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*Corso di Laurea Magistrale in GEOSCIENZE*

***Metodi Elettromagnetici in Geofisica (6 CFU)  
- MEMAG -***

**UD-2A: Campi EM**

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# Some mathematical background and formalism

The NABLA operator (or DEL) is, by definition:

$$\nabla = i\nabla_1 + j\nabla_2 + k\nabla_3 = i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz}$$

Notice that all symbols **in bold** are related to **vectors**

In which **i, j, k** are unit vector

Strictly speaking is not a specific operator, but rather a convenient mathematical notation for those three operators, that makes many equations easier to write.

In fact, by using  $\nabla$  it is possible to make operations such as:

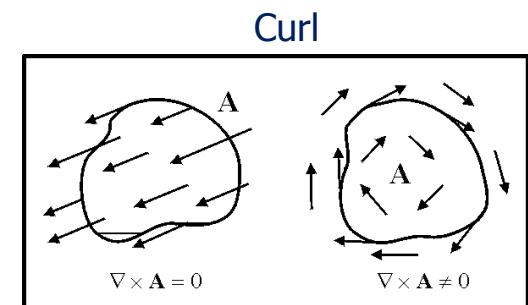
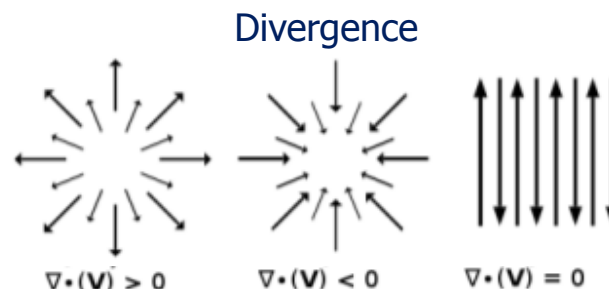
$$\text{Gradient: } \text{grad } s = \frac{\partial s}{\partial x} \mathbf{e}_x + \frac{\partial s}{\partial y} \mathbf{e}_y + \frac{\partial s}{\partial z} \mathbf{e}_z = \nabla s$$

$$\text{Divergence: } \text{div } \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \mathbf{v}$$

$$\text{Curl (rotore): } \text{curl } \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \mathbf{e}_x + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \mathbf{e}_y + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \mathbf{e}_z = \nabla \times \mathbf{v}$$

The **divergence** is roughly a measure of a vector field's increase in the direction it points; but more accurately, it is a measure of that field's tendency to converge toward or repel from a point.

The **curl** at a point is proportional to the on-axis torque at that point.



# Electromagnetism and Maxwell equations

Maxwell's equations mathematically describe the physics of electromagnetic fields. Constitutive relationships quantify material properties. Combining the two provides the foundations for electromagnetism in the media.

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}}{\partial t}(\mathbf{r}, t) \implies \text{Maxwell-Faraday eq. (Faraday law of induction)}$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \left( \mathbf{J}(\mathbf{r}, t) + \frac{\partial \mathbf{D}}{\partial t}(\mathbf{r}, t) \right) \implies \text{Ampere-Maxwell eq.}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = q \implies \text{Gauss eq.}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0 \implies \text{Gauss eq. for magnetism}$$

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial q}{\partial t}(\mathbf{r}, t) \implies \text{Electric charge is conserved}$$

Notice that all symbols **in bold** are related to **vectors**

In which:

**E** is the electric field strength vector (V/m)

**B** is the magnetic flux density vector (T)

**D** is the electric displacement vector  
(or dipole moment density) (C/m<sup>2</sup>)

**q** is the electric charge density (C/m<sup>2</sup>)

**J** is the current density vector (A/m<sup>2</sup>)

**r** is a generic space vector (m); **t** is time (s)

From these building blocks, all classic electromagnetics (e.g. induction, radio waves, resistivity, circuit theory, ...) can be derived after we characterize material electrical properties.

