Esercizi
13 ottobre 2021

La serie

$$\sum_{n=0}^{+\infty} \frac{c^n}{n!}$$

converge per ogni $c \in \mathbb{R}$

Con il criterio del rapporto

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{\frac{c^{n+1}}{(n+1)!}}{\frac{c^n}{n!}}\right| = \lim_{n\to+\infty}\left|\frac{c^{n+1}}{(n+1)!}\cdot\frac{n!}{c^n}\right| =$$

$$= \lim_{n\to+\infty}|c|\frac{1}{(n+1)n!}\cdot n! = \lim_{n\to+\infty}\frac{|c|}{n+1} = 0$$

Con il criterio della radice

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \sqrt[n]{\left|\frac{c^n}{n!}\right|} = |c| \lim_{n\to+\infty} \left(\sqrt[n]{n!}\right)^{-1}$$

Osserviamo che:

$$\sqrt[n]{n!} = e^{\frac{\log n!}{n}} = e^{\frac{\log 1 + \log 2 + \ldots \log n}{n}}$$

- Se n è pari allora:

$$\sqrt[n]{n!} = e^{\frac{\log n!}{n}} = e^{\frac{\log 1 + \log 2 + \dots \log n}{n}}$$

$$= e^{\left(\frac{\log 1 + \log 2 + \dots + \log\left(\frac{n}{2} - 1\right)}{n}\right)} \cdot e^{\left(\frac{\log\left(\frac{n}{2}\right) + \log\left(\frac{n}{2} + 1\right) + \dots + \log n}{n}\right)}$$

$$\geq e^{\left(\frac{\log\left(\frac{n}{2}\right) + \log\left(\frac{n}{2} + 1\right) + \dots + \log n}{n}\right)}$$

$$\geq e^{\frac{(n+2)}{2}\left(\frac{\log\left(\frac{n}{2}\right)}{n}\right)} = e^{\frac{(n+2)}{2n}\log\left(\frac{n}{2}\right)}$$

$$\geq e^{\frac{n}{2n}\log\left(\frac{n}{2}\right)} =$$

$$e^{\log\left(\frac{n}{2}\right) \cdot \frac{1}{2}} = \sqrt{\frac{n}{2}} \to +\infty$$

- Se n è dispari allora:

$$
\begin{aligned}
\sqrt[n]{n!} = e^{\frac{\log n!}{n}} &= e^{\frac{\log 1+\log 2+\cdots+\log n}{n}} \\
&= e^{\left(\frac{\log 1+\log 2+\cdots \log\left(\frac{n-1}{2}\right)}{n}\right)} \cdot e^{\left(\frac{\log\left(\frac{n+1}{2}\right)+\log\left(\frac{n+1}{2}+1\right)+\cdots+\log n}{n}\right)} \\
&\geq e^{\left(\frac{\log\left(\frac{n+1}{2}\right)+\log\left(\frac{n+1}{2}+1\right)+\cdots+\log n}{n}\right)} \\
&\geq e^{\frac{(n+1)}{2}\left(\frac{\log\left(\frac{n+1}{2}\right)}{n}\right)} = e^{\frac{(n+1)}{2n}\log\left(\frac{n+1}{2}\right)} \\
&\geq e^{\frac{1}{2}\log\left(\frac{n}{2}\right)} = \sqrt{\frac{n}{2}} \to +\infty
\end{aligned}
$$

Oppure....

Osservo che:

$$\sum_{n=0}^{+\infty} \frac{c^n}{n!} = e^c$$

Calcolare la somma della serie

$$S = \sum_{n=0}^{+\infty} \frac{6}{n(n+1)(n+2)}$$

Primo metodo

Osservo che

$$\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{n+2-n}{n(n+1)(n+2)} = \frac{2}{n(n+1)(n+2)}$$

Quindi:

$$S = 3\sum_{n=1}^{+\infty} \frac{2}{n(n+1)(n+2)} = 3\sum_{n=1}^{+\infty}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$
$$= 3\left[\sum_{n=1}^{+\infty}\left(\frac{1}{n(n+1)}\right) - \sum_{n=1}^{+\infty}\left(\frac{1}{(n+1)(n+2)}\right)\right]$$
$$= 3\left[\sum_{n=1}^{+\infty}\left(\frac{1}{n(n+1)}\right) - \sum_{n=2}^{+\infty}\left(\frac{1}{n(n+1)}\right)\right] = \frac{3}{2}$$

