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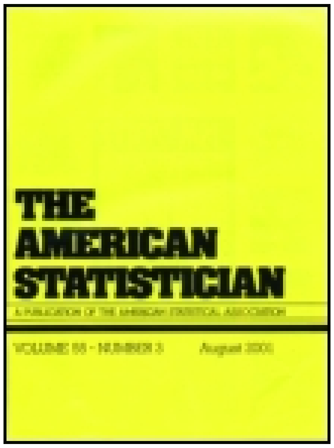

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Some Useful Moment Results in Sampling Problems

B. O'NEILL

We consider the standard sampling problem involving a finite population of N objects and a sample of n objects taken from this population using simple random sampling without replacement. We consider the relationship between the moments of the sampled and unsampled parts and show how these are related to the population moments. We derive expectation, variance, and covariance results for the various quantities under consideration and use these to obtain standard sampling results with an extension to variance estimation with a "finite population correction." This clarifies and extends standard results in sampling theory for the estimation of the mean and variance of a population.

KEY WORDS: Finite population correction; Population moments; Ratio of nested variance estimators; Sample moments; Variance estimator.

In this article, we examine the interrelation between sample and population moments arising in basic sampling problems, with a view to allowing estimation of a finite population variance. The relationship between the sample mean and variance has previously been investigated in Zhang (2007) and Sen (2012) giving moment results in terms of the unknown parameters in the problem. In the present article, we extend this analysis to look at the relationship between the sample moments and finite population moments. These wider results allow us to obtain a confidence interval for the finite population variance.

We consider a standard case in sampling problems where we sample from a finite population using simple random sampling without replacement. We consider a finite population vector $\boldsymbol{X}_N = (X_1, X_2, \ldots, X_N)$ and a sample vector $\boldsymbol{X}_n = (X_1, X_2, \ldots, X_n)$ with $n \leq N$ taken from this population. We have n sampled values and $N - n$ unsampled values, giving a total of N population values. (For the values $X_1, X_2, \ldots, X_N$ in the population, we will use the standard convention of representing random variables by upper-case letters and their realized values by corresponding lower-case letters. In our analysis, we will consider the population values to be random variables. Note that n and N are separate variables, not following this convention.)

The population is presumed to be infinitely exchangeably extendible, meaning that it can be embedded within an exchangeable series of values called the "superpopulation." Since the population vector is exchangeable this means that the first n values included in the sample vector implicitly give us a simple random sample without replacement (which is why the model of an exchangeable superpopulation can adequately capture this sampling process).

From the representation theorem of de Finetti, the condition of infinite exchangeability is equivalent to saying that the random variables in the population are independent and identically distributed conditional on the underlying superpopulation distribution (see O'Neill 2009 for discussion). This common distribution gives a common mean and variance for the individual random variables in the population, which we denote by μ and σ^2 respectively. These underlying parameters represent the true mean and variance of the superpopulation distribution, but we will actually be concerned with inferences about the mean and variance of the finite population.

1. DESCRIPTIVE QUANTITIES FOR THE POPULATION AND ITS PARTS

To analyze this standard case, we define mean and variance quantities for the sampled part, the unsampled part, and the whole population:

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$

$$\bar{X}_{n:N} = \frac{1}{N-n}\sum_{i=n+1}^{N} X_i \quad S_{n:N}^2 = \frac{1}{N-n-1}\sum_{i=n+1}^{N}(X_i - \bar{X}_{n:N})^2,$$

$$\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i \qquad S_N^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X}_N)^2.$$

(Note that we incorporate Bartlett's correction into the population variance, which is contrary to the approach taken in some texts. This differs from some other treatments of sampling, which use the number of data points as the denominator

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in the sample variance fraction, without using Bartlett's correction. The reason we choose to incorporate Bartlett's correction is that it makes sense to consider the finite population variance as an estimator of a larger infinite superpopulation variance in the context of a superpopulation model. With this consideration, Bartlett's correction ensures that the sample variance and population variance both have the same expected value and therefore function as unbiased estimators of the superpopulation variance parameter.)

We also define a distance measure comparing the means of the sampled and unsampled parts:

$$D_N^2 = \frac{n(N-n)}{N}\left(\bar{X}_n - \bar{X}_{n:N}\right)^2.$$

(Discussion of interpretation of the distance measure is set out in Appendix B.) The mean and variance for the population can be decomposed into statistics for the sampled and unsampled parts, according to the following result. (All proofs are in Appendix A.)

Result 1. The population mean and variance can be decomposed into

$$N\bar{X}_N = n\bar{X}_n + (N-n)\,\bar{X}_{n:N},$$
$$(N-1)S_N^2 = (n-1)\,S_n^2 + (N-n-1)S_{n:N}^2 + D_N^2.$$

It is easily shown that the mean statistics all have expected value μ and the variance statistics and distance measure all have expected value σ^2. This means that the above formulas turn into simply arithmetic decompositions when we take the expectation of both sides. The result gives us decompositions for the population mean and variance, which can be used in inference problems to derive confidence intervals subject to "finite population correction." (We will see more on this later.) The decompositions in the above result are represented graphically in Figure 1, which shows how the descriptive quantities for the population are formed by the descriptive quantities for the sampled and unsampled parts.

In addition to the above, it will be useful to be able to refer directly to the part of the variance decomposition containing information from the unsampled part. We will refer to this as the out-of-sample variability measure and denote it by

$$C_N^2 = \frac{(N-n-1)\,S_{n:N}^2 + D_N^2}{N-n}.$$

This allows us to write the variance decomposition as

$$(N-1)\,S_N^2 = (n-1)\,S_n^2 + (N-n)\,C_N^2.$$

To allow us to use the descriptive quantities effectively, we will need to know a bit about their marginal behavior, and how they are related to each other. Specifically, it will be useful to find the mean and variance of each of these quantities and the covariances between them. These results will extend the well-known moment results for the sample mean and variance, to look at the moment results for all parts of the above decomposition. To determine the various moments of interest we will begin with specification of the relevant moments of the underlying

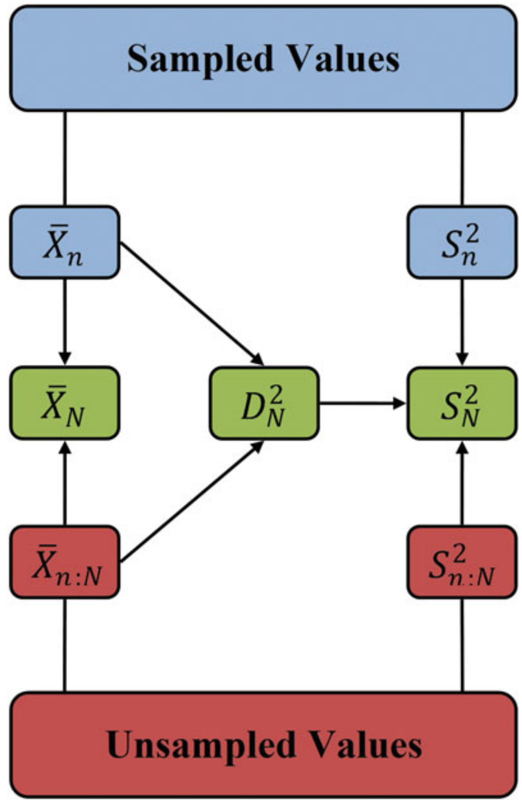

Figure 1. Decomposition of descriptive quantities for the population.

distribution for the values in the population. For our purposes, we will require values for the mean, variance, skewness, and kurtosis of the distribution. (We do not assume that these values are known.) Following standard notation we take the central moments to be

$$\mathbb{E}((X_i-\mu)^2) = \sigma^2,$$
$$\mathbb{E}((X_i-\mu)^3) = \gamma\sigma^3 \quad \mathbb{E}((X_i-\mu)^4) = \kappa\sigma^4.$$

This means that σ^2 is the variance, γ is the skewness, and κ is the kurtosis for the underlying distribution of the superpopulation. To facilitate our results, we also define the quantity

$$\phi = \frac{\gamma}{\sqrt{\kappa - 1}}.$$

This measure adjusts the skewness based on the kurtosis. One of the properties of skewness and kurtosis is that $\gamma^2 \leq \kappa - 1$ so that $-1 \leq \phi \leq 1$. (See Sen 2012, and note that Sen used the notation κ to refer to the *excess* kurtosis, so the result he presented is that $\gamma^2 \leq \kappa + 2$. This is equivalent to the result we present here once the difference in notation is accounted for.) This means that the adjusted measure gives us a simple bounded measure of skewness.

This parameter specification for the moments leads to results for the central moments of the various quantities of interest. Our interest will be in making inferences about the mean and variance parameters, but we will need to consider the higher-order moments to describe the behavior of our estimators. (All proofs are in Appendix A.)

Result 2. The expected values and variances of the mean quantities are

$$\mathbb{E}\left(\bar{X}_n\right) = \mu \qquad \mathbb{V}\left(\bar{X}_n\right) = \frac{\sigma^2}{n},$$
$$\mathbb{E}\left(\bar{X}_{n:N}\right) = \mu \qquad \mathbb{V}\left(\bar{X}_{n:N}\right) = \frac{\sigma^2}{N-n},$$
$$\mathbb{E}\left(\bar{X}_N\right) = \mu \qquad \mathbb{V}\left(\bar{X}_N\right) = \frac{\sigma^2}{N}.$$

Result 3. The expected values and variances of the variance quantities are

$$\mathbb{E}\left(S_n^2\right) = \sigma^2 \qquad \mathbb{V}\left(S_n^2\right) = \left(\kappa - \frac{n-3}{n-1}\right)\frac{\sigma^4}{n},$$
$$\mathbb{E}\left(S_{n;N}^2\right) = \sigma^2 \qquad \mathbb{V}\left(S_{n:N}^2\right) = \left(\kappa - \frac{N-n-3}{N-n-1}\right)\frac{\sigma^4}{N-n},$$
$$\mathbb{E}\left(S_N^2\right) = \sigma^2 \qquad \mathbb{V}\left(S_N^2\right) = \left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N}.$$

Result 4. The expected value and variance of the distance measure are

$$\mathbb{E}\left(D_N^2\right) = \sigma^2$$
$$\mathbb{V}\left(D_N^2\right) = \left(2 + (\kappa - 3)\left(\frac{N}{n\,(N-n)} - \frac{3}{N}\right)\right)\sigma^4.$$

Result 5. The expected value and variance of the out-of-sample variability measure are

$$\mathbb{E}\left(C_N^2\right) = \sigma^2$$
$$\mathbb{V}\left(C_N^2\right) = \left(2 + (\kappa - 3)\left(1 - \frac{2}{N} + \frac{1}{Nn}\right)\right)\frac{\sigma^4}{N-n}.$$

Result 6. The covariances with the distance measure are

$$\mathbb{C}\left(\bar{X}_n, D_N^2\right) = \frac{N-n}{N}\cdot\frac{\gamma\sigma^3}{n},$$
$$\mathbb{C}\left(S_n^2, D_N^2\right) = (\kappa - 3)\,\frac{N-n}{N}\cdot\frac{\sigma^4}{n},$$
$$\mathbb{C}\left(\bar{X}_{n:N}, D_N^2\right) = \frac{n}{N}\cdot\frac{\gamma\sigma^3}{N-n},$$
$$\mathbb{C}\left(S_{n:N}^2, D_N^2\right) = (\kappa - 3)\,\frac{n}{N}\cdot\frac{\sigma^4}{N-n},$$
$$\mathbb{C}\left(\bar{X}_N, D_N^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(S_N^2, D_N^2\right) = \left(\frac{2N}{N-1} + (\kappa - 3)\right)\frac{\sigma^4}{N}.$$

Result 7. The covariances within the mean and variance quantities are

$$\mathbb{C}\left(\bar{X}_n, \bar{X}_{n:N}\right) = 0,$$
$$\mathbb{C}\left(\bar{X}_n, \bar{X}_N\right) = \frac{\sigma^2}{N},$$
$$C\left(S_n^2, S_N^2\right) = \left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N},$$
$$\mathbb{C}\left(S_n^2, S_{n:N}^2\right) = 0,$$
$$\mathbb{C}\left(\bar{X}_{n:N}, \bar{X}_N\right) = \frac{\sigma^2}{N},$$
$$C\left(S_{n:N}^2, S_N^2\right) = \left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N}.$$