# Constitutive equations

In addition to the Maxwell's equations additional equations have to be introduced to relating the material physical properties to EM fields. For homogeneous and isotropic media they can be written as linear equations:

$$1) \mathbf{J}(r, t) = \tilde{\sigma} \mathbf{E}(r, t)$$

$$2) \mathbf{D}(r, t) = \tilde{\epsilon} \mathbf{E}(r, t)$$

$$3) \mathbf{B}(r, t) = \tilde{\mu} \mathbf{H}(r, t)$$

In which  $\tilde{\sigma}$ ,  $\tilde{\epsilon}$ ,  $\tilde{\mu}$  represent the **ELECTRICAL CONDUCTIVITY**, the **DIELECTRIC PERMITTIVITY** and the **MAGNETIC PERMEABILITY**, respectively.

Constitutive equations provide a macroscopic (or average behavior) description of how electrons/atoms/molecules/ions respond to the application of an EM field.

By simplifying:

ELECTRICAL CONDUCTIVITY describes how free electric charges flow to form a current when an electric field is present;

DIELECTRIC PERMITTIVITY describes how constrained electric charges are displaced in response to an electric field;

MAGNETIC PERMEABILITY describes how intrinsic atomic and molecular magnetic moments respond to a magnetic field.

# Constitutive equations

In free-space values of these 3 parameters are constants and equal to:

$$\varepsilon = \varepsilon_0 = 10^{-9}/36\pi = 8.85 \times 10^{-12} \text{ [F/m]} \approx \varepsilon_{\text{air}} \quad \mu = \mu_0 = 4\pi 10^{-7} \text{ [H/m]} \approx \mu_{\text{air}} \quad \sigma = \sigma_0 = 0 \text{ [S/m]} \approx \sigma_{\text{air}}$$

The electromagnetic fields in free space can be completely described either by **E** and **H**, or by **D** and **B**, because  $\varepsilon = \varepsilon_0$  and  $\mu = \mu_0$  while in regions where the permittivity and permeability values vary with position (like in almost all geological materials!) both sets of vectors are needed to completely describe the electromagnetic fields.

It is convenient to introduce some additional parameters, namely:

The **relative dielectric permittivity** (or dielectric constant)  $\rightarrow \varepsilon_r = \frac{\varepsilon_m}{\varepsilon_0}$  [dimensionless]

The **relative magnetic permeability**  $\rightarrow \mu_r = \frac{\mu_m}{\mu_0}$  [dimensionless]

Considering the two previous relations, both parameters are pure numbers and are not “constant” in nature, but at the opposite the exhibit, in the general case, a dependency by the frequency (f) of the oscillating EM field, so:

$$\varepsilon_r(f) = \frac{\varepsilon_m(f)}{\varepsilon_0} \quad \mu_r(f) = \frac{\mu_m(f)}{\mu_0}$$

For increasing frequencies both parameters have a decreasing non linear and complex trend

# Constitutive equations

In general,  $\tilde{\sigma}$ ,  $\tilde{\varepsilon}$ ,  $\tilde{\mu}$  are tensors and can also be nonlinear (for instance,  $\tilde{\sigma} = \tilde{\sigma}(\mathbf{E})$ ). In this case the response of the material can be in a different direction from the exciting field and dependent of field strength, frequency,....

Moreover, an additional feature of the properties is that they can depend on **the history of the incident field (hysteresis)**. To be fully correct, we should therefore write the constitutive equations in the form:

$$\mathbf{J}(r, t) = \int_0^{\infty} \tilde{\sigma}(\beta) \mathbf{E}(t - \beta) d\beta$$

$$\mathbf{D}(r, t) = \int_0^{\infty} \tilde{\varepsilon}(\beta) \mathbf{E}(t - \beta) d\beta$$

$$\mathbf{B}(r, t) = \int_0^{\infty} \tilde{\mu}(\beta) \mathbf{H}(t - \beta) d\beta$$

that must be used when the considered physical parameters are DISPERSIVE (i.e. frequency dependent) and, as already stated the 3 parameters are:

$$\tilde{\sigma} = \tilde{\sigma}(f); \quad \tilde{\varepsilon} = \tilde{\varepsilon}(f); \quad \tilde{\mu} = \tilde{\mu}(f)$$

By extension, the adjective "dispersive" is used for all the other dependencies (for instance when the parameters are dependent by the velocity, the phase,...