Secondo metodo

Cerchiamo A, B, C tali che:

$$\frac{6}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{(n+1)} + \frac{C}{(n+2)}$$

Si ha che:

$$\begin{aligned} &\frac{A}{n} + \frac{B}{(n+1)} + \frac{C}{(n+2)} \\ =&\frac{A(n+1)(n+2) + Bn(n+2) + Cn(n+1)}{n(n+1)(n+2)} \\ =&\frac{(A+B+C)n^2 + (3A+2B+C)n + (2A)}{n(n+1)(n+2)} \end{aligned}$$

Pertanto

$$\begin{cases} A+B+C=0 \\ 3A+2B+C=0 \\ 2A=6 \end{cases} \Leftrightarrow \begin{cases} A=3 \\ B+C=-3 \\ 2B+C=-9 \end{cases} \Leftrightarrow \begin{cases} A=3 \\ B=-6 \\ C=3 \end{cases}$$

Definiamo: $S_k = \sum_{n=1}^{k} \frac{6}{n(n+1)(n+2)}$

$$\begin{aligned} S_k &= \sum_{n=1}^{k} \left(\frac{3}{n} - \frac{6}{(n+1)} + \frac{3}{(n+2)} \right) \\ &= \sum_{n=1}^{k} \left(\frac{3}{n} - \frac{3}{(n+1)} - \frac{3}{(n+1)} + \frac{3}{(n+2)} \right) \\ &= \sum_{n=1}^{k} \frac{3}{n} - \sum_{n=1}^{k} \frac{3}{(n+1)} - \sum_{n=1}^{k} \frac{3}{(n+1)} + \sum_{n=1}^{k} \frac{3}{(n+2)} \end{aligned}$$

$$= \sum_{n=1}^{k} \frac{3}{n} - \sum_{n=2}^{k+1} \frac{3}{n} - \sum_{n=1}^{k} \frac{3}{(n+1)} + \sum_{n=2}^{k+1} \frac{3}{(n+1)}$$

$$= 3 - \frac{3}{k+1} - \frac{3}{2} + \frac{3}{k+2} = \frac{3}{2} - \frac{3}{k+1} + \frac{3}{k+2}$$

$$S_k \to \frac{3}{2}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n+1}{n}\binom{4n}{2n-1}}{\binom{3n}{n}}$$

$$\sum_{n=0}^{\infty} \frac{\binom{2n+1}{n}\binom{4n}{2n-1}}{\binom{3n}{n}} = \sum_{n=0}^{\infty} \frac{(2n+1)! \cdot (4n)! \cdot n! \cdot (2n)!}{n! \cdot (n+1)! \cdot (2n+1)! \cdot (2n-1)! \cdot (3n)!}$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(2n+1)!} \cdot (4n)! \cdot \cancel{n!} \cdot (2n)!}{\cancel{n!} \cdot (n+1)! \cdot \cancel{(2n+1)!} \cdot (2n-1)! \cdot (3n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(4n)! \cdot (2n)!}{(n+1)! \cdot (2n-1)! \cdot (3n)!} = \sum_{n=0}^{\infty} \frac{(4n)! \cdot (2n-1)! \cdot (2n)}{(n+1)! \cdot (3n)! \cdot (2n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(4n)! \cdot (2n)}{(n+1)! \cdot (3n)!}$$

Applichiamo il criterio del rapporto trovando

$$\frac{a_{n+1}}{a_n} = \frac{(4n+4)! \cdot (2n+2)}{(n+2)! \cdot (3n+3)!} \cdot \frac{(n+1)! \cdot (3n)!}{(4n)! \cdot (2n)}$$

$$= \frac{(4n+4)(4n+3)(4n+2)(4n+1)\cancel{(4n)}! \cdot (2n+2)}{(n+2)\cancel{(n+1)}! \cdot (3n+3)(3n+2)(3n+1)\cancel{(3n)}!} \cdot$$

$$= \frac{\cancel{(n+1)}! \cdot \cancel{(3n)}!}{\cancel{(4n)}! \cdot (2n)}$$

$$= \frac{16(n+1)(4n+3)(2n+1)(4n+1)(n+1)}{6n(n+1) \cdot (3n+2)(3n+1)(n+2)}$$

$$= \frac{8}{3} \frac{(4n+3)(2n+1)(4n+1)(n+1)}{n(n+2) \cdot (3n+2)(3n+1)} \to \frac{8 \cdot 32}{3 \cdot 9} = \frac{256}{27} > 1$$

Quindi la serie diverge.