Result 8. The covariances between the mean and variance quantities are

$$\mathbb{C}\left(\bar{X}_n, S_n^2\right) = \frac{\gamma\sigma^3}{n},$$
$$\mathbb{C}\left(\bar{X}_{n:N}, S_n^2\right) = 0,$$
$$\mathbb{C}\left(\bar{X}_N, S_n^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(\bar{X}_n, S_{n:N}^2\right) = 0,$$
$$\mathbb{C}\left(\bar{X}_{n:N}, S_{n:N}^2\right) = \frac{\gamma\sigma^3}{N-n},$$
$$\mathbb{C}\left(\bar{X}_N, S_{n:N}^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(\bar{X}_n, S_N^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(\bar{X}_{n:N}, S_N^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(\bar{X}_N, S_N^2\right) = \frac{\gamma\sigma^3}{N}.$$

Result 9. The covariances with the out-of-sample variability measure are

$$\mathbb{C}\left(\bar{X}_n, C_N^2\right) = \frac{1}{N}\cdot\frac{\gamma\sigma^3}{n},$$
$$\mathbb{C}\left(S_n^2, C_N^2\right) = (\kappa - 3)\,\frac{1}{N}\cdot\frac{\sigma^4}{n},$$
$$\mathbb{C}\left(\bar{X}_{n:N}, C_N^2\right) = \frac{N-1}{N}\cdot\frac{\gamma\sigma^3}{N-n},$$
$$\mathbb{C}\left(S_{n:N}^2, C_N^2\right) = \left(\frac{2}{N} + (\kappa - 3)\,\frac{N-1}{N}\right)\frac{\sigma^4}{N-n},$$
$$\mathbb{C}\left(\bar{X}_N, C_N^2\right) = \frac{\gamma\sigma^3}{N},$$
$$\mathbb{C}\left(S_N^2, C_N^2\right) = \left(\frac{2N}{N-1} + (\kappa - 3)\right)\frac{\sigma^4}{N}.$$

Remark 1. To ensure the existence of the relevant moments we assume in our analysis that $N > 3$ and $n > 1$. This means that the above results are all properly defined for the sample and population and moments. (Values for the unsampled part are properly defined if $N - n > 1$ so that there is more than one unsampled value.)

This gives us all the relevant moment results for looking at the location, scale, and covariance for the various quantities of interest. With a substantial amount of additional algebra, it can be shown that the moment results fit together using the rules for linear functions using the previous decomposition results (see Appendix B). This can be used as a useful check on working, or as an alternative derivation of some of the results.

2. ASYMPTOTIC CORRELATION AND DISTRIBUTIONAL APPROXIMATIONS

Correlation results follow trivially from the covariance and variance results for the various quantities. They are not particularly interesting in their own right, but they are of interest asymptotically, since we would like to know if these quantities are asymptotically linearly related or not. This will be valuable in attempting to derive asymptotic distributions.

To obtain asymptotic results, we will need to determine what happens as $n \to \infty$. Because we are dealing with problems with a finite population size, we must deal with the fact that the sample size is bounded by the population size, so that the former can only tend toward infinity if the latter also tends to infinity (at least as fast). To deal with this, we will want to consider the limiting case for the sample based on some limiting value for the unsampled proportion, which we define by

$$u \equiv \frac{N-n}{N}.$$

We will assume that $n \to \infty$ in such a way that a limiting value for u exists. This allows us to refer to limits pertaining to the sample size without the population size entering explicitly into our analysis. (For simplicity, we will not introduce any new notation for the limiting value of u. However, it should be understood that anytime we refer to this value in a limiting context, we mean to refer to the limiting value.) This allows us to undertake an asymptotic analysis to find the limiting correlation between the quantities.

Result 10. As $n \to \infty$, the correlations with the distance measure are

$$\begin{aligned} &\mathrm{Corr}\left(\bar{X}_n, D_N^2\right) \to 0 && \mathrm{Corr}\left(S_n^2, D_N^2\right) \to 0,\\ &\mathrm{Corr}\left(\bar{X}_{n:N}, D_N^2\right) \to 0 && \mathrm{Corr}\left(S_{n:N}^2, D_N^2\right) \to 0,\\ &\mathrm{Corr}\left(\bar{X}_N, D_N^2\right) \to 0 && \mathrm{Corr}\left(S_N^2, D_N^2\right) \to 0. \end{aligned}$$

Result 11. As $n \to \infty$, the correlations within the mean and variance quantities are

$$\begin{aligned} &\mathrm{Corr}\left(\bar{X}_n, \bar{X}_{n:N}\right) = 0 && \mathrm{Corr}\left(\bar{X}_n, \bar{X}_N\right) = \sqrt{1-u},\\ &\mathrm{Corr}\left(S_n^2, S_N^2\right) \to \sqrt{1-u} && \mathrm{Corr}\left(S_n^2, S_{n:N}^2\right) = 0,\\ &\mathrm{Corr}\left(\bar{X}_{n:N}, \bar{X}_N\right) = \sqrt{u} && \mathrm{Corr}\left(S_{n:N}^2, S_N^2\right) \to \sqrt{u}. \end{aligned}$$

Result 12. As $n \to \infty$, the correlations between the mean and variance quantities are

$$\begin{aligned} &\mathrm{Corr}\left(\bar{X}_n, S_n^2\right) \to \phi && \mathrm{Corr}\left(\bar{X}_{n:N}, S_n^2\right) = 0,\\ &\mathrm{Corr}\left(\bar{X}_N, S_n^2\right) \to \sqrt{1-u}\cdot\phi && \mathrm{Corr}\left(\bar{X}_n, S_{n:N}^2\right) = 0,\\ &\mathrm{Corr}\left(\bar{X}_{n:N}, S_{n:N}^2\right) \to \phi && \mathrm{Corr}\left(\bar{X}_N, S_{n:N}^2\right) \to \sqrt{u}\cdot\phi,\\ &\mathrm{Corr}\left(\bar{X}_n, S_N^2\right) \to \sqrt{1-u}\cdot\phi && \mathrm{Corr}\left(\bar{X}_{n:N}, S_N^2\right) \to \sqrt{u}\cdot\phi.\\ &\mathrm{Corr}\left(\bar{X}_N, S_N^2\right) \to \phi \end{aligned}$$

Result 13. As $n \to \infty$, the correlations with the out-of-sample variability measure are

$$\begin{aligned} &\mathrm{Corr}\left(\bar{X}_n, C_N^2\right) \to 0 && \mathrm{Corr}\left(S_n^2, C_N^2\right) \to 0,\\ &\mathrm{Corr}\left(\bar{X}_{n:N}, C_N^2\right) \to \phi && \mathrm{Corr}\left(S_{n:N}^2, C_N^2\right) \to \frac{\kappa-3}{\kappa-1},\\ &\mathrm{Corr}\left(\bar{X}_N, C_N^2\right) \to \sqrt{u}\cdot\phi && \mathrm{Corr}\left(S_N^2, C_N^2\right) \to \sqrt{u}. \end{aligned}$$

These results show us that many of the quantities are asymptotically uncorrelated as $n \to \infty$. For those pairs that are correlated, most of the limiting correlations depend on the adjusted skewness parameter ϕ and the unsampled proportion u. For unskewed distributions, the pairs between the mean and variance quantities are asymptotically uncorrelated and so are the pairs between the means and the out-of-sample variability measure. In particular, Result 12 shows that the sample mean is correlated with the sample variance through the adjusted skewness parameter and this correlation remains when $n \to \infty$. This result is already known in the literature and has been given previously in Zhang (2007) and Sen (2012).

In addition to determining the asymptotic correlations between the quantities, we will also be interested in their asymptotic distributions. To do this, we define the degrees of freedom:

$$\begin{aligned} \mathrm{DF}_n &\equiv \frac{2\sigma^4}{\mathbb{V}(S_n^2)} = \frac{2n}{\kappa-(n-3)/(n-1)},\\ \mathrm{DF}_{n:N} &\equiv \frac{2\sigma^4}{\mathbb{V}(S_{n:N}^2)} = \frac{2(N-n)}{\kappa-(N-n-3)/(N-n-1)},\\ \mathrm{DF}_N &\equiv \frac{2\sigma^4}{\mathbb{V}(S_N^2)} = \frac{2N}{\kappa-(N-3)/(N-1)},\\ \mathrm{DF}_D &\equiv \frac{2\sigma^4}{\mathbb{V}(D_N^2)} = \frac{2}{2+(\kappa-3)(N/n(N-n)-3/N)},\\ \mathrm{DF}_C &\equiv \frac{2\sigma^4}{\mathbb{V}(C_N^2)} = \frac{2(N-n)}{2+(\kappa-3)(1-2/N+1/Nn)}. \end{aligned}$$

Result 14. If σ is finite then, as $n \to \infty$ we have asymptotic distribution:

$$\bar{X}_n - \bar{X}_N \sim \sqrt{\frac{N-n}{Nn}} \cdot \mathrm{N}(0, \sigma^2).$$

If κ and σ are both finite then, as $n \to \infty$, we have asymptotic distributions:

$$\begin{aligned} &S_n^2/\sigma^2 \sim \mathrm{ChiSq}(\mathrm{DF}_n)/\mathrm{DF}_n, && D_N^2/\sigma^2 \sim \mathrm{ChiSq}(\mathrm{DF_D})/\mathrm{DF}_D,\\ &S_{n:N}^2/\sigma^2 \sim \mathrm{ChiSq}(\mathrm{DF}_{n:N})/\mathrm{DF}_{n:N}, && C_N^2/\sigma^2 \sim \mathrm{ChiSq}(\mathrm{DF_C})/\mathrm{DF}_C.\\ &S_N^2/\sigma^2 \sim \mathrm{ChiSq}(\mathrm{DF}_N)/\mathrm{DF}_N, \end{aligned}$$

(Strictly speaking these are not asymptotic distributions because n appears in the distributional form. Though it is possible to state a constant asymptotic distributional form for transformed versions of these quantities, it is more useful and intuitive to write the asymptotic form in this way. A more strict approach with proper regard to technical niceties is shown in the proof in Appendix A.)

Result 15. If κ and σ are both finite, then for large n we have the approximate distributions:

$$\frac{\bar{X}_n - \bar{X}_N}{S_n^2} \overset{\text{Approx}}{\sim} \sqrt{\frac{N-n}{Nn}} \cdot St\,(\mathrm{DF}_n)$$
$$\frac{S_N^2}{S_n^2} \overset{\text{Approx}}{\sim} \frac{n-1}{N-1} + \frac{N-n}{N-1} \cdot \frac{1}{F\,(\mathrm{DF}_n, \mathrm{DF}_C)}.$$

(The St in the first result refers to the Student's T distribution. Note that this approximation ignores the fact that $\mathrm{Corr}(\bar{X}_n - \bar{X}_N, S_n^2) \to \sqrt{u} \cdot \phi$, and it is based on treating the numerator and denominator as being independent. It is possible to adjust the asymptotic distribution to account for skewness in the superpopulation distribution but we do not pursue that here; see Chen 1995 for more on this issue.)

The two quantities in Result 15 are "quasi-pivotal" in the sense that their distribution depends on the parameter κ only through the degrees of freedom. The distributional results should be quite robust to estimation of this parameter so they are almost—but not quite—pivotal. Both of the distributions of the quantities are based on the asymptotic marginal distributions of the parts, which we gave in Result 14, albeit ignoring the asymptotic correlation in the first case. This now gives us asymptotic distributional results, which can be used in the calculation of interval estimates for the population mean and variance.

Special case. The general results shown above give rise to some interesting special cases, some of which have well-known properties. A mesokurtic distribution occurs when $\kappa = 3$, in which case we have $DF_n = n-1$ and $DF_C = N-n$. An important special case of this is the commonly used case of a normal superpopulation. In this case, it is well known that the means are independent of the variances, giving rise to the exact distributions of the above form—that is, they are not just approximations in this special case.

3. CONFIDENCE INTERVALS FOR THE POPULATION MEAN AND VARIANCE

One of the main uses of the above results is to allow us to form confidence intervals for the population mean and variance, giving results that include a "finite population correction" and that are also able to take account of the skewness and kurtosis of the distribution. The moment results we have derived allow us to deal with a finite population, and this allows us to generalize the standard intervals for the mean and variance parameters to obtain analogous intervals for the finite population case. (Interval estimation for a finite population mean is already well understood in the literature; see, for example, Cochran (1963) and Särndal, Swensson, and Wretman (1992). To the knowledge of the present author, the present interval result for a finite population variance is new.)