# Constitutive equations: electrical permittivity $\epsilon$

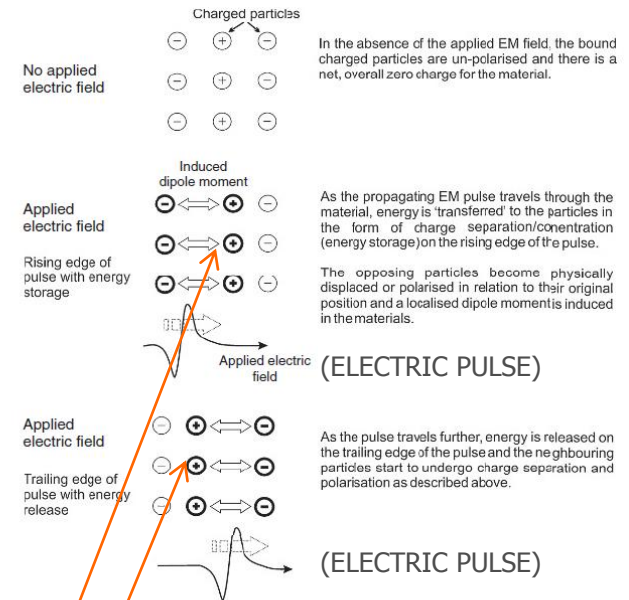
The permittivity of subsurface materials can vary noticeably (but electric conductivity more and more!), especially in the presence of free and bound water, and **dielectric permittivity is usually a complex, frequency-dependent quantity with real (storage) and imaginary (loss) components.**

**Table 2.1** Typical values of relative permittivity (real component) and static conductivity for common subsurface materials at an antenna frequency of 100 MHz

Material	Static conductivity, $\sigma_s$ (mS/m)	Relative permittivity, $\epsilon_{ave}$
Air	0	1
Clay – dry	1–100	2–20
Clay – wet	100–1000	15–40
Concrete – dry	1–10	4–10
Concrete – wet	10–100	10–20
Freshwater	0.1–10	78 (25 °C)–88
Freshwater ice	1–0.000001	3
Seawater	4000	81–88
Seawater ice	10–100	4–8
Permafrost	0.1–10	2–8
Granite – dry	0.001–0.00001	5–8
Granite – fractured and wet	1–10	5–15
Limestone – dry	0.001–0.0000001	4–8
Limestone – wet	10–100	6–15
Sandstone – dry	0.001–0.0000001	4–7
Sandstone – wet	0.01–0.001	5–15
Shale – saturated	10–100	6–9
Sand – dry	0.0001–1	3–6
Sand – wet	0.1–10	10–30
Sand – coastal, dry	0.01–1	5–10
Soil – sandy, dry	0.1–100	4–6
Soil – sandy, wet	10–100	15–30
Soil – loamy, dry	0.1–1	4–6
Soil – loamy, wet	10–100	10–20
Soil – clayey, dry	0.1–100	4–6
Soil – clayey, wet	100–1000	10–15
Soil – average	5	16

Jol, 2009

POLARIZATION occurs at both the local atomic scale and at the edges of the material where there are no neighbouring charges to balance the effect. On the rising edge of the incident pulse, energy is “transferred” to the particles in the form of **charge separation (energy storage)** and released in the trailing edge. A dipole moment is induced and a net dipole moment density **D** is generated across the polarised charges. In homogeneous, isotropic, non dispersive materials  $\rightarrow \mathbf{D} = \epsilon \mathbf{E}$



**Figure 2.1** Conceptual diagram illustrating the process of energy storage/release, of polarisation and the development of a dipole moment occurring when an electromagnetic (EM) wave propagates through a material.

