2}}\, \Gamma\left(\frac{\gamma}{2}+1\right)}$$

Teor Sia $s \in (0, \frac{d}{2})$ e $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$

Allora $\exists$ $C = C_{s\,d}$ t.c.

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C \, \|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

Dim

$$f(x) = (2\pi)^{-\frac{d}{2}} \int e^{i\xi x} \overbrace{\left(\hat{f}(\xi)|\xi|^s\right)}^{\hat{g}(\xi)} |\xi|^{-s} \, d\xi$$

$$= c \int g(x-y) \; |y|^{-(d-s)} \, dy$$

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C \; \|g\|_{L^2(\mathbb{R}^d)} = \|\hat{g}\|_{L^2} = \|f\|_{\dot{H}^s}$$

se

$$\frac{1}{q} = \frac{1}{2} - \frac{\cancel{d} - (\cancel{d} - s)}{d} = \frac{1}{2} - \frac{s}{d}$$

Lemma $\forall\ s \in [0,1]$, e se $k = s k_1 + (1-s) k_2$

allora

$$\| f \|_{\dot H^k} \le \| f \|_{\dot H^{k_1}}^{s} \ \| f \|_{\dot H^{k_2}}^{(1-s)}$$

$s = s \cdot 1 + (1-s) 0$

In particolare per $s \in [0,1]$ ho

$$\| f \|_{\dot H^s} \le \| f \|_{L^2}^{1-s} \ \| f \|_{\dot H^1}^{s}$$

$$= \| f \|_{L^2}^{1-s} \ \| \nabla f \|_{L^2}^{s}$$

$k = s k_1 + (1-s) k_2$

$$\| f \|_{\dot H^k}^2 = \int |\xi|^{2k} |\hat f(\xi)|^2$$

$$= \int |\xi|^{2 s k_1} |\hat f(\xi)|^{2s} \ |\hat f|^{2(1-s)} \ |\xi|^{2(1-s) k_2} \, d\xi$$

$$= \int \left( |\xi|^{2k_1} |\hat f(\xi)|^2 \right)^s \ \left( |\hat f|^2 |\xi|^{2k_2} \right)^{1-s} \, d\xi$$

$\frac{1}{s}$ $\quad$ $\frac{1}{1-s}$

$$\le \left\| \left( |\xi|^{k_1} |\hat f| \right)^{2s} \right\|_{L^{\frac{1}{s}}} \ \left\| \left( |\xi|^{k_2} |\hat f| \right)^{2(1-s)} \right\|_{L^{\frac{1}{1-s}}}$$

$$= \left\| |\xi|^{k_1} |\hat f| \right\|_{L^{\cancel{2s/s}}}^{2s} \ \left\| |\xi|^{k_2} |\hat f| \right\|_{L^2}^{2(1-s)}$$

$$= \| f \|_{\dot H^{k_1}}^{2s} \ \| f \|_{\dot H^{k_2}}^{2(1-s)}$$

$$\| f \|_{\dot H^k}^{\cancel{2}} \le \| f \|_{\dot H^{k_1}}^{\cancel{2}s} \ \| f \|_{\dot H^{k_2}}^{\cancel{2}(1-s)} \ .$$

**Lemma** Siano $0 < r < \frac{d}{2} < s$. Allora

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq \|u\|_{\dot{H}^r(\mathbb{R}^d)}^{\frac{s-\frac{d}{2}}{s-r}} \; \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{\frac{d}{2}-r}{s-r}}$$

Esempio

$$\boxed{\|u\|_{L^\infty(\mathbb{R}^3)} \leq \|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} \|\nabla^2 u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)}}$$

$\|u\|_{\dot{H}^{\frac{3}{2}}}$

Supponiamo che

$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C \, \|u\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}$ è falsa

$d \geq 1$

$$W^{d,1}(\mathbb{R}^d) \not\leq L^\infty(\mathbb{R}^d)$$

$$W^{\frac{d}{p},p}(\mathbb{R}^d) \leq L^\infty(\mathbb{R}^d)$$

$s > \frac{d}{2}$ $\qquad H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$

$$u(x) = \int_{\mathbb{R}^d} e^{ix\xi} \, \hat{u}(\xi)\, d\xi$$

$$|u(x)| \le \int_{\mathbb{R}^d} \langle \xi \rangle^{-s} \langle \xi \rangle^{s} \, |\hat{u}(\xi)|\, d\xi$$

$$\le \left| \langle \xi \rangle^{-s} \right|_{L^2} \left| \langle \xi \rangle^{s} \hat{u} \right|_{L^2}$$

$$\le C_{sd} \, |u|_{H^s}$$

$$|u(x)| \le \int_{\mathbb{R}^d} |\hat{u}|\, d\xi = \int_{|\xi| \le R} |\hat{u}|\, |\xi|^{r} \, |\xi|^{-r} d\xi$$

$$+ \int_{|\xi| \ge R} |\hat{u}| \, |\xi|^{s} \, |\xi|^{-s} d\xi$$

$$\le \Big( \int_{|\xi| \le R} \overset{|\xi|^{-2r+d-1}\, d\xi}{|\xi|^{-2r} d\xi} \Big)^{\frac{1}{2}} |u|_{\dot{H}^r} + \Big( \int_{|\xi| \ge R} \overset{|\xi|^{-2s+d-1}\, d|\xi|}{|\xi|^{-2s} d\xi} \Big)^{\frac{1}{2}} |u|_{\dot{H}^s}$$

$$\lesssim R^{\frac{d}{2} - r} |u|_{\dot{H}^r} + R^{\frac{d}{2} - s} |u|_{\dot{H}^s}$$

$$|u|_{L^\infty} \lesssim R^{\frac{d}{2} - r} |u|_{\dot{H}^r} + R^{\frac{d}{2} - s} |u|_{\dot{H}^s}$$

$$R^{\cancel{\frac{d}{2}} - r} |u|_{\dot{H}^r} = R^{\cancel{\frac{d}{2}} - s} |u|_{\dot{H}^s}$$

$$R^{s-r} = \frac{|u|_{\dot{H}^s}}{|u|_{\dot{H}^r}} \qquad R = \frac{|u|_{\dot{H}^s}^{\frac{1}{s-r}}}{|u|_{\dot{H}^r}^{\frac{1}{s-r}}}$$

$$R^{\frac{d}{2}-r}\,\|u\|_{\dot H^r} \qquad \uparrow n \qquad R=\frac{\|u\|_{\dot H^s}^{\frac{1}{s-r}}}{\|u\|_{\dot H^r}^{\frac{1}{s-r}}}$$

$$= \frac{\|u\|_{\dot H^s}^{\frac{\frac{d}{2}-r}{s-r}}}{\|u\|_{\dot H^r}^{\frac{\frac{d}{2}-r}{s-r}}}\,\|u\|_{\dot H^r} =$$

$$= \|u\|_{\dot H^s}^{\frac{\frac{d}{2}-r}{s-r}}\;\|u\|_{\dot H^r}^{\frac{s-\cancel{r}}{s-r}-\frac{\frac{d}{2}-\cancel{r}}{s-r}}$$

$$= \|u\|_{\dot H^s}^{\frac{\frac{d}{2}-r}{s-r}}\;\|u\|_{\dot H^r}^{\frac{s-\frac{d}{2}}{s-r}}$$

Stein - Weiss 69

Grafakos

Keel - Tao disuguaglianza di Strichartz

endpoint