Both confidence intervals can be formed by using the "quasi-pivotal" quantities in Result 15 to creating corresponding probability intervals with random upper and lower bounds. To form the confidence interval for the population mean, we let $t_{\alpha/2,k}$ be the $1-\alpha/2$ percentile of the Student's T distribution with k degrees of freedom (i.e., the area in the right tail is $\alpha/2$). Assuming n is large we then have

$$\begin{aligned}
1-\alpha &\approx \mathbb{P}\left(-\frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \le \frac{\bar{X}_n - \bar{X}_N}{S_n^2} \le \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}}\right) \\
&= \mathbb{P}\left(\bar{X}_n - \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \cdot S_n \le \bar{X}_N \le \bar{X}_n + \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \cdot S_n\right) \\
&= \mathbb{P}\left(\bar{X}_N \in \left[\bar{X}_n \pm \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \cdot S_n\right]\right).
\end{aligned}$$

To form the confidence interval for the population variance, we choose some $0 \le \theta \le \alpha$ and we let $F^*_{1-\theta,k_1,k_2}$ be the θ percentile of the F-distribution with k_1 and k_2 degrees of freedom (i.e., the area in the right tail is $1-\theta$). Assuming n is large we then have

$$\begin{aligned}
1-\alpha &\approx \mathbb{P}\left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C}} \le \frac{S_N^2}{S_n^2} \le \frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right) \\
&= \mathbb{P}\left(\left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)S_n^2 \le S_N^2 \le \left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)S_n^2\right) \\
&= \mathbb{P}\left(S_N^2 \in \left[\left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)S_n^2, \times \left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)S_n^2\right]\right).
\end{aligned}$$

The above probability statements give us corresponding confidence intervals once the actual observed values $\bar{x}_n$ and s_n^2 are substituted:

$$\begin{aligned}
\mathrm{CI}_N^{\mathrm{Mean}}(1-\alpha) &= \left[\bar{x}_n \pm \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{\frac{N-n}{N}} \cdot s_n\right], \\
\mathrm{CI}_N^{\mathrm{var}}(1-\alpha) &= \left[\left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)s_n^2, \times \left(\frac{n-1}{N-1} + \frac{N-n}{N-1}\frac{1}{F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C}}\right)s_n^2\right].
\end{aligned}$$

With a little algebra, these can be rewritten in terms of n and u as

$$\mathrm{CI}_N^{\mathrm{Mean}}(1-\alpha) = \left[\bar{x}_n \pm \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}}\sqrt{u} \cdot s_n\right],$$

$$\mathrm{CI}_N^{\mathrm{var}}(1-\alpha) = \left[\left(1 - \frac{nu\left(1 - 1/F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C}\right)}{n-(1-u)}\right) s_n^2, \right.$$
$$\left. \times \left(1 - \frac{nu\left(1 - 1/F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C}\right)}{n-(1-u)}\right) s_n^2 \right].$$

The first of these intervals is similar to the standard confidence interval formula for inference about a finite population mean. The only difference from the standard interval is that we have taken account of the kurtosis in the degrees of freedom calculation. This confidence interval includes the finite population correction term given by the square root of the unsampled proportion. The result is well known for mesokurtic distributions (e.g., a normal distribution) and is presented in standard texts on sampling theory (e.g., Cochran 1963, pp. 20–24) as well as introductory textbooks in statistics (see, e.g., Kault 2003, pp. 227–229; Dorofeev and Grant 2006, pp. 42–43; Wieres 2011, p. 299).

The second of these confidence intervals gives us a generalized interval for the population variance, which can be used in cases where we have a finite population. This interval also takes account of the kurtosis in the degrees of freedom calculation. The finite population correction in this case comes into the interval calculation directly, and also comes in through the degrees of freedom calculation. From the general form for this confidence interval, we can form an equal-tail interval by choosing $\theta = \alpha/2$ or we can form a minimum length interval by choosing $0 \le \theta \le \alpha$ to minimize the interval length (see Appendix B).

Both of the intervals can be applied when the kurtosis of the distribution is known. If the kurtosis is unknown (as will usually be the case), it can be replaced with a consistent estimator of some kind (see Appendix B). Some practitioners may wish to follow the standard method of applying the mean interval with the assumption that they are dealing with a mesokurtic distribution ($\kappa = 3$) for the degrees of freedom calculation.

We have been concerned in this analysis with deriving confidence intervals for the mean and variance of a finite population. However, these results also give confidence intervals for the mean and variance of the superpopulation as a special case. To obtain the latter, all we need to do is take $N \to \infty$ so that the population is the infinite superpopulation. In this case, we have convergence in probability to

$$\bar{X}_{n:N} \to \bar{X}_N \to \mu \quad S^2_{n:N} \to S^2_N \to \sigma^2 \quad D^2_N \to n\left(\bar{X}_n - \mu\right)^2.$$

This means that the population mean is the parameter μ and the population variance is the parameter σ^2. We also have $\mathrm{DF}_C \to \infty$ so that $F^*_{\theta,\mathrm{DF}_n,\mathrm{DF}_C} \to \chi^2_{\theta,\mathrm{DF}_n}/\mathrm{DF}_n$, which means that our confidence intervals for μ and σ^2 are given by

$$\mathrm{CI}_\infty^{\mathrm{Mean}}(1-\alpha) = \left[\bar{x}_n \pm \frac{t_{\alpha/2,\mathrm{DF}_n}}{\sqrt{n}} \cdot s_n\right],$$
$$\mathrm{CI}_\infty^{\mathrm{var}}(1-\alpha) = \left[\frac{\mathrm{DF}_n}{\chi^2_{\alpha-\theta,\mathrm{DF}_n}} \cdot s_n^2, \frac{\mathrm{DF}_n}{\chi^2_{1-\theta,\mathrm{DF}_n}} \cdot s_n^2\right].$$

This gives us confidence intervals in accordance with the standard intervals already in the literature for the mean and variance parameters. (Our intervals use the more general formula for the degrees of freedom, taking account of the kurtosis.) Again, we can form an equal-tail interval for the variance by choosing $\theta = \alpha/2$ or we can form a minimum length interval by choosing $0 \le \theta \le \alpha$ to minimize the interval length (see Appendix B).

4. CONCLUDING REMARKS

The standard sample mean and variance quantities in simple random sampling problems can be treated using a decomposition that decomposes the overall mean and variance into parts attributable to the sampled and unsampled parts of the population. Using standard moment techniques it is possible to derive the means, variances, and covariances of these quantities to obtain a good understanding of their behavior up to the second moment.

In this article, we have derived these moment results and used them to get simple confidence interval formulas for the population mean and variance, including terms for finite population correction. This extends present results for the confidence interval for a population variance and also gives another approach to derivation of the interval for the population mean. The results we have derived can be used in elementary sampling problems or introductory courses to allow inferences in finite populations.

In this article, we have mostly sidestepped complications relating to estimating the unknown skewness and kurtosis for use in the interval calculations. Substitution of a consistent estimator would give good long-run properties in our confidence interval formulas, but the exact effect on the intervals for small samples has not been considered here. This would be an appropriate avenue for further research.

APPENDIX A: PROOFS OF MOMENT RESULTS

In this appendix, we set out proofs of the various lemmas and results in the main body of the article, concerning moments of quantities of interest in our analysis. We introduce some new lemmas where necessary to break up the proofs into simpler pieces.

Lemma A.1. The distance measure can be written as

$$D^2_N = n\left(\bar{X}_n - \bar{X}_N\right)^2 + (N-n)\left(\bar{X}_{n:N} - \bar{X}_N\right)^2.$$

Proof of Lemma A.1. We have

$$\begin{aligned}\bar{X}_n - \bar{X}_N &= \frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{N}\sum_{i=1}^{N} X_i \\ &= \left(\frac{1}{n} - \frac{1}{N}\right)\sum_{i=1}^{n} X_i - \frac{1}{N}\sum_{i=n+1}^{N} X_i \\ &= \frac{N-n}{nN}\sum_{i=1}^{n} X_i - \frac{1}{N}\sum_{i=n+1}^{N} X_i \\ &= \frac{N-n}{N}\left(\bar{X}_n - \bar{X}_{n:N}\right),\end{aligned}$$

and

$$\begin{aligned}\bar{X}_{n:N} - \bar{X}_N &= \frac{1}{N-n}\sum_{i=n+1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{N} X_i \\ &= \left(\frac{1}{N-n} - \frac{1}{N}\right)\sum_{i=n+1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{n} X_i \\ &= \frac{n}{N(N-n)}\sum_{i=n+1}^{N} X_i - \frac{1}{N}\sum_{i=1}^{n} X_i \\ &= \frac{n}{N}\left(\bar{X}_{n:N} - \bar{X}_n\right).\end{aligned}$$

This means that

$$\begin{aligned} & n\left(\bar{X}_n - \bar{X}_N\right)^2 + (N-n)\left(\bar{X}_{n:N} - \bar{X}_N\right)^2 \\ &= \left[n\left(\frac{N-n}{N}\right)^2 + (N-n)\left(\frac{n}{N}\right)^2\right]\left(\bar{X}_n - \bar{X}_{n:N}\right)^2 \\ &= \frac{n}{N^2}[(N-n)^2 + (N-n)n]\left(\bar{X}_n - \bar{X}_{n:N}\right)^2 \\ &= \frac{n}{N^2}(N-n)N\left(\bar{X}_n - \bar{X}_{n:N}\right)^2 \\ &= \frac{n(N-n)}{N}\left(\bar{X}_n - \bar{X}_{n:N}\right)^2 = D_N^2,\end{aligned}$$

which was to be shown. □

Proof of Result 1. For the mean decomposition, we have

$$N\bar{X}_N = \sum_{i=1}^{N} X_i = \sum_{i=1}^{n} X_i + \sum_{i=N+1}^{N} X_i = n\bar{X}_n + (N-n)\,\bar{X}_{n:N}.$$

For the variance decomposition, we have

$$\begin{aligned}(N-1)\,S_N^2 &= \sum_{i=1}^{N}\left(X_i - \bar{X}_N\right)^2 \\ &= \sum_{i=1}^{n}\left(X_i - \bar{X}_N\right)^2 + \sum_{i=n+1}^{N}\left(X_i - \bar{X}_N\right)^2 \\ &= \sum_{i=1}^{n}\left(\left(X_i - \bar{X}_n\right) - \left(\bar{X}_N - \bar{X}_n\right)\right)^2 \\ &\quad + \sum_{i=n+1}^{N}\left(\left(X_i - \bar{X}_{n:N}\right) - \left(\bar{X}_N - \bar{X}_{n:N}\right)\right)^2 \\ &= \sum_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2 + n\left(\bar{X}_N - \bar{X}_n\right)^2 \\ &\quad + \sum_{i=n+1}^{N}\left(X_i - \bar{X}_{n:N}\right)^2 + (N-n)\left(\bar{X}_N - \bar{X}_{n:N}\right)^2 \\ &= (n-1)\,S_n^2 + (N-n-1)\,S_{n:N}^2 + D_N^2.\end{aligned}$$

(The second last step follows from the fact that $\sum_{i=1}^{n}(X_i - \bar{X}_n) = \sum_{i=n+1}^{N}(X_i - \bar{X}_{n:N}) = 0$.) □

To prove the first few moment results of interest, we will work with values that have had the mean removed. This makes it easier to derive the results in a succinct way, without dealing with large numbers of mean terms that cancel out in the final calculations. To assist with our analysis, we define the values $Y_i = X_i - \mu$, which are adjusted to remove the mean value of the values. We also define the quantities:

$$\begin{aligned}\bar{Y}_n &= \frac{1}{n}\sum_{i=1}^{n} Y_i & \bar{\bar{Y}}_n &= \frac{1}{n}\sum_{i=1}^{n} Y_i^2, \\ \bar{Y}_{n:N} &= \frac{1}{N-n}\sum_{i=n+1}^{N} Y_i & \bar{\bar{Y}}_{n:N} &= \frac{1}{N-n}\sum_{i=n+1}^{N} Y_i^2, \\ \bar{Y}_N &= \frac{1}{N}\sum_{i=1}^{N} Y_i & \bar{\bar{Y}}_N &= \frac{1}{N}\sum_{i=1}^{N} Y_i^2.\end{aligned}$$

Using these quantities it can be shown that

$$\begin{aligned}\bar{X}_n &= \bar{Y}_n + \mu & S_n^2 &= \frac{n}{n-1}\left(\bar{\bar{Y}}_n - \bar{Y}_n^2\right), \\ \bar{X}_{n:N} &= \bar{Y}_{n:N} + \mu & S_{n:N}^2 &= \frac{N-n}{N-n-1}\left(\bar{\bar{Y}}_{n:N} - \bar{Y}_{n:N}^2\right), \\ \bar{X}_N &= \bar{Y}_N + \mu & S_N^2 &= \frac{N}{N-1}\left(\bar{\bar{Y}}_N - \bar{Y}_N^2\right), \\ D_N^2 &= \frac{n(N-n)}{N}\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^2. & &\end{aligned}$$

Lemma A.2. We have the following moment results:

$$\begin{aligned}\mathbb{E}\left(\bar{Y}_n\right) &= 0 & \mathbb{E}\left(\bar{\bar{Y}}_n\right) &= \sigma^2, \\ \mathbb{E}\left(\bar{Y}_n^2\right) &= \frac{\sigma^2}{n} & \mathbb{E}\left(\bar{\bar{Y}}_n \bar{Y}_n\right) &= \frac{\gamma\sigma^3}{n}, \\ \mathbb{E}\left(\bar{Y}_n^3\right) &= \frac{\gamma\sigma^3}{n^2} & \mathbb{E}\left(\bar{\bar{Y}}_n \bar{Y}_n^2\right) &= \frac{(\kappa+n-1)\,\sigma^4}{n^2}, \\ \mathbb{E}\left(\bar{Y}_n^4\right) &= \frac{(\kappa+3n-3)\,\sigma^4}{n^3} & \mathbb{E}\left(\bar{\bar{Y}}_{n^2}\right) &= \frac{(\kappa+n-1)\,\sigma^4}{n}.\end{aligned}$$

Analogous results hold for the quantities for the unsampled part and the population.