Jol, 2009

shifting  $\rightarrow$  POLARIZATION

# Constitutive equations: electrical permittivity $\epsilon$

If the charges are free to move and can physically interact (as in the dipolar molecules of free water), then the displacement and polarisation process ( $f$ ) converts some of the EM energy into heat during the particle interactions. As such, a component of energy loss is introduced into the polarisation process that acts out of phase with the energy storage and release mechanism.

This phenomenon occurs in most materials, and therefore, the **permittivity** is usually described as a **complex quantity( $f$ )**, with the **real component** representing the **"instantaneous" energy storage – release mechanism** and **the imaginary component** representing the **energy dissipation**.

The frequency dependence of the polarisation process is a manifestation of the permittivity **relaxation phenomena** where the time-dependent displacement mechanism is acting at different rates to the alternating, applied electric field.

Below the relaxation frequency, the particles are able to 'react quickly' to the applied field and stay in phase with its changes. **At, and above, the relaxation frequency**, they cannot keep up with the rapidly changing field and spend most of their time in motion, therefore, producing **significant loss of energy** as heat to the surrounding matrix.

Most materials display a range of permittivity **relaxation** → **overall decrease in the value of the real component of the permittivity and a peaking in the imaginary component as frequency increases**. The **peak value** of this distribution is called the **relaxation frequency**.

There are more than one relaxation frequency due to different relaxation mechanism involving electrons, atoms, dipole molecules,... giving a complex overall response different for each material. Mixtures of materials and/or inhomogeneous ones exhibit an even more complex behaviour.

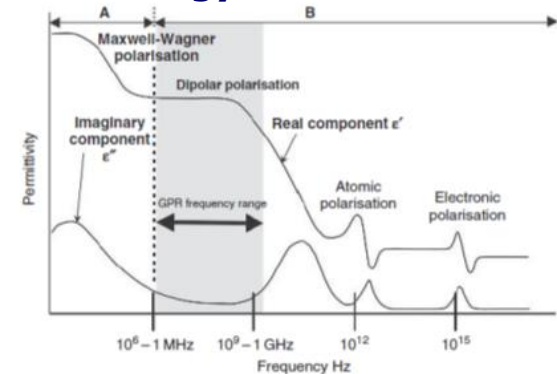


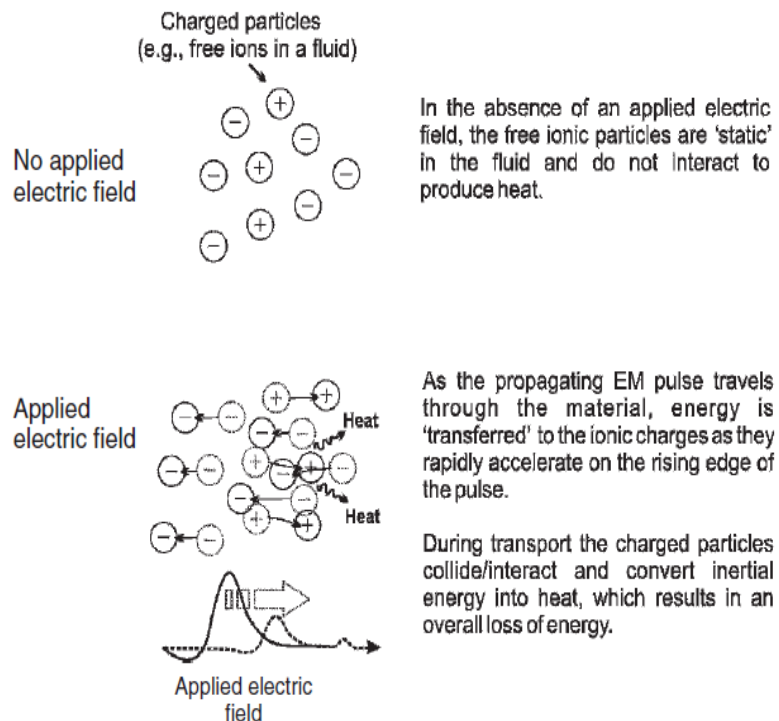
Figure 2.2 Permittivity relaxation phenomena associated with an idealised, lossy dielectric material. Region A represents the polarisations associated with free-charge and material interactions whilst region B represents molecular and atomic polarisations associated with bound charges. (Adapted from Daniels, 1996).



