

TEORIE DI GAUGE NON-ABELIANE

Prendiamo una teoria inv. sotto una transf. globale
data da un gruppo di Lie G semplice (semi-semplice)

Prendiamo un campo $\varphi \rightarrow \varphi(x)$ in una rapp. R ,
che transf. sotto G cioè è un campo a valori
in V_R (sezione del $\mathbb{P}h$
VETT. con fibre $E_x \cong V_R$)

Es: $G = SU(2)$ $R = \underline{2}$

$\varphi \in V_R$ $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$

Se φ è "uno scalare
nella rep. $\underline{2}$ " vuol
dire che φ_1 e φ_2
sono scalari!

(φ prende valori in
 $V_R \otimes V_{\text{Lorentz}}$)

Lagrangiana invariante (φ scalare \mathbb{C})

$L = \partial_\mu \varphi_1^* \partial^\mu \varphi_1 + \partial_\mu \varphi_2^* \partial^\mu \varphi_2 - m^2 |\varphi_1|^2 - m^2 |\varphi_2|^2$

← prodotto tensoriale
di sp. di rep. di
gruppi diversi $\underline{2}$

$= \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi$ vale anche k
 $SU(N)$ $N \otimes \bar{N} = 1 \oplus \text{Adj}$

↑ c'è singoletto

$\xrightarrow{U \in SU(2)}$ $\partial_\mu \varphi^\dagger U^\dagger U \partial^\mu \varphi - m^2 \varphi^\dagger U^\dagger U \varphi = L$

qta forma
rende L
manifestamente
invariante

$e \partial \varphi \mapsto U \partial \varphi$

Ora abbiamo

$L(\varphi, \partial \varphi)$ invariante sotto $\varphi \mapsto U \varphi$ (U matrice
costante)

$U \in G$

$U = e^{-i \alpha^a t_a^R}$

φ trasforma in rep. R

cioè $L(U\varphi, U\partial\varphi) = L(\varphi, \partial\varphi)$

Passiamo a trasf. locali (di gauge)

$$U \mapsto U(x) = e^{-i\alpha^a(x)t_a}$$

$\partial_\mu \psi$ non trasforma più
come $U \partial_\mu \psi$

$$\partial_\mu \psi \mapsto \partial_\mu (U\psi) = \underline{(\partial_\mu U)\psi} + U \partial_\mu \psi$$

Introduciamo la DERIVATA COVARIANTE (vorremo che $D\psi \mapsto U D\psi$)

$$D_\mu \psi = (\partial_\mu + i A_\mu) \psi$$

↑
matrice

analogo
in GR

$$D_\mu v^s = \partial_\mu v^s + \Gamma_{\mu\nu}^s v^\nu$$

↑
matrice
con indici s, v

$$(D_\mu \psi)_i = \partial_\mu \psi_i + i (A_\mu)_i^j \psi_j$$

A_μ è definita in modo tale che

$$D_\mu \psi \mapsto U D_\mu \psi$$

$$D_\mu \psi \mapsto (\partial_\mu + i A'_\mu) \psi' = (\partial_\mu + i A'_\mu) U \psi =$$

$$= (\partial_\mu U) \psi + \underline{U \partial_\mu \psi} + i A'_\mu U \psi$$

$$+ \underline{i U A_\mu \psi} - i U A_\mu \psi$$

$$= \underline{U D_\mu \psi} + (\partial_\mu U + i A'_\mu U - i U A_\mu) \psi$$

⇒ Otteniamo $D_\mu \psi \mapsto U D_\mu \psi$ se

$$A'_\mu = U A_\mu U^{-1} + i (\partial_\mu U) U^{-1}$$

$$[= -i U \partial_\mu U^{-1} \text{ perché } U U^{-1} = 1]$$

$$\Rightarrow \partial_\mu (U U^{-1}) = 0$$

$\mathcal{L}(\psi, D_\mu \psi)$ è invariante sotto transf. locali:

$$\begin{cases} \psi' = U\psi \\ A'_\mu = U A_\mu U^{-1} + i(\partial_\mu U)U^{-1} \end{cases}$$

perché
 $\mathcal{L}(X, Y) = \mathcal{L}(UX, UY)$

Transf. INFINITESIME $\iff \alpha^a(x) \ll 1$

$$U(x) = e^{-i \alpha^a(x) t_R^a} \simeq 1 - i \alpha^a(x) t_R^a \equiv 1 - i \underbrace{\alpha^a(x)}_{\text{oggetto nelle}}$$