Proof of Lemma A.2. We first consider the general form

$$\mathbb{E}\left(\bar{Y}_n^a \bar{\bar{Y}}_n^b\right),$$

for nonnegative integers a and b. Substituting in and expanding out the power sums, we have

$$\begin{aligned}\mathbb{E}\left(\bar{Y}_n^a \bar{\bar{Y}}_n^b\right) &= \frac{1}{n^{a+b}}\mathbb{E}\left(\left(\sum_{i=1}^{n} Y_i\right)^a \left(\sum_{i=1}^{n} Y_i^2\right)^b\right) \\ &= \frac{1}{n^{a+b}}\sum_{\boldsymbol{\alpha}}\mathbb{E}\left(Y_1^{\alpha_1}\ldots Y_n^{\alpha_n}\right) \\ &= \frac{1}{n^{a+b}}\sum_{\boldsymbol{\alpha}}\prod_{i=1}^{n}\mathbb{E}\left(Y_i^{\alpha_i}\right),\end{aligned}$$

where the summation in the last two lines is taken over all vectors $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ composed of nonnegative integers consistent with the previous power sums. This requires the indices in the vector $\boldsymbol{\alpha}$ to have the following properties:

$$\sum_{i=1}^{n}\alpha_i = a + 2b \qquad \sum_{i=1}^{n}\frac{\alpha_i}{2} \geq b$$
$$\alpha_i \neq 1 \text{ for all } i = 1, \ldots n.$$

(The last requirement follows from the fact that $\mathbb{E}(Y_i) = 0$ for all $i = 1, \ldots n$, which means any term of this kind can be dropped out of the sum.) We are now able to obtain the required moments specified in the lemma, which use particular values of a and b.

Now, using the preliminary result above, we are able to obtain the required moments specified in the lemma. With $a = 2$ and $b = 0$, we have

$$\mathbb{E}\left(\bar{Y}_n^2\right) = \frac{1}{n^2} n \mathbb{E}\left(Y_i^2\right) = \frac{1}{n}\mathbb{E}\left(Y_i^2\right) = \frac{\sigma^2}{n}.$$

With $a = 3$ and $b = 0$, we have

$$\mathbb{E}\left(\bar{Y}_n^3\right) = \frac{1}{n^3} n \mathbb{E}\left(Y_i^3\right) = \frac{1}{n^2}\mathbb{E}\left(Y_i^3\right) = \frac{\gamma\sigma^3}{n^2}.$$

With $a = 4$ and $b = 0$, we have

$$\begin{aligned}\mathbb{E}\left(\bar{Y}_n^4\right) &= \frac{1}{n^4}\left(n\mathbb{E}\left(Y_i^4\right) + 3n(n-1)\mathbb{E}\left(Y_i^2\right)^2\right)\\ &= \frac{1}{n^4}\left(n\kappa\sigma^4 + 3n(n-1)\sigma^4\right) = \frac{\kappa + 3n - 3}{n^3}\sigma^4.\end{aligned}$$

With $a = 0$ and $b = 1$, we have

$$\mathbb{E}\left(\overline{\overline{Y}}_n\right) = \frac{1}{n} n \mathbb{E}\left(Y_i^2\right) = \mathbb{E}\left(Y_i^2\right) = \sigma^2.$$

With $a = 1$ and $b = 1$, we have

$$\mathbb{E}\left(\bar{Y}_n \overline{\overline{Y}}_n\right) = \frac{1}{n^2} n \mathbb{E}\left(Y_i^3\right) = \mathbb{E}\left(Y_i^3\right) = \frac{\gamma\sigma^3}{n}.$$

With $a = 2$ and $b = 1$, we have

$$\begin{aligned}\mathbb{E}\left(\bar{Y}_n^2 \overline{\overline{Y}}_n\right) &= \frac{1}{n^3}\left(n\mathbb{E}\left(Y_i^4\right) + n(n-1)\mathbb{E}\left(Y_i^2\right)^2\right)\\ &= \frac{1}{n^3}\left(n\kappa\sigma^4 + n(n-1)\sigma^4\right)\\ &= \frac{(\kappa + n - 1)\sigma^4}{n^2}.\end{aligned}$$

With $a = 0$ and $b = 2$, we have

$$\begin{aligned}\mathbb{E}\left(\overline{\overline{Y}}_n^2\right) &= \frac{1}{n^2}\left(n\mathbb{E}\left(Y_i^4\right) + n(n-1)\mathbb{E}\left(Y_i^2\right)^2\right)\\ &= \frac{1}{n^2}\left(n\kappa\sigma^4 + n(n-1)\sigma^4\right) = \frac{(\kappa + n - 1)\sigma^4}{n}.\end{aligned}$$

Analogous results hold for the mean of the unsampled part and the mean of the population, and these are found in the same way. □

Proof of Result 2. Since $\mathbb{E}(\bar{Y}_n) = 0$ and $\mathbb{V}(\bar{Y}_n) = \sigma^2/n$, we have

$$\begin{aligned}\mathbb{E}\left(\bar{X}_n\right) &= \mathbb{E}\left(\bar{Y}_n + \mu\right) = \mathbb{E}\left(\bar{Y}_n\right) + \mu = \mu,\\ \mathbb{V}\left(\bar{X}_n\right) &= \mathbb{V}\left(\bar{Y}_n + \mu\right) = \mathbb{V}\left(\bar{Y}_n\right) = \frac{\sigma^2}{n}.\end{aligned}$$

Analogous results hold for the mean of the unsampled part and the mean of the population, and these are found in the same way. □

Proof of Result 3. Using the results in Lemma A.2 (including analogous results for the unsampled part), we have

$$\begin{aligned}\mathbb{E}\left(S_n^2\right) &= \frac{n}{n-1}\mathbb{E}\left(\overline{\overline{Y}}_n - \bar{Y}_n^2\right) = \frac{n}{n-1}\left(\mathbb{E}\left(\overline{\overline{Y}}_n\right) - \mathbb{E}\left(\bar{Y}_n^2\right)\right)\\ &= \frac{n}{n-1}\left(\sigma^2 - \frac{\sigma^2}{n}\right) = \frac{n}{n-1}\left(1 - \frac{1}{n}\right)\sigma^2 = \sigma^2.\end{aligned}$$

To obtain the second raw moment, we first obtain

$$\begin{aligned}\mathbb{E}\left(\left(\overline{\overline{Y}}_n - \bar{Y}_n^2\right)^2\right) &= \mathbb{E}\left(\overline{\overline{Y}}_n^2 - 2\bar{Y}_n^2\overline{\overline{Y}}_n + \bar{Y}_n^4\right)\\ &= \mathbb{E}\left[\left(\overline{\overline{Y}}_n^2\right) - 2\mathbb{E}\left(\bar{Y}_n^2\overline{\overline{Y}}_n\right) + \mathbb{E}\left(\bar{Y}_n^4\right)\right]\\ &= \left[\frac{(\kappa + n - 1)}{n} - 2\frac{(\kappa + n - 1)}{n^2}\right.\\ &\quad \left. + \frac{\kappa + n - 1}{n^3} + \frac{2(n-1)}{n^3}\right]\sigma^4\\ &= \frac{\kappa + n - 1}{n^3}\left(n^2 - 2n + 1\right)\sigma^4 + \frac{2(n-1)}{n^3}\sigma^4\\ &= \frac{(n-1)^2}{n^3}(\kappa + n - 1)\sigma^4 + \frac{2(n-1)}{n^3}\sigma^4\\ &= \frac{n-1}{n^3}\left[(n-1)(\kappa + n - 1) + 2\right]\sigma^4.\end{aligned}$$

The second raw moment is then given by

$$\begin{aligned}\mathbb{E}\left(S_n^4\right) &= \frac{n^2}{(n-1)^2}\mathbb{E}\left(\left(\overline{\overline{Y}}_n - \bar{Y}_n^2\right)^2\right)\\ &= \frac{1}{n(n-1)}\left[(n-1)(\kappa + n - 1) + 2\right]\sigma^4\\ &= \frac{(n-1)(\kappa + n - 1) + 2}{n(n-1)}\sigma^4.\end{aligned}$$

Hence, the variance is given by

$$\begin{aligned}\mathbb{V}\left(S_n^2\right) &= \mathbb{E}\left(S_n^4\right) - \mathbb{E}\left(S_n^2\right)^2\\ &= \frac{(n-1)(\kappa + n - 1) + 2}{n(n-1)}\sigma^4 - \sigma^4\\ &= \frac{(n-1)(\kappa + n - 1) + 2 - n(n-1)}{n(n-1)}\sigma^4\\ &= \frac{(n-1)\kappa - (n-3)}{n(n-1)}\sigma^4\\ &= \left(\kappa - \frac{n-3}{n-1}\right)\frac{\sigma^4}{n}.\end{aligned}$$

This gives the stated results. □

Proof of Result 4. Using the results in Lemma A.2 (including analogous results for the unsampled part), we have mean given by

$$\begin{aligned}
\mathbb{E}(D_N^2) &= \frac{n\,(N-n)}{N}\mathbb{E}\left(\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^2\right) \\
&= \frac{n\,(N-n)}{N}\left(\mathbb{E}(\bar{Y}_n^2) - 2\mathbb{E}\left(\bar{Y}_n\right)\mathbb{E}\left(\bar{Y}_{n:N}\right) + \mathbb{E}\left(\bar{Y}_{n:N}^2\right)\right) \\
&= \frac{n\,(N-n)}{N}\left(\mathbb{E}(\bar{Y}_n^2) + \mathbb{E}\left(\bar{Y}_{n:N}^2\right)\right) \\
&= \frac{n\,(N-n)}{N}\left(\frac{\sigma^2}{n} + \frac{\sigma^2}{N-n}\right) \\
&= \frac{n\,(N-n)}{N}\left(\frac{1}{n} + \frac{1}{N-n}\right)\sigma^2 \\
&= \frac{n\,(N-n)}{N}\frac{N}{n\,(N-n)}\sigma^2 = \sigma^2.
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathbb{E}\left(\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^4\right) \\
&= \mathbb{E}\left(\bar{Y}_n^4\right) + 6\mathbb{E}\left(\bar{Y}_n^2\right)\mathbb{E}\left(\bar{Y}_{n:N}^2\right) + \mathbb{E}\left(\bar{Y}_{n:N}^4\right) \\
&= \frac{(\kappa + 3n - 3)\,\sigma^4}{n^3} + 6\frac{\sigma^2}{n}\frac{\sigma^2}{N-n} + \frac{(\kappa + 3\,(N-n) - 3)\,\sigma^4}{(N-n)^3} \\
&= \left[\frac{(\kappa-3) + 3n}{n^3} + \frac{6}{n\,(N-n)} + \frac{(\kappa-3) + 3\,(N-n)}{(N-n)^3}\right]\sigma^4 \\
&= \left[3\left(\frac{1}{n^2} + \frac{2}{n\,(N-n)} + \frac{1}{(N-n)^2}\right) + (\kappa-3)\left(\frac{1}{n^3} + \frac{1}{(N-n)^3}\right)\right]\sigma^4 \\
&= \left[\frac{3N^2}{n^2\,(N-n)^2} + (\kappa-3)\,\frac{N^3 - 3N^2n + 3Nn^2}{n^3\,(N-n)^3}\right]\sigma^4 \\
&= \frac{N^2}{n^2\,(N-n)^2}\left[3 + (\kappa-3)\,\frac{N^2 - 3Nn + 3n^2}{Nn\,(N-n)}\right]\sigma^4. \\
&= \frac{N^2}{n^2\,(N-n)^2}\left[3N + (\kappa-3)\left(\frac{N^2}{n\,(N-n)} - 3\right)\right]\frac{\sigma^4}{N}.
\end{aligned}$$