# Constitutive equations: electrical conductivity $\sigma$

Simplifying, **electrical conductivity**  $\sigma$  describes the ability of a material to pass free electric charges under the influence of an applied field. In metals, these charges relate to the free electrons of the metal atoms, whilst in fluids they are represented by the charges of dissolved anions and cations. These charge carriers rapidly accelerate to a terminal velocity generating internal **conduction currents**. As they propagate, they randomly collide against other atoms, ions or electrons, which produces energy loss in the form of heat.

Depending by the frequency of the EM field, the parameter can assume peculiar behavior.



At **low frequencies** ( $< \sim 1\text{kHz}$ ) the conductivity is basically a measure of the movement of charge carriers in response to an applied electric field, as described by the Ohm's law (i.e. the 1<sup>st</sup> Constitutive eq.  $\mathbf{J} = \sigma \mathbf{E}$ ) for "ohmic conductors":

$$\sigma_{eff} = \sigma' = \sigma_{DC}$$

**Figure 2.5** Conceptual diagram illustrating the process of conduction in a material containing free charges such as anions and cations.

# Constitutive equations: electrical conductivity $\sigma$

For **intermediate frequencies** ( $>0.1 \text{ MHz} < \sim 10 \text{ GHz}$ ) the charge response is almost “instantaneous” and the conduction current is in phase with the electric field  $\mathbf{E}$  so that:  
which is a real value (almost) frequency independent often referred as “STATIC CONDUCTIVITY”.

$$\sigma = \sigma_{DC}$$

At **very high frequencies** ( $>10 \text{ GHz}$ ) the inertial effect of the accelerating charges produces a lag in the physical response and a conduction current that is out of phase with the electric field variations. The conductivity must now be described by a complex, valued quantity where the imaginary component represents the out-of-phase component of the current. This typically increases with frequency and adds to the energy storage effect of the permittivity.

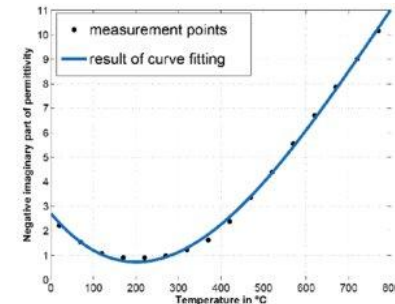
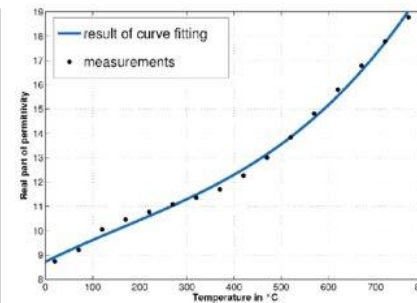
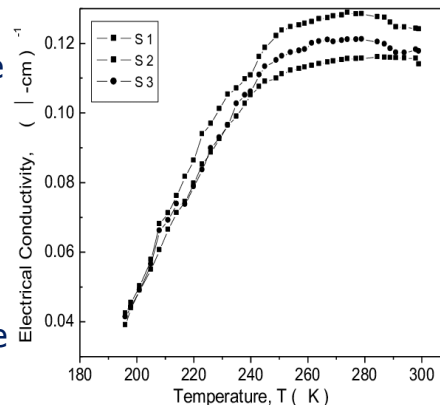
$$\sigma(\omega) = \sigma' - i\sigma''(\omega)$$

**Depending by which is the dominant behavior different geophysical methods have been exploited thus actually analyzing different and complementary parameters of the subsurface.**

Let remember that all the approximations are valid only when the hypotheses on which they are based are matched.