* off. in rep. Adj
 gli altri: combinat. lin.
 di generatori dell'Algebra
 (in questi rep t_R^a sono)

$$\delta\psi = -i \alpha^a(x) \psi \quad (\delta\psi \equiv \psi' - \psi)$$

RAPP. AGGIUNTA
 (abuso di linguaggio, *)

$$\delta A_\mu = A'_\mu - A_\mu = (1 - i\alpha) A_\mu (1 + i\alpha) + i(\partial_\mu (1 - i\alpha))(1 + i\alpha) - A_\mu$$

$$= A_\mu - i\alpha A_\mu + i A_\mu \alpha + \partial_\mu \alpha - A_\mu$$

$$= \partial_\mu \alpha + i[A_\mu, \alpha] \quad \leftarrow D_\mu \alpha \quad (*) \text{ vedi sotto}$$

$$= \partial_\mu \alpha^a t_R^a - i\alpha^a [t_R^a, A_\mu]$$

Se prendiamo α^a cost. $\rightsquigarrow \delta A_\mu = -i[\alpha^a t_R^a, A_\mu]$

$\Rightarrow A_\mu$ transf. nelle rep Adj

$$\rightarrow A_\mu = A_\mu^a t_R^a \quad (*)$$

La connessione che opera su un vett. nella rapp. R è data da (*)

(Se allora $\psi \in R$ e $X \in \tilde{R}$
 $D_\mu \psi = \partial_\mu \psi + i A_\mu^a t_R^a \psi$ $D_\mu X = \partial_\mu X + i A_\mu^a t_R^a X$)

$$\begin{aligned}
\delta A_\mu &= \partial_\mu \alpha^a t_R^a - i \alpha^a A_\mu^b [t_R^a, t_R^b] = \\
&= \partial_\mu \alpha^c t_R^c - i \alpha^a A_\mu^b f^{abc} t_R^c \\
&= (\partial_\mu \alpha^c + f^{cab} \alpha^a A_\mu^b) t_R^c \equiv \delta A_\mu^c t_R^c \\
&\quad \Downarrow \\
\delta A_\mu^a &= \partial_\mu \alpha^a - f^{abc} A_\mu^b \alpha^c
\end{aligned}$$

$$(*) \quad D_\mu \alpha = \partial_\mu \alpha + i A_\mu^a t_{Ad}^a \alpha = \partial_\mu \alpha + i A_\mu^a [t^a, \alpha] = \\
= \partial_\mu \alpha + i [A_\mu, \alpha]$$

TERMINI CINETICO per i bosoni di gauge $A_\mu^a(x)$

↑ INV. sotto Lorentz, gauge e quadratico nelle derivate.

Finora introdotto campo nuovo, ma scritto solo termini interatt.

$$\left[\text{In QED avevamo } -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \right]$$

Partiamo calcolando $[D_\mu, D_\nu]$

$$\begin{aligned}
& [(\partial_\mu + i A_\mu), (\partial_\nu + i A_\nu)] \psi = \\
& = \underbrace{[\partial_\mu, \partial_\nu]}_{=0} \psi + i [\partial_\mu, A_\nu] \psi - i [\partial_\nu, A_\mu] \psi \\
& \quad - \underbrace{[A_\mu, A_\nu]}_{\neq 0 \text{ in teor. di gauge non-ab}} \psi
\end{aligned}$$

$$\begin{aligned}
& = i \partial_\mu (A_\nu \psi) - i A_\nu \partial_\mu \psi - (\mu \leftrightarrow \nu) \\
& \quad - [A_\mu, A_\nu] \psi
\end{aligned}$$

$$= i(\partial_\mu A_\nu) \varphi + i \cancel{A_\nu} \partial_\mu \varphi - i \cancel{A_\nu} \partial_\mu \varphi - (\mu \leftrightarrow \nu) - [A_\mu, A_\nu] \varphi$$

$$= i \left[\partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] \right] \varphi$$

$$\equiv i F_{\mu\nu} \varphi$$

$$F_{\mu\nu} \equiv -i [D_\mu, D_\nu]$$

Vediamo come trasforma $F_{\mu\nu}$

$$F_{\mu\nu} \varphi = [D_\mu, D_\nu] \varphi \mapsto U [D_\mu, D_\nu] \varphi = U F_{\mu\nu} \varphi =$$

$$= U F_{\mu\nu} U^{-1} (U \varphi)$$

$$\mapsto F'_{\mu\nu} \varphi' = F'_{\mu\nu} (U \varphi)$$

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}$$

trasforma in maniera covariante (come un'obj. nelle rep. Adj) con $U(x)$.

possiamo espandere

$$F_{\mu\nu} = F_{\mu\nu}^a t_a^a \quad \leftarrow \text{CURVATURA (della connessione } A_\mu)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] = \partial_\mu A_\nu^a t_a^a - \partial_\nu A_\mu^a t_a^a + i A_\mu^b A_\nu^c [t_b^b, t_c^c] =$$

$$= (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c) t_a^a$$

$$\Rightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$$

Il termine cinetico dev'essere Lorentz inv. e gauge inv.