Now, the second raw moment of the distance measure is given by

$$\begin{aligned}
\mathbb{E}(D_N^4) &= \frac{n^2\,(N-n)^2}{N^2}\mathbb{E}(\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^4) \\
&= \left(3 + (\kappa-3)\left(\frac{N}{n\,(N-n)} - \frac{3}{N}\right)\right)\sigma^4.
\end{aligned}$$

The variance of the distance measure is given by

$$\begin{aligned}
\mathbb{V}(D_N^2) &= \mathbb{E}\left(D_N^4\right) - \mathbb{E}\left(D_N^2\right)^2 \\
&= \left(2 + (\kappa-3)\left(\frac{N}{n\,(N-n)} - \frac{3}{N}\right)\right)\sigma^4.
\end{aligned}$$

This gives the stated results. □

Proof of Result 5. The expected value follows trivially from the fact that the unsampled variance and distance measures are unbiased estimators of the variance parameter (see Results 3 and 4). For the variance, we can use the variance decomposition to obtain

$$\begin{aligned}
&\mathbb{V}\Big((N-n-1)\,S_{n:N}^2 + D_N^2\Big) \\
&= \mathbb{V}\Big((N-1)\,S_N^2 - (n-1)\,S_n^2\Big) \\
&= (N-1)^2\,\mathbb{V}\Big(S_N^2\Big) + (n-1)^2\,\mathbb{V}\left(S_n^2\right) - 2\,(N-1)\,(n-1)\,\mathbb{C}\left(S_N^2, S_n^2\right) \\
&= (N-1)^2\,\mathbb{V}\Big(S_N^2\Big) + (n-1)^2\,\mathbb{V}\left(S_n^2\right) - 2\,(N-1)\,(n-1)\,\mathbb{V}\Big(S_N^2\Big) \\
&= (N-1)\,[(N-1) - 2\,(n-1)]\,\mathbb{V}\Big(S_N^2\Big) + (n-1)^2\,\mathbb{V}\left(S_n^2\right) \\
&= (N-1)\,(N-2n+1)\,\mathbb{V}\left(S_N^2\right) + (n-1)^2\,\mathbb{V}\left(S_n^2\right) \\
&= (N-1)\,(N-2n+1)\left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N} + (n-1)^2\left(\kappa - \frac{n-3}{n-1}\right)\frac{\sigma^4}{n} \\
&= \begin{bmatrix} \left(n\,(N-1)\,(N-2n+1) + N\,(n-1)^2\right)\kappa \\ -\,(n\,(N-2n+1)\,(N-3) + N\,(n-1)\,(n-3)) \end{bmatrix}\frac{\sigma^4}{Nn} \\
&= \begin{bmatrix} \left(n\left(N^2 - 2Nn + 2n - 1\right) + N\left(n^2 - 2n + 1\right)\right)\kappa \\ -\left(n\left(N^2 - 2Nn - 2N + 6n - 3\right) + N\left(n^2 - 4n + 3\right)\right) \end{bmatrix}\frac{\sigma^4}{Nn} \\
&= \begin{bmatrix} \left(N^2n - 2Nn^2 + 2n^2 - n + Nn^2 - 2Nn + N\right)\kappa \\ -\left(N^2n - 2Nn^2 - 2Nn + 6n^2 - 3n + Nn^2 - 4Nn + 3N\right) \end{bmatrix}\frac{\sigma^4}{Nn} \\
&= \begin{bmatrix} \left(N^2n - Nn^2 + 2n^2 - 2Nn + N - n\right)\kappa \\ -\left(N^2n - Nn^2 - 6Nn + 6n^2 + 3N - 3n\right) \end{bmatrix}\frac{\sigma^4}{Nn} \\
&= \begin{bmatrix} (Nn - 2n + 1)\,\kappa \\ -\,(Nn - 6n + 3) \end{bmatrix}(N-n)\,\frac{\sigma^4}{Nn} \\
&= \left[2Nn + (\kappa-3)\,(Nn - 2n + 1)\right](N-n)\,\frac{\sigma^4}{Nn} \\
&= \left(2 + (\kappa-3)\,\left(1 - \tfrac{2}{N} + \tfrac{1}{Nn}\right)\right)(N-n)\,\sigma^4.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{V}(C_N^2) &= \mathbb{V}\left(\frac{(N-n-1)\,S_{n:N}^2 + D_N^2}{N-n}\right) \\
&= \left(2 + (\kappa-3)\left(1 - \frac{2}{N} + \frac{1}{Nn}\right)\right)\frac{\sigma^4}{N-n},
\end{aligned}$$

which was to be shown. □

Proof of Result 6. Using Lemma A.2 (with analogous results for the unsampled part) and noting that $\mathbb{E}(\bar{Y}_n) = \mathbb{E}(\bar{Y}_{n:N}) = 0$, we have

$$\begin{aligned}
\mathbb{C}\left(\bar{X}_n, D_N^2\right) &= \mathbb{C}\left(\bar{X}_n, \frac{n\,(N-n)}{N}\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^2\right) \\
&= \frac{n\,(N-n)}{N}\mathbb{C}\left(\bar{Y}_n, \left(\bar{Y}_n^2 - 2\bar{Y}_n\bar{Y}_{n:N} + \bar{Y}_{n:N}\right)\right) \\
&= \frac{n\,(N-n)}{N}\left[\mathbb{C}(\bar{Y}_n, \bar{Y}_n^2) - 2\mathbb{C}\left(\bar{Y}_n, \bar{Y}_n\bar{Y}_{n:N}\right)\right] \\
&= \frac{n\,(N-n)}{N}\Big[\mathbb{E}\left(\bar{Y}_n^3\right) - \mathbb{E}\left(\bar{Y}_n\right)\mathbb{E}\left(\bar{Y}_n^2\right) - 2\mathbb{E}\left(\bar{Y}_n^2\right)\mathbb{E}\left(\bar{Y}_{n:N}\right) \\
&\quad + 2\mathbb{E}\left(\bar{Y}_n\right)^2\mathbb{E}\left(\bar{Y}_{n:N}\right)\Big] \\
&= \frac{n\,(N-n)}{N}\mathbb{E}\left(\bar{Y}_n^3\right) = \frac{N-n}{N}\cdot\frac{\gamma\sigma^3}{n}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\mathbb{C}\left(S_n^2, D_N^2\right) = \mathbb{C}\left(\frac{n}{n-1}\left(\overline{\overline{Y}}_n - \bar{Y}_n^2\right), \frac{n\,(N-n)}{N}\left(\bar{Y}_n - \bar{Y}_{n:N}\right)^2\right) \\
&= \frac{n^2\,(N-n)}{N\,(n-1)}\mathbb{C}\left(\left(\overline{\overline{Y}}_n - \bar{Y}_n^2\right), \left(\bar{Y}_n^2 - 2\bar{Y}_n\bar{Y}_{n:N} + \bar{Y}_{n:N}^2\right)\right) \\
&= \frac{n^2\,(N-n)}{N\,(n-1)}\Big[\mathbb{C}\left(\overline{\overline{Y}}_n, \bar{Y}_n^2\right) - 2\mathbb{C}\left(\overline{\overline{Y}}_n, \bar{Y}_n\bar{Y}_{n:N}\right) - \mathbb{C}\left(\bar{Y}_n^2, \bar{Y}_n^2\right) \\
&\quad + 2\mathbb{C}\left(\bar{Y}_n^2, \bar{Y}_n\bar{Y}_{n:N}\right)\Big] \\
&= \frac{n^2\,(N-n)}{N\,(n-1)}\begin{bmatrix} E\left(\overline{\overline{Y}}_n\bar{Y}_n^2\right) - E\left(\overline{\overline{Y}}_n\right)E\left(\bar{Y}_n^2\right) \\ -2E\left(\overline{\overline{Y}}_n\bar{Y}_n\right)E\left(\bar{Y}_{n:N}\right) + 2E\left(\overline{\overline{Y}}_n\right)E\left(\bar{Y}_n\right)E\left(\bar{Y}_{n:N}\right) \\ -E\left(\bar{Y}_n^4\right) + \mathbb{E}\left(\bar{Y}_n^2\right)^2 \\ +2E\left(\bar{Y}_n^3\right)E\left(\bar{Y}_{n:N}\right) - 2E\left(\bar{Y}_n^2\right)E\left(\bar{Y}_n\right)E\left(\bar{Y}_{n:N}\right) \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n^2 (N-n)}{N(n-1)} \left[\mathbb{E}\left(\overline{\overline{Y}}_n \bar{Y}_n^2\right) - \mathbb{E}\left(\overline{\overline{Y}}_n\right) \mathbb{E}\left(\bar{Y}_n^2\right) - \mathbb{E}\left(\bar{Y}_n^4\right) + \mathbb{E}\left(\bar{Y}_n^2\right)^2 \right] \\
&= \frac{n^2 (N-n)}{N(n-1)} \left[\frac{(\kappa+n-1)}{n^2} - \frac{1}{n} - \frac{(\kappa+3n-3)}{n^3} + \frac{1}{n^2} \right] \sigma^4 \\
&= \frac{(N-n)}{nN(n-1)} [(\kappa+n)n - n^2 - (\kappa+3n-3)] \sigma^4 \\
&= \frac{(N-n)}{nN(n-1)} (\kappa-3)(n-1)\sigma^4 \\
&= (\kappa-3) \cdot \frac{N-n}{N} \cdot \frac{\sigma^4}{n}.
\end{aligned}
$$

It is a simple matter to construct analogous proofs for the covariances between the unsampled parts with the distance measure. (The only difference is that n and $N-n$ are swapped in the resulting formulas.) For brevity, we omit these proofs here. Once we have these results, we can then obtain the covariances for the population quantities with the distance measure using the mean and variance decompositions. For the population mean, we have

$$
\begin{aligned}
\mathbb{C}\left(\bar{X}_N, D_N^2\right) &= \frac{n}{N}\mathbb{C}\left(\bar{X}_n, D_N^2\right) + \frac{N-n}{N}\mathbb{C}\left(\bar{X}_{n:N}, D_N^2\right) \\
&= \frac{n}{N}\frac{N-n}{N} \cdot \frac{\gamma\sigma^3}{n} + \frac{N-n}{N}\frac{n}{N} \cdot \frac{\gamma\sigma^3}{N-n} \\
&= \frac{N-n}{N} \cdot \frac{\gamma\sigma^3}{N} + \frac{n}{N} \cdot \frac{\gamma\sigma^3}{N} = \frac{\gamma\sigma^3}{N}.
\end{aligned}
$$