There are peculiar materials (e.g. ice, water, soils as mixtures of solids, liquid and gaseous components) for which the general approximations are no longer valid.

Moreover, there are cases in which some additional parameters which usually can be neglected must be considered (e.g. pressure, temperature,...) → For instance in geothermal studies!



# Constitutive equations: magnetic permeability $\mu$

$$\mu_r = \frac{\mu_m}{\mu_0} \rightarrow \mu_m = \mu_0(1 + \chi)$$

**Magnetic susceptibility**  $\chi$  [dimensionless] can be anisotropic, but for weakly to moderately magnetic materials, the anisotropy is generally negligible.

Water and most soil-forming minerals are nonferromagnetic (i.e. the real relative magnetic permeability  $\mu_r$  is  $\approx 1$ ).

When ferromagnetic impurities are present (especially iron, nickel, cobalt and their alloys and compounds like oxides) the magnetic permeability of the soil mass is proportional to the volume fraction of impurities.

**Total magnetization** of a material is a vector such as:  $\mathbf{M}_t = \mathbf{M}_r + \mathbf{M}_i$  [A/m]

The induced component  $\mathbf{M}_i$  (**induced magnetization**) is produced in response to an applied field, which in geophysical investigations is the Earth's Magnetic Field (EMF).

The remanent component  $\mathbf{M}_r$  (**remanent magnetization**) is the "permanent" magnetization that remains when applied fields are removed.

For weakly to moderately magnetic materials (such as most earth materials), induced magnetization is aligned with and proportional to  $\mathbf{H}$ , **the earth's magnetic field** (the earth's field is normally given in nT, units of the magnetic induction  $\mathbf{B}$ ) so:

$$\mathbf{M}_i = \chi \mathbf{H}$$

**In this course we will not consider magnetic properties of materials, which are the base of MAGNETIC METHOD here not described.**

# Electrical permittivity $\varepsilon$ and conductivity $\sigma$

$$\varepsilon_T(\omega) = \varepsilon'_T(\omega) - i\varepsilon''_T(\omega)$$

In the general case  $\varepsilon$  and  $\sigma$  should be expressed by:

$$\sigma_T(\omega) = \sigma'_T(\omega) - i\sigma''_T(\omega)$$

In which real parts are noted by ( ' ) while imaginary ones by ( " ). Pedex (T) refers to the "total complex"  $\varepsilon$  and  $\sigma$

**Real** and **imaginary** parts *are also referred to* as the **in-phase** and **quadrature** components, respectively.

The two total parameters are related so that:  $\varepsilon_T = \frac{\sigma_T}{i\omega}$

Therefore, both  $\varepsilon_T$  and  $\sigma_T$  contain THE SAME INFORMATION are simply two different ways of expressing what is measured.

In fact, from the physical point of view conductivity and polarization effects CANNOT be DIVIDED, so:

$$\varepsilon_{eff}(\omega) = \varepsilon'(\omega) + \frac{\sigma''(\omega)}{\omega}$$

$$\sigma_{eff}(\omega) = \sigma'(\omega) + \omega\varepsilon''(\omega)$$

and in terms of "Total effect":

$$\varepsilon_T(\omega) = \left[ \varepsilon'(\omega) + \frac{\sigma''(\omega)}{\omega} \right] - i \left[ \varepsilon''(\omega) + \frac{\sigma'(\omega)}{\omega} \right]$$

$$\sigma_T(\omega) = [\sigma'(\omega) + \omega\sigma\varepsilon''(\omega)] + i[\sigma''(\omega) + \omega\sigma\varepsilon'(\omega)]$$

This way is apparent that:

- 1) there is more than dielectric polarization contributing to what we measure as the effective permittivity or stored energy in the system;
- 2) there is more than pure (ohmic) conduction contributing to what we measure as electrical conductivity or energy loss in the system.

# Electrical permittivity $\varepsilon$ and conductivity $\sigma$

From the previously discussed concepts about permittivity and conductivity (dependence of frequency, complex nature, impossibility to “divide” the two components) we can consider different approximations taking into account WHICH mechanisms are more or less efficient in one specific medium and for a specific EM field.