$$= -k \operatorname{tr} (F_{\mu\nu} F^{\mu\nu})$$

$$\operatorname{tr}(FF) \rightarrow \operatorname{tr}(U F U^{-1} U F U^{-1}) =$$

$$= \operatorname{tr}(FF)$$

↑
ciclicità della traccia

$$= -k F_{\mu\nu}^a F^{b\mu\nu} \operatorname{tr}(t_R^a t_R^b) =$$

$$= -k F_{\mu\nu}^a F^{b\mu\nu} C(R) \delta^{ab} = -k C(R) F_{\mu\nu}^a F^{a\mu\nu}$$

$$\rightarrow = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \quad \Leftarrow \quad k = \frac{1}{4C(R)}$$

↑
termine
cinetico normalizzato
canonico.

$$\downarrow$$

$G = SU(N) \quad e \quad R = N \quad C(N) = 1/2$

$$\rightarrow k = \frac{1}{2}$$

[Se G non è compatto, allora $\operatorname{tr}(t_R^a t_R^b) \neq \delta^{ab}$,
ma sulla diagonale ci saranno $+1$ e -1
 \Rightarrow presenza dei termini cinetici negativi.]

$$-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

contiene termini CUBICI e QUARTICI in A_μ^a

\rightarrow termini di interazione tra i
bosoni vettori

La lagrangiana $\mathcal{L}(\varphi, A)$ inv. sotto transf. gauge

$$\mathcal{L}(\varphi, A) = \underbrace{\mathcal{L}(\varphi, D_\mu \varphi)}_{\text{bepk}} - \frac{1}{4g^2} F_{\mu\nu}^a F^{a\mu\nu}$$

Per i calcoli perturbativi, spesso è più utile risolvere

$$A_\mu^a \mapsto g A_\mu^a \quad (\text{ridef. di campo})$$

$$F_{\mu\nu}^a \mapsto g \partial_\mu A_\nu^a - g \partial_\nu A_\mu^a - g^2 f^{abc} A_\mu^b A_\nu^c$$

$$= g F_{\mu\nu}^a$$

g^2 sparisce dal termine c.m.

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$$

$$D_\mu \psi \mapsto (\partial_\mu + ig A_\mu) \psi$$

$$\mathcal{L} = \mathcal{L}_{\text{ferm}}(\psi, D_\mu \psi) - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

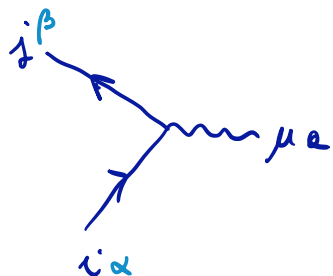
g è la COSTANTE DI ACCOPPIAMENTO

REGOLE DI FEYNMAN (massless)

Scegliamo ψ spinore^v e lo diciamo ψ

$$\begin{aligned} \mathcal{L}_{\text{ferm}}(\psi, D_\mu \psi) &= i \bar{\psi} \gamma^\mu D_\mu \psi = \\ &= i \bar{\psi} \gamma^\mu \partial_\mu \psi - \underbrace{g \bar{\psi} \gamma^\mu A_\mu \psi}_{\text{molteplice}} \end{aligned}$$

$$-g \bar{\psi} \gamma^\mu A_\mu^a t_R^a \psi = -g \bar{\psi}_{i\alpha} \gamma^\mu_{\alpha\beta} (t_R^a)_{ij} \psi_{j\beta} A_\mu^a$$



$$-ig \gamma^\mu_{\alpha\beta} (t_R^a)_{ij}$$

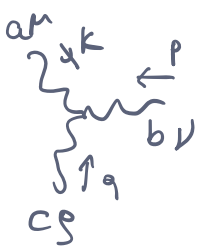
$$L_{kin}(A) = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c) \cdot (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu} - g f^{ade} A^\mu_d A^\nu_e)$$

$$= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \frac{1}{2} g f^{abc} A_\mu^b A_\nu^c (\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}) - \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^\mu_d A^\nu_e$$

propagator
(?)

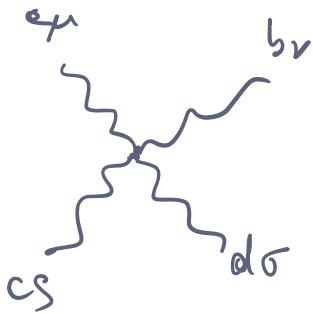
Vertex

$$-i g \int d^4x \tilde{A}_\alpha^h(x) \tilde{A}_\beta^r(y) K^{\alpha\beta} \tilde{A}_\gamma^m(z) \eta^{\beta\gamma}$$



$$\frac{\delta}{\delta \tilde{A}_\mu^a} \frac{\delta}{\delta \tilde{A}_\nu^b} \frac{\delta}{\delta \tilde{A}_\rho^c}$$

$$g f^{abc} [\eta^{\mu\nu} (p-k)^\rho + \eta^{\nu\rho} (q-p)^\mu + \eta^{\mu\rho} (k-q)^\nu]$$



$$-i g^2 [f^{abe} f^{cde} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ace} f^{bde} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ade} f^{bce} (\eta^{\mu\nu} \eta^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})]$$

[PESKIN]