For the population variance, we have

$$
\begin{aligned}
\mathbb{C}\left(S_N^2, D_N^2\right) &= \frac{1}{N-1}\Big[(n-1)\mathbb{C}\left(S_n^2, D_N^2\right) \\
&\quad + (N-n-1)\,\mathbb{C}\left(S_{n:N}^2, D_N^2\right) + \mathbb{V}\left(D_N^2\right)\Big] \\
&= \left[\begin{array}{c} (\kappa-3)\,\frac{N-n}{N} \cdot \frac{n-1}{n} + (\kappa-3)\,\frac{n}{N} \cdot \frac{N-n-1}{N-n} \\ + \left(2 + (\kappa-3)\left(\frac{N}{n(N-n)} - \frac{3}{N}\right)\right) \end{array}\right] \frac{\sigma^4}{N-1} \\
&= \Big[2 + \Big(\frac{(n-1)(N-n)}{Nn} + \frac{(N-n-1)\,n}{N(N-n)} \\
&\quad + \frac{N}{n(N-n)} - \frac{3}{N}\Big)(\kappa-3)\Big] \frac{\sigma^4}{N-1} \\
&= (2 + (\text{Part A})\,(\kappa-3))\,\frac{\sigma^4}{N-1}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&(\text{Part A}) \\
&= \frac{(n-1)(N-n)}{Nn} + \frac{(N-n-1)\,n}{N(N-n)} + \frac{N}{n(N-n)} - \frac{3}{N} \\
&= \frac{(n-1)(N-n)^2 + (N-n-1)\,n^2 + N^2 - 3n(N-n)}{Nn(N-n)} \\
&= \frac{(n-1)\left(N^2 - 2Nn + n^2\right) + (N-n-1)\,n^2 + N^2 - 3n(N-n)}{Nn(N-n)} \\
&= \frac{N^2 n - 2Nn^2 + n^3 - N^2 + 2Nn - n^2 + Nn^2 - n^3 - n^2 + N^2 - 3nN + 3n^2}{Nn(N-n)} \\
&= \frac{N^2 n - Nn^2 - Nn + n^2}{Nn(N-n)} = \frac{Nn(N-n) - n(N-n)}{Nn(N-n)} = \frac{N-1}{N}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathbb{C}\left(S_N^2, D_N^2\right) &= \left(2 + \frac{N-1}{N}(\kappa-3)\right)\frac{\sigma^4}{N-1} \\
&= \left(\frac{2N}{N-1} + (\kappa-3)\right)\frac{\sigma^4}{N}.
\end{aligned}
$$

This gives the stated results. □

Proof of Result 7. The sample mean and unsampled mean are independent, since their underlying values are independent. This means that the covariance between these quantities is zero. For the other two covariance results, we can use the decomposition for the population mean to obtain

$$
\begin{aligned}
\mathbb{C}\left(\bar{X}_n, \bar{X}_N\right) &= \frac{1}{N}\mathbb{C}\left(\bar{X}_n, N\bar{X}_N\right) \\
&= \frac{1}{N}\mathbb{C}\left(\bar{X}_n, n\bar{X}_n + (N-n)\,\bar{X}_{n:N}\right) \\
&= \frac{n}{N}\mathbb{V}\left(\bar{X}_n\right) = \frac{\sigma^2}{N}, \\
\mathbb{C}\left(\bar{X}_{n:N}, \bar{X}_N\right) &= \frac{1}{N}\mathbb{C}\left(\bar{X}_n, N\bar{X}_N\right) \\
&= \frac{1}{N}\mathbb{C}\left(\bar{X}_n, n\bar{X}_n + (N-n)\,\bar{X}_{n:N}\right) \\
&= \frac{N-n}{N}\mathbb{V}\left(\bar{X}_{n:N}\right) = \frac{\sigma^2}{N}.
\end{aligned}
$$

We now show the covariances for the variance quantities. As with the mean quantities, the sample variance and unsampled variance are independent, since their underlying values are independent. For the other two covariance results, we can use the decomposition for the population variance to obtain

$$
\begin{aligned}
&\mathbb{C}\left(S_n^2, (N-1)\,S_N^2\right) \\
&= \mathbb{C}\left(S_n^2, (n-1)\,S_n^2 + (N-n-1)\,S_{n:N}^2 + D_N^2\right) \\
&= (n-1)\,\mathbb{V}\left(S_n^2\right) + \mathbb{C}\left(S_n^2, D_N^2\right) \\
&= (n-1)\left(\kappa - \frac{n-3}{n-1}\right)\frac{\sigma^4}{n} + (\kappa-3)\,\frac{N-n}{N}\frac{\sigma^4}{n} \\
&= \left[(n-1)\left(\kappa - \frac{n-3}{n-1}\right) + (\kappa-3)\,\frac{N-n}{N}\right]\frac{\sigma^4}{n} \\
&= \left[\kappa\,(n-1) - (n-3) + \kappa\frac{N-n}{N} - \frac{3N-3n}{N}\right]\frac{\sigma^4}{n} \\
&= \left[\kappa\left(n - 1 + \frac{N-n}{N}\right) - \left((n-3) + \frac{3N-3n}{N}\right)\right]\frac{\sigma^4}{n} \\
&= \left[\kappa n\frac{N-1}{N} - n\left(\frac{N-3}{N}\right)\right]\frac{\sigma^4}{n} \\
&= (N-1)\left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N} \\
&= (N-1)\,\mathbb{V}\left(S_N^2\right).
\end{aligned}
$$

Using Result 3, we therefore have

$$
\begin{aligned}
\mathbb{C}\left(S_n^2, S_N^2\right) &= \frac{1}{N-1}\mathbb{C}\left(S_n^2, (N-1)\,S_N^2\right) \\
&= \frac{1}{N-1}(N-1)\,\mathbb{V}\left(S_N^2\right) \\
&= \mathbb{V}\left(S_N^2\right) = \left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N}.
\end{aligned}
$$

The covariance result for the unsampled variance and population variance follows directly by analogy. This gives the stated results. □

Proof of Result 8. We begin with the simplest of these results. Since the underlying values in the sampled and unsampled parts are independent, this gives us

$$\mathbb{C}\left(\bar{X}_n, S^2_{n:N}\right) = 0 \qquad \mathbb{C}\left(\bar{X}_{n:N}, S^2_n\right) = 0.$$

Now, for the covariance between the sample mean and sample variance, we can reexpress this using the mean-adjusted quantities:

$$\begin{aligned}\mathbb{C}\left(\bar{X}_n, S^2_n\right) &= \mathbb{C}\left(\bar{Y}_n, S^2_n\right)\\ &= \mathbb{E}\left(\bar{Y}_n S^2_n\right) - \mathbb{E}\left(\bar{Y}_n\right)\mathbb{E}\left(S^2_n\right) = \mathbb{E}\left(\bar{Y}_n S^2_n\right).\end{aligned}$$

Using the results in Lemma A.2 (including analogous results for the unsampled part), we have

$$\begin{aligned}\mathbb{C}\left(\bar{X}_n, S^2_n\right) &= \mathbb{E}\left(\bar{Y}_n S^2_n\right)\\ &= \mathbb{E}\left(\bar{Y}_n\left(\overline{\overline{Y}}_n - \bar{Y}^2_n\right)\right)\\ &= \frac{n}{n-1}\left(\mathbb{E}\left(\bar{Y}_n\overline{\overline{Y}}_n\right) - \mathbb{E}\left(\bar{Y}^3_n\right)\right)\\ &= \frac{n}{n-1}\left(\frac{\gamma\sigma^3}{n} - \frac{\gamma\sigma^3}{n^2}\right)\\ &= \frac{1}{n-1}\left(1-\frac{1}{n}\right)\gamma\sigma^3\\ &= \frac{1}{n-1}\left(\frac{n-1}{n}\right)\gamma\sigma^3 = \frac{\gamma\sigma^3}{n}.\end{aligned}$$

We then have

$$\mathbb{C}\left(\bar{X}_N, S^2_n\right) = \frac{n}{N}\mathbb{C}\left(\bar{X}_n, S^2_n\right) = \frac{n}{N}\frac{\gamma\sigma^3}{n} = \frac{\gamma\sigma^3}{N},$$

and

$$\begin{aligned}\mathbb{C}\left(\bar{X}_n, S^2_N\right) &= \frac{1}{N-1}\left[\left((n-1)\,\mathbb{C}\left(\bar{X}_n, S^2_n\right) + \mathbb{C}\left(\bar{X}_n, D^2_N\right)\right)\right]\\ &= \frac{1}{N-1}\left[(n-1)\,\frac{\gamma\sigma^3}{n} + \frac{N-n}{N}\cdot\frac{\gamma\sigma^3}{n}\right]\\ &= \frac{1}{N-1}\left[(n-1) + \frac{N-n}{N}\right]\frac{\gamma\sigma^3}{n}\\ &= \frac{1}{N-1}\left(\frac{Nn-n}{N}\right)\frac{\gamma\sigma^3}{n} = \frac{n}{N}\frac{\gamma\sigma^3}{n} = \frac{\gamma\sigma^3}{N}.\end{aligned}$$

Analogous results hold for the unsampled part and the population, and these are found in the same way. This gives the stated results. □

Proof of Result 9. For the sample quantities, we have

$$\begin{aligned}\mathbb{C}\left(\bar{X}_n, C^2_N\right) &= \frac{1}{N-n}\mathbb{C}\left(\bar{X}_n, D^2_N\right)\\ &= \frac{1}{N-n}\frac{N-n}{N}\cdot\frac{\gamma\sigma^3}{n} = \frac{\gamma\sigma^3}{Nn},\\ \mathbb{C}\left(S^2_n, C^2_N\right) &= \frac{1}{N-n}\mathbb{C}\left(S^2_n, D^2_N\right)\\ &= \frac{1}{N-n}(\kappa-3)\cdot\frac{N-n}{N}\cdot\frac{\sigma^4}{n} = (\kappa-3)\cdot\frac{\sigma^4}{Nn}.\end{aligned}$$

For the unsampled quantities, we have

$$\begin{aligned}&\mathbb{C}\left(\bar{X}_{n:N}, C^2_N\right)\\ &= \frac{1}{N-n}\left[(N-n-1)\,\mathbb{C}\left(\bar{X}_{n:N}, S^2_{n:N}\right) + \mathbb{C}\left(\bar{X}_{n:N}, D^2_N\right)\right]\\ &= \frac{1}{N-n}\left[(N-n-1)\,\frac{\gamma\sigma^3}{N-n} + \frac{n}{N}\cdot\frac{\gamma\sigma^3}{N-n}\right]\\ &= \frac{1}{(N-n)^2}\left[(N-n-1) + \frac{n}{N}\right]\gamma\sigma^3\\ &= \frac{1}{(N-n)^2}\left[N^2 - Nn - N + n\right]\frac{\gamma\sigma^3}{N}\\ &= \frac{1}{(N-n)^2}\left[(N-n)^2 + Nn - N + n - n^2\right]\frac{\gamma\sigma^3}{N}\\ &= \frac{1}{(N-n)^2}\left[(N-n)^2 + (N-n)(n-1)\right]\frac{\gamma\sigma^3}{N}\\ &= \frac{N-1}{N-n}\cdot\frac{\gamma\sigma^3}{N},\end{aligned}$$

$$\begin{aligned}&\mathbb{C}\left(S^2_{n:N}, C^2_N\right)\\ &= \frac{1}{N-n}\left[(N-n-1)\,\mathbb{V}\left(S^2_{n:N}\right) + \mathbb{C}\left(S^2_{n:N}, D^2_N\right)\right]\\ &= \frac{1}{N-n}\left[(N-n-1)\left(\kappa - \frac{N-n-3}{N-n-1}\right)\frac{\sigma^4}{N-n}\right.\\ &\quad\left. + (\kappa-3)\cdot\frac{n}{N}\cdot\frac{\sigma^4}{N-n}\right]\\ &= \left[\begin{array}{c}\left(N-n-1+\frac{n}{N}\right)(\kappa-3)\\ +(N-n-1)\left(3-\frac{N-n-3}{N-n-1}\right)\end{array}\right]\frac{\sigma^4}{(N-n)^2}\\ &= \left[\begin{array}{c}\left(N^2-Nn-N+n\right)(\kappa-3)\\ +\left(3(N-n-1)-(N-n-3)\right)\end{array}\right]\frac{\sigma^4}{N(N-n)^2}\\ &= \left[\begin{array}{c}\left(N^2-Nn-N+n\right)(\kappa-3)\\ +(3N-3n-3-N+n+3)\end{array}\right]\frac{\sigma^4}{N(N-n)^2}\\ &= \left[\begin{array}{c}(N-n)(N-1)(\kappa-3)\\ +2(N-n)\end{array}\right]\frac{\sigma^4}{N(N-n)^2}\\ &= \left(2+(\kappa-3)(N-1)\right)\frac{\sigma^4}{N(N-n)}.\end{aligned}$$