One of the most important parameters is the FREQUENCY of oscillation of **E** and **H**.

At **frequencies above approximately 100 kHz** (Knigh and Endres, 2005) it is commonly assumed that:

$$\frac{\sigma''(\omega)}{\omega} = 0 \quad \sigma'(\omega) = \sigma_{DC}$$

The expressions for the effective parameters then become:

$$\varepsilon_{eff}(\omega) = \varepsilon'(\omega)$$

$$\sigma_{eff}(\omega) = \sigma_{DC}(\omega) + \omega\varepsilon''(\omega)$$

At the opposite, when  $\sigma' \gg \omega\varepsilon''$  (i.e. for **low frequencies**):

$$\sigma_{eff} \cong \sigma'$$

Such approximations can be a source of confusion in comparing studies of low-frequency and high-frequency electrical properties: at low frequencies  $\varepsilon_{eff}$  is considered to be governed by the parameter that is completely neglected in defining  $\varepsilon_{eff}$  at high frequencies while at high frequencies  $\varepsilon_{eff}$  is considered to be equivalent to the parameter that is completely neglected in defining  $\varepsilon_{eff}$  at low frequencies.

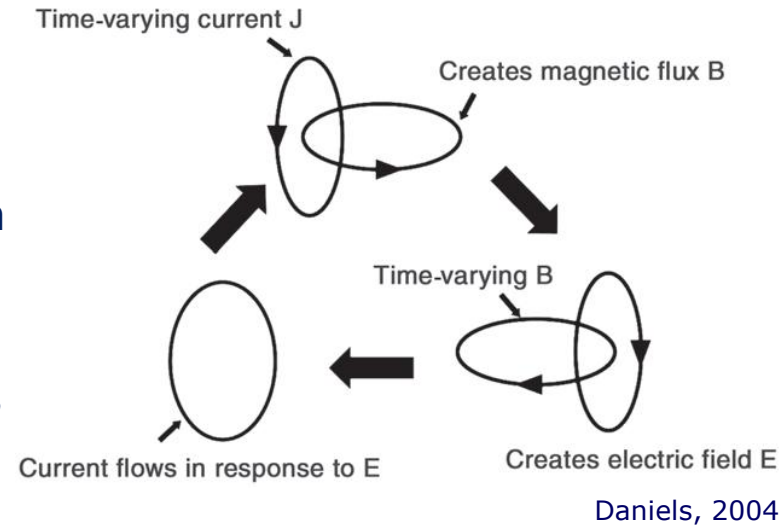
**The assumptions are valid for the frequency range of the studies** but result in two sets of expressions containing different sets of parameters. As we start to consider the value of broadband measurements of electrical properties to extract information about the near surface of the earth, the resulting **artificial division into “high frequency” and “low frequency”** can impede our ability to understand the underlying mechanisms governing the measured response.

# EM fields characteristics

Maxwell equations describe a coupled set of **E** and **B** fields varying with time as a corollary it is apparent that changing electric fields create magnetic fields which in turn induce electric fields.

For instance, *if we start considering a time-varying current  $J \rightarrow$  it in turn creates a magnetic field  $B$ ,  $\rightarrow$  which in turn induces an electric field  $E$ ,  $\rightarrow$  which in turn causes the electric charge to move thus producing a current  $J$ ,  $\rightarrow$  ...*

This continuing succession of one field inducing the other, results in fields that move through the medium: how?



Daniels, 2004

*It depends on the relative magnitude of losses, and the fields may **diffuse** or **propagate as waves***

Mathematically, the wave character is seen by rewriting Maxwell's equations to eliminate either the electric or magnetic field. Combining Faraday's law and Ampere's law and considering the constitutive equations we obtain the transverse vector wave equations:

$$\nabla \times \nabla \times \mathbf{E} + \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and

$$\nabla \times \nabla \times \mathbf{H} + \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

When material properties are assumed isotropic, frequency-independent (i.e. not dispersive), and linear..

In cartesian coordinates



# EM fields characteristics

Let now consider for simplicity Maxwell equations in the free space and without sources:

$$\nabla \cdot \mathbf{E} = 0 \quad \longrightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0; \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \longrightarrow \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0; \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \longrightarrow \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \quad (4)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \quad (3)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}; \quad (5)$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \longrightarrow \quad \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}, \quad (6)$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}, \quad (7)$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 \varepsilon_0 \frac{\partial E_z}{\partial t}; \quad (8)$$

Let now search for their solutions in which (for instance)  $\mathbf{E}$  is along  $y$  and  $\mathbf{B}$  along  $z$ :

$$\vec{E} = E_y \hat{y}, \quad (9)$$

$$\vec{B} = B_z \hat{z}. \quad (10)$$

Since  $\mathbf{E}_x, \mathbf{E}_z$  and  $\mathbf{B}_x, \mathbf{B}_y$  are **null**, (1) and (2) will become:

$$\frac{\partial E_y}{\partial y} = 0, \quad (11)$$

$$\frac{\partial B_z}{\partial z} = 0; \quad (12)$$

eq. (4) does not give any contribution, while (3) and (5) become, respectively:

$$\frac{\partial E_y}{\partial z} = 0, \quad (13)$$

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}; \quad (14)$$

Moreover, eq. (8) does not give any contribution, while (6) and (7) become, respectively:

$$\frac{\partial B_z}{\partial y} = 0, \quad (15)$$

$$-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}. \quad (16)$$

# EM fields characteristics

Let now analyze in detail the consequences of such equations:

From eq. (11), (13) and (14) we conclude that **E** does not change in the  $yz$  plane while it **varies along axis  $x$  when a magnetic field varying with time is present.**

From eq. (12), (15) and (16) we conclude that even **B** does not change in the plane  $xy$ , but it **varies along axis  $x$  when an electric field varying with time is present.**

Let now derive both members of eq. (14) for coordinate  $x$ :

$$(14) \quad \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t} \quad \rightarrow \quad \frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x \partial t}$$

And similarly both members of eq. (16) for  $t$

$$(16) \quad -\frac{\partial B_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} \quad \rightarrow \quad -\frac{\partial^2 B_z}{\partial t \partial x} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

By comparing such two eq., rewriting  $\partial^2 B_z / \partial x \partial t$  as  $\partial^2 B_z / \partial t \partial x$  we obtain:

$$\frac{\partial^2 E_y}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2} = 0 \quad (17)$$

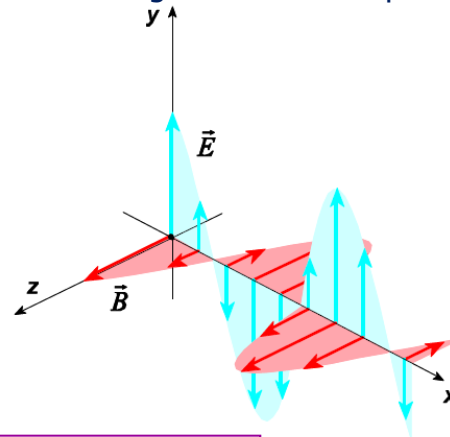
Doing the same for the magnetic field **B** we obtain:

$$\frac{\partial^2 B_z}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 B_z}{\partial t^2} = 0 \quad (18)$$

Both equations (17) and (18) satisfy the general wave eq. (in this case mono-dimensional):

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \psi(x, t)}{\partial t^2} = 0$$

So both **E** and **B** oscillate on planes perpendicular to axis  $x$  along which they propagate with a **PLANAR WAVE**. Waves are **POLARIZED** (in our example **E** along  $y$  and **B** along  $z$ ), but we can generalize the equations as:



$$\frac{\partial^2 \vec{E}}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \quad \frac{\partial^2 \vec{B}}{\partial x^2} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0,$$

Where:  $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$



***Questions?***