For the population quantities, we have

$$\begin{aligned}&\mathbb{C}\left(\bar{X}_N, C^2_N\right)\\ &= \frac{n}{N}\mathbb{C}\left(\bar{X}_n, C^2_N\right) + \frac{N-n}{N}\mathbb{C}\left(\bar{X}_{n:N}, C^2_N\right)\\ &= \frac{n}{N}\frac{\gamma\sigma^3}{Nn} + \frac{N-n}{N}\frac{N-1}{N-n}\cdot\frac{\gamma\sigma^3}{N}\\ &= \frac{\gamma\sigma^3}{N^2} + (N-1)\cdot\frac{\gamma\sigma^3}{N^2} = \frac{\gamma\sigma^3}{N}.\end{aligned}$$

$$\begin{aligned}&\mathbb{C}\left(S^2_N, C^2_N\right)\\ &= \frac{1}{N-n}\mathbb{C}\left(S^2_N, (N-n-1)\,S^2_{n:N} + D^2_N\right)\\ &= \frac{1}{N-n}\left[(N-n-1)\,\mathbb{C}\left(S^2_N, S^2_{n:N}\right) + \mathbb{C}\left(S^2_N, D^2_N\right)\right]\\ &= \frac{1}{N-n}\left[(N-n-1)\left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N}\right.\end{aligned}$$

$$+\left(\frac{2N}{N-1}+(\kappa-3)\right)\frac{\sigma^4}{N}\Bigg]$$

$$=\left[(N-n-1)\left(\kappa-\frac{N-3}{N-1}\right)+\frac{2N}{N-1}+(\kappa-3)\right]$$

$$\times\frac{\sigma^4}{N(N-n)}$$

$$=\Bigg[(N-n-1)\left(\kappa-3+3-\frac{N-3}{N-1}\right)$$

$$+\frac{2N}{N-1}+(\kappa-3)\Bigg]\frac{\sigma^4}{N(N-n)}$$

$$=\Bigg[(N-n)(\kappa-3)+(N-n-1)\left(3-\frac{N-3}{N-1}\right)$$

$$+\frac{2N}{N-1}\Bigg]\frac{\sigma^4}{N(N-n)}$$

$$=(\kappa-3)\frac{\sigma^4}{N}+[(N-n-1)(3(N-1)-(N-3))+2N]$$

$$\times\frac{\sigma^4}{N(N-n)(N-1)}$$

$$=(\kappa-3)\frac{\sigma^4}{N}+[2N(N-n)]\frac{\sigma^4}{N(N-n)(N-1)}$$

$$=(\kappa-3)\frac{\sigma^4}{N}+\frac{2\sigma^4}{N-1}=\left(\frac{2N}{N-1}+(\kappa-3)\right)\frac{\sigma^4}{N}.$$

This gives the stated results. □

Proof of Results 10–13. The proofs of these results all follow along the same lines. Each of the correlation expressions follows trivially from the previous moment results and these can then be reframed in terms of n and u using the fact that

$$N=\frac{n}{1-u}\qquad N-n=\frac{nu}{1-u}.$$

Once the correlation results are in this form, it is then a simple matter to take limits as $n\to\infty$ noting that we refer implicitly to the limiting value of u, which is a proportion. This allows us to obtain each of the asymptotic correlation results. We give one example here:

$$\mathrm{Corr}\left(\bar{X}_n,D_N^2\right)=\frac{\mathbb{C}\left(\bar{X}_n,D_N^2\right)}{\sqrt{\mathbb{V}\left(\bar{X}_n\right)\mathbb{V}\left(D_N^2\right)}}$$

$$=\frac{\frac{N-n}{N}\cdot\gamma}{\sqrt{2n+(\kappa-3)\left(\frac{N}{N-n}-3\frac{n}{N}\right)}}$$

$$=\frac{u\cdot\gamma}{\sqrt{2n+(\kappa-3)(1/u-3(1-u))}}\to 0.$$

This shows the first result in Result 10. The remaining proofs are omitted, but follow along the same lines. □

Lemma A.3. If κ and σ are both finite then, as $n\to\infty$ we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2-1\right)\xrightarrow{\text{Dist}}\mathrm{N}(0,\kappa-1).$$

Proof. We define the standardized values:

$$Z_i=\frac{X_i-\mu}{\sigma}.$$

We now look at the sequence of iid values $Z_1^2,\ldots,Z_n^2$ which have moments:

$$\mathbb{E}\left(Z_i^2\right)=1\qquad\mathbb{V}\left(Z_i^2\right)=\kappa-1.$$

The quantity under analysis in the lemma is the sample mean of the Z_i^2 quantities, minus their true mean, and multiplied by $\sqrt{n}$. The distribution therefore follows directly from the central limit theorem for iid random variables (see, e.g., Bartoszyński and Niewiadomska-Bugaj 1996, pp. 431–432). □

For convenience, we split the proof of Result 14 into two parts, first looking at the distribution of the mean difference, and then looking at the scaled variance quantities. Both of these results appeal to the central limit theorem.

Proof of Result 14 (Mean difference). In this proof, we will look only at the asymptotic distribution of the mean difference quantity. With a little algebra, it can easily be shown that

$$\bar{X}_n-\bar{X}_N=\frac{1}{n}\left[u\sum_{i=1}^{n}X_i-(1-u)\sum_{i=n+1}^{N}X_i\right].$$

The random variables $uX_1,\ldots,uX_n$ are iid and so are $(1-u)X_{n+1},\ldots,(1-u)X_N$. From the CLT this means that both sums converge to independent normal random variables as $n\to\infty$ and $N-n\to\infty$ (which is what occurs in our limiting analysis). Since sums of independent normal random variables are also normally distributed this means that the mean difference is asymptotically normal. It remains only to note that

$$\mathbb{E}\left(\bar{X}_n-\bar{X}_N\right)=0\qquad\mathbb{V}\left(\bar{X}_n-\bar{X}_N\right)=\frac{N-n}{Nn}\cdot\sigma^2.$$

By taking the first part out of the distribution as a scaling constant, this then gives the required distributional result. □

Proof of Result 14 (Scaled variances). We now look at the distributions of the various scaled variance quantities. For brevity, we will do this only for the sample variance, since the other proofs are all analogous. With a little algebra it is easy to show that

$$(n-1)S_n^2=\sum_{i=1}^{n}\left(X_i-\bar{X}_n\right)^2=\sum_{i=1}^{n}(X_i-\mu)^2-n\left(\bar{X}_n-\mu\right)^2.$$

Rearranging, we obtain

$$S_n^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2-\left(\bar{X}_n-\mu\right)^2+\frac{1}{n}S_n^2.$$

This means that

$$\sqrt{n}\cdot\left(\frac{S_n^2}{\sigma^2}-1\right)=\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2-1\right]$$

$$-\frac{1}{\sqrt{n}}\left(\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}}\right)^2+\frac{1}{\sqrt{n}}\frac{S_n^2}{\sigma^2}.$$

Using Slutsky's theorem (see, e.g., Bartoszyński and Niewiadomska-Bugaj 1996, pp. 421–422), it is easy to show that the second term converges in distribution to zero and the third term converges in probability to zero, which means that the limiting distribution of the quantity of interest is equal to the limiting distribution of the first term in the expansion. From Lemma A.4 this means that

$$\sqrt{n} \cdot \left(\frac{S_n^2}{\sigma^2} - 1 \right) \xrightarrow{\text{Dist}} \text{N}(0, \kappa - 1).$$

Hence, we have

$$\frac{S_n^2}{\sigma^2} \overset{\text{Asymp}}{\sim} \text{N}\left(1, \frac{\kappa - 1}{n}\right).$$

Now, to complete the proof it remains only to note that the chi-squared distribution is asymptotically normal with variance equal to twice its mean. Hence, as $\text{DF} \to \infty$ we have

$$\frac{\text{ChiSq}(\text{DF})}{\text{DF}} \to \frac{1}{\text{DF}} \text{N}(\text{DF}, 2\text{DF}) = \text{N}\left(1, \frac{2}{\text{DF}}\right).$$

This means that we can reexpress the asymptotic normal form as a scaled chi-squared. Since $\text{DF}_n \to 2n/(\kappa - 1)$ the asymptotic distribution above can equivalently be written as

$$\frac{S_n^2}{\sigma^2} \overset{\text{Asymp}}{\sim} \frac{\text{ChiSq}(\text{DF}_n)}{\text{DF}_n}.$$

(In fact, we know from Result 8 that the mean and variance in this asymptotic distribution are exact for all n.) Although both forms are asymptotically equivalent, it is more sensible to use the chi-squared distribution rather than a normal distribution, since the latter quantity is nonnegative and is exact in the case of a normal superpopulation. The remaining proofs for the other scaled variance quantities are all analogous to the above proof. □

Lemma A.4. As $n \to \infty$, we have $\text{Corr}(\bar{X}_n - \bar{X}_N, S_n^2) \to \sqrt{u}\phi$.

Proof. The variance of the mean difference is given by

$$\begin{aligned} \mathbb{V}\left(\bar{X}_n - \bar{X}_N\right) &= \mathbb{V}\left(\bar{X}_n\right) - 2\mathbb{C}\left(\bar{X}_n, \bar{X}_N\right) + \mathbb{V}\left(\bar{X}_N\right) \\ &= \frac{\sigma^2}{n} - 2\frac{\sigma^2}{N} + \frac{\sigma^2}{N} = \frac{\sigma^2}{n} - \frac{\sigma^2}{N} = u\frac{\sigma^2}{n}. \end{aligned}$$

We therefore have

$$\begin{aligned} \text{Corr}\left(\bar{X}_n - \bar{X}_N, S_n^2\right) &= \frac{\mathbb{C}\left(\bar{X}_n - \bar{X}_N, S_n^2\right)}{\sqrt{\mathbb{V}\left(\bar{X}_n - \bar{X}_N\right)\mathbb{V}\left(S_n^2\right)}} \\ &= \frac{\mathbb{C}\left(\bar{X}_n, S_N^2\right) - \mathbb{C}\left(\bar{X}_N, S_N^2\right)}{\sqrt{\mathbb{V}\left(\bar{X}_n - \bar{X}_N\right)\mathbb{V}\left(S_n^2\right)}} \\ &= \frac{\frac{\gamma\sigma^3}{n} - \frac{\gamma\sigma^3}{N}}{\sqrt{u\frac{\sigma^2}{n}\left(\kappa - \frac{n-3}{n-1}\right)\frac{\sigma^4}{n}}} \\ &= \frac{\left(\frac{1}{n} - \frac{1}{N}\right)\gamma}{\frac{1}{n}\sqrt{u\left(\kappa - \frac{n-3}{n-1}\right)}} \\ &= \frac{\frac{N-n}{N}\gamma}{\sqrt{u\left(\kappa - \frac{n-3}{n-1}\right)}} = \sqrt{u} \cdot \frac{\gamma}{\sqrt{\kappa - \frac{n-3}{n-1}}}. \end{aligned}$$

As $n \to \infty$ we have $\text{Corr}\left(\bar{X}_n - \bar{X}_N, S_n^2\right) \to \sqrt{u} \cdot \phi$. □

As with Result 14, we split the proof of Result 15 into two parts, first looking at the approximate distribution of the studentized mean difference, and then looking at the variance ratio quantity. Both of these results appeal to the previous asymptotic marginal distributions of the parts going into the quantity.

Proof of Result 15 (Studentized mean difference). The approximate distribution in this result is based on appeal to the asymptotic marginal distributions in Result 14 when $n \to \infty$. From Lemma A.4, we know that as $n \to \infty$ we have $\text{Corr}(\bar{X}_n - \bar{X}_N, S_n^2) \to \sqrt{u} \cdot \phi$. If we ignore this dependence for purposes of our approximation, we obtain

$$\begin{aligned} \frac{\bar{X}_n - \bar{X}_N}{S_n^2} &= \frac{\bar{X}_n - \bar{X}_N}{\sigma^2} \Bigg/ \frac{S_n^2}{\sigma^2} \overset{\text{Approx}}{\sim} \sqrt{\frac{N-n}{Nn}} \\ &\quad \cdot \frac{\text{N}(0,1)}{\text{ChiSq}(\text{DF}_n)/\text{DF}_n} = \sqrt{\frac{N-n}{Nn}} \cdot St(\text{DF}_n). \end{aligned}$$

As stated in the result, this approximation ignores the correlation between the numerator and denominator in the expression. This gives us the approximate distribution for the studentized mean difference between the sample and population. □

Proof of Result 15 (Variance ratio). The approximate distribution in this result is also based on appeal to the asymptotic marginal distributions in Result 14 when $n \to \infty$. Using the decomposition for the population variance, we have

$$\begin{aligned} \frac{S_N^2}{S_n^2} &= \frac{(N-1)S_N^2}{(N-1)S_n^2} = \frac{(n-1)S_n^2 + (N-n)C_N^2}{(N-1)S_n^2} \\ &= \frac{n-1}{N-1} + \frac{N-n}{N-1} \cdot \frac{C_N^2/\sigma^2}{S_n^2/\sigma^2} \\ &\overset{\text{Approx}}{\sim} \frac{n-1}{N-1} + \frac{N-n}{N-1} \cdot \frac{\text{ChiSq}(\text{DF}_C)/\text{DF}_C.}{\text{ChiSq}(\text{DF}_n)/\text{DF}_n} \\ &= \frac{n-1}{N-1} + \frac{N-n}{N-1} \cdot \frac{1}{F(\text{DF}_n, \text{DF}_C)}. \end{aligned}$$

This approximation makes use of the fact that the numerator and denominator in this case are asymptotically uncorrelated (though not necessarily independent). □

APPENDIX B: FURTHER MATERIAL

In this appendix, we set out some extensions to the material in the body of the article. This material is ancillary to the main results in the article, but may be of interest to practitioners in giving some further detail on the meaning and interpretations of quantities, and some additional issues that arise in applications of the present material.

The distance measure. In the body of the article, we defined the distance measure:

$$D_N^2 = \frac{n(N-n)}{N}\left(\bar{X}_n - \bar{X}_{n:N}\right)^2.$$

The easiest way to understand the meaning of the distance measure is by considering a population with $N = 2$ objects, giving us population values X_1 and X_2. In this case, we have population variance given by

$$\begin{aligned} S_N^2 &= \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}_N\right)^2 \\ &= \left(X_1 - \frac{X_1+X_2}{2}\right)^2 + \left(X_2 - \frac{X_1+X_2}{2}\right)^2 \\ &= \left(\frac{X_1-X_2}{2}\right)^2 + \left(\frac{X_2+X_1}{2}\right)^2 \\ &= 2\left(\frac{X_1-X_2}{2}\right)^2 = \frac{1}{2}(X_1-X_2)^2. \end{aligned}$$

If $N = 2$ and $n = 1$, we then have

$$D_N^2 = \frac{1}{2}(X_1 - X_2)^2 = S_N^2.$$

In this special case, the distance measure is equivalent to the two-point population variance. In the more general case, the distance measure extends this idea to allow the comparison of two *sets* of points, compared by using their means. As with the sample and population variances, we have $\mathbb{E}(D_N^2) = \sigma^2$ so that the distance measure is centered around the variance of the superpopulation. The variance of the distance measure is quite high, owing to the fact that it is similar to a two-point estimator.

(It is actually possible to use a distance measure of this form to estimate variance in the pathological situation where you have a sample of values where the only known information is the mean of two groups partitioning the sample. This yields a very inaccurate interval estimate, but that is not surprising given that it is a two-point estimator.) □

Minimum length variance intervals. In the body of the article, we derived the variance interval by choosing a value $0 \leq \theta \leq \alpha$ to obtaining the confidence interval. The general form of the resulting confidence interval allows any value of θ in this range. One obvious way to proceed is to choose θ to minimize the length of the interval. To do this, we will let Q be the quantile function of the F-distribution with the appropriate number of degrees of freedom. This gives us

$$F^*_{1-\theta,\mathrm{DF}_n,\mathrm{DF}_C} = Q(1-\theta) \qquad F^*_{\alpha-\theta,\mathrm{DF}_n,\mathrm{DF}_C} = Q(\alpha-\theta).$$

To minimize the interval length, we want to minimize the objective function:

$$H_\alpha(\theta) = \frac{1}{Q(\alpha-\theta)} - \frac{1}{Q(1-\theta)}.$$

Differentiating with respect to θ we obtain

$$\frac{dH_\alpha}{d\theta}(\theta) = -\frac{Q'(\alpha-\theta)}{Q(\alpha-\theta)^2} + \frac{Q'(1-\theta)}{Q(1-\theta)^2}.$$

We therefore obtain the critical point equation:

$$\frac{Q'(\alpha-\hat{\theta})}{Q'(1-\hat{\theta})} = \left(\frac{Q(\alpha-\hat{\theta})}{Q(1-\hat{\theta})}\right)^2.$$

Solving for $\hat{\theta}$ in the allowable range gives us the appropriate value to form the minimum length interval. Since the quantile function for the F-distribution cannot be written in closed form, in practice this will require numerical solution. □

Variance interval with unknown kurtosis. The application of the variance interval in this article requires us to estimate the kurtosis of the underlying superpopulation distribution, to obtain the appropriate degrees of freedom for the interval. By way of reminder, we note that we have $\mathbb{E}((X_i-\mu)^4) = \kappa\sigma^4$, so that

$$\kappa = \frac{\mathbb{E}\left((X_i-\mu)^4\right)}{\left(\mathbb{E}((X_i-\mu)^2)\right)^2}.$$

(Note that we have *not* adjusted to measure "excess kurtosis." This means that $\kappa = 3$ for a normal distribution.) The kurtosis κ gives a scale-adjusted measure of the heaviness of the tails of the superpopulation distribution (see Dodge and Rousson 1999).

In this article, we do not give lengthy consideration to estimates of kurtosis. An examination of various estimators of skewness and kurtosis can be found in Joanes and Gill (1998). For our purposes, it will suffice to set out some examples of these estimators, which are taken from that article. A simple estimator that is unbiased for normal samples is

$$K_n = \frac{n(n+1)}{(n-1)}\frac{\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^4}{\left(\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2\right)^2}.$$

More complicated estimators are used in various statistical software packages:

$$\begin{aligned} K_n^{\mathrm{MINITAB}} = K_n^{\mathrm{BMDP}} &= \frac{(n-1)^2}{n}\frac{\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^4}{\left(\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2\right)^2}, \\ K_n^{\mathrm{SAS}} = K_n^{\mathrm{SPSS}} = K_n^{\mathrm{EXCEL}} &= \frac{n(n+1)(n-1)}{(n-2)(n-3)} \\ &\quad\times\frac{\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^4}{\left(\sum_{i=1}^n\left(X_i-\bar{X}_n\right)^2\right)^2} - \frac{9(n-5/3)}{(n-2)(n-3)}. \end{aligned}$$

Joanes and Gill (1998) looked at the mean square error of these kurtosis estimators for various distributions. They found that the estimators used in MINITAB and BMDP have lower mean square error for a normal distribution, but the estimator used in SAS, SPSS, and EXCEL has a lower mean square error for highly skewed distributions. It is easy to see that as $n \to \infty$ all of these estimators converge to the same limiting value. All are weakly consistent, in the sense that they converge in probability to the true kurtosis parameter κ.

Any of these estimators can be substituted into the confidence interval formulas set out in this article, and all should perform adequately for large n. The optimal estimator will depend on the particular distribution one is working with, which of course, is unknown. □

Fitting the moment results together. We noted in the main body of this article that the moment results fit together according to the rules for moments of linear functions using our decomposition results for the population mean and variance. Although this is quite cumbersome to do, we set out the details here as a useful check on our working, or as an alternative derivation of some of the moment results. The simplest of the two cases is the decomposition for the population mean, which is

$$N\bar{X}_N = n\bar{X}_n + (N-n)\,\bar{X}_{n:N}.$$

Taking the variance of both sides of this equation should establish a relationship between the various variance and covariance quantities applicable to the terms in the decomposition. Taking the variance of the left-hand side, we have

$$\mathbb{V}\,(\text{LHS}) = \mathbb{V}\left(N\bar{X}_N\right) = N^2\mathbb{V}\left(\bar{X}_N\right) = N^2\frac{\sigma^2}{N} = N\sigma^2.$$

Taking the variance of the right-hand side, we have

$$\begin{aligned}\mathbb{V}\,(\text{RHS}) &= \mathbb{V}\left(n\bar{X}_n + (N-n)\,\bar{X}_{n:N}\right) = n^2\mathbb{V}\left(\bar{X}_n\right) \\ &\quad + (N-n)^2\,\mathbb{V}\left(\bar{X}_{n:N}\right) \\ &= n^2\frac{\sigma^2}{n} + (N-n)^2\,\frac{\sigma^2}{N-n} \\ &= n\sigma^2 + (N-n)\,\sigma^2 \\ &= N\sigma^2 = \mathbb{V}(\text{LHS}).\end{aligned}$$

The more complicated case is the decomposition for the population variance, which is

$$(N-1)\,S_N^2 = (n-1)\,S_n^2 + (N-n-1)\,S_{n:N}^2 + D_N^2.$$

Taking the variance of both sides of this equation should establish a relationship between the various variance and covariance quantities applicable to the terms in the decomposition. Taking the variance of the left-hand side, we have

$$\begin{aligned}\mathbb{V}(\text{LHS}) &= \mathbb{V}\left((N-1)\,S_N^2\right) \\ &= (N-1)^2\,\mathbb{V}\left(S_N^2\right) = (N-1)^2\left(\kappa - \frac{N-3}{N-1}\right)\frac{\sigma^4}{N}.\end{aligned}$$

Taking the variance of the right-hand side, we have

$$\begin{aligned}&\mathbb{V}\,(\text{RHS}) \\ &= \mathbb{V}\left((n-1)\,S_n^2 + (N-n-1)\,S_{n:N}^2 + D_N^2\right) \\ &= (n-1)^2\,\mathbb{V}\left(S_n^2\right) + (N-n-1)^2\,\mathbb{V}\left(S_{n:N}^2\right) + \mathbb{V}\left(D_N^2\right) \\ &\quad + 2\,(n-1)\,\mathbb{C}\left(S_n^2, D_N^2\right) + 2\,(N-n-1)\,\mathbb{C}\left(S_{n:N}^2, D_N^2\right) \\ &= \begin{bmatrix} (n-1)^2\left(\kappa - \frac{n-3}{n-1}\right)\frac{N}{n} + (N-n-1)^2\left(\kappa - \frac{N-n-3}{N-n-1}\right)\frac{N}{N-n} \\ +2N + (\kappa-3)\left(\frac{N^2}{n(N-n)} - 3\right) \\ +2\,(n-1)\,(\kappa-3)\,\frac{N-n}{n} + 2\,(N-n-1)\,(\kappa-3)\,\frac{n}{N-n} \end{bmatrix}\frac{\sigma^4}{N} \\ &= \begin{bmatrix} \begin{pmatrix} (n-1)^2\,\frac{N}{n} + (N-n-1)^2\,\frac{N}{N-n} + \frac{N^2}{n(N-n)} - 3 \\ +2\,(n-1)\,\frac{N-n}{n} + 2\,(N-n-1)\,\frac{n}{N-n} \end{pmatrix}\kappa \\ -\begin{pmatrix} (n-1)\,(n-3)\,\frac{N}{n} + (N-n-1)\,(N-n-3)\,\frac{N}{N-n} \\ -2N + 3\frac{N^2}{n(N-n)} - 9 \\ +6\,(n-1)\,\frac{N-n}{n} + 6\,(N-n-1)\,\frac{n}{N-n} \end{pmatrix} \end{bmatrix}\frac{\sigma^4}{N}. \\ &= [(\text{Part A})\cdot\kappa - (\text{Part B})]\,\frac{\sigma^4}{N}.\end{aligned}$$

With a little algebra it can be shown that

$$\begin{aligned}&(\text{Part A}) \\ &= \begin{pmatrix} (n-1)^2\,\frac{N}{n} + (N-n-1)^2\,\frac{N}{N-n} + \frac{N^2}{n\,(N-n)} - 3 \\ +2\,(n-1)\,\frac{N-n}{n} + 2\,(N-n-1)\,\frac{n}{N-n} \end{pmatrix} \\ &= (N-1)^2\,,\end{aligned}$$

$$\begin{aligned}&(\text{Part B}) \\ &= \begin{pmatrix} (n-1)\,(n-3)\,\frac{N}{n} + (N-n-1)\,(N-n-3)\,\frac{N}{N-n} \\ -2N + 3\frac{N^2}{n\,(N-n)} - 9 \\ +6\,(n-1)\,\frac{N-n}{n} + 6\,(N-n-1)\,\frac{n}{N-n} \end{pmatrix} \\ &= (N-1)\,(N-3)\,,\end{aligned}$$

which gives us

$$\begin{aligned}\mathbb{V}(\text{RHS}) &= [(\text{Part A})\cdot\kappa + (\text{Part B})]\frac{\sigma^4}{N} \\ &= [(N-1)^2\kappa + (N-1)(N-3)]\frac{\sigma^4}{N} \\ &= (N-1)^2(\kappa - \frac{N-3}{N-1})\frac{\sigma^4}{N} = \mathbb{V}(\text{LHS}).\end{aligned}$$

This confirmation of the moment rules for linear functions operates as a check on our moment results, to confirm that they fit together as they should.

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