Esercizi su processi MA ed AR

funzioni di autocorrelazione

spettro

equazioni di Yule-Walker

Stima della funzione di autocorrelazione

Equazioni di Yule-Walker

$\{y(t)\}$ processo stocastico stazionario

$E[y(t)] = 0$

k	0	1	2	3	4	5	6	7
y	3,211	2,316	3,447	4,751	3,447	0,722	0,568	-0,712

(a) $\hat{\gamma}_y(0) = ?$ $\hat{\gamma}_y(1) = ?$ $\hat{\gamma}_y(2) = ?$

(b) $\boxed{MA(1)}$ oppure $\boxed{AR(1)}$?

(c) $\boxed{\hat{\sigma} = ?}$

$$\hat{\gamma}(k) = \frac{1}{N-|k|} \sum_{i=0}^{N-|k|-1} y(i)\,y(i+k) \qquad |k|<N$$

$$\sigma_y^2 = E[y^2] - (E[y])^2$$

$$\hat{\gamma}(0) = \frac{1}{8} \sum_{i=0}^{7} [y(i)]^2 \simeq \boxed{7{,}357}$$

$$\hat{\gamma}(1) = \frac{1}{7} \sum_{i=0}^{6} y(i)\,y(i+1) \simeq 6{,}746$$

$$\hat{\gamma}(2) = \frac{1}{6} \sum_{i=0}^{5} y(i)\, y(i+2) \simeq 6{,}218$$

$$\hat{\gamma}(0) \simeq 7.357$$

$$\hat{\gamma}(1) \simeq 6.746$$

$$\hat{\gamma}(2) \simeq \boxed{6.218}$$

$\hat{\gamma}(2) \neq 0$

NON può essere MA(1)

$\hat{\gamma}(2) \neq 0 \Rightarrow$ il processo è $AR(1)$

$$\mathcal{M} \quad y(t) = a\, y(t-1) + \eta(t)$$

$$\eta(\cdot) \sim WN(0, \lambda^2)$$

$\hat{a} = ?$

$\hat{\lambda}^2 = ?$

equazioni di Yule-Walker

$$\gamma(1) = a\,\gamma(0)$$

$$\gamma(2) = a\,\gamma(1)$$

$$\gamma(0) = a\,\gamma(1) + \lambda^2$$

$$\hat{\gamma}(\tau) \xrightarrow[N\to\infty]{} \gamma(\tau)$$

$$\hat{\gamma}(1) = \hat{a}\,\hat{\gamma}(0) \Longrightarrow \hat{a} = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \approx 0{,}92$$

$$\hat{\gamma}(2) = \hat{a}\,\hat{\gamma}(1) \Longrightarrow \hat{a} = \frac{\hat{\gamma}(2)}{\hat{\gamma}(0)} \approx 0{,}92$$

$$\hat{\gamma}(0) = \hat{a}\,\hat{\gamma}(1) + \hat{\sigma}^2 \Longrightarrow \hat{\sigma}^2 = \hat{\gamma}(0) - \hat{a}\,\hat{\gamma}(1)$$

$$\approx 1{,}15$$

$v(t)$ può star stazionario $E[v(t)] = 0$

$$\begin{cases} \gamma(0) = 5 \\ \gamma(\pm 1) = 2 \\ \gamma(\pm \tau) = 0 \quad |\tau| \geq 2 \end{cases}$$

$MA(1)$

$m = ?$

$WN(0, \lambda^2)$

$$v(t) = C(z)\eta(t)$$

$$C(z) = ?$$
$$\lambda^2 = ?$$

$$MA(1) \Longrightarrow C(z) = (1 + c z^{-1})$$

$$var(v) = \gamma(0) = (1^2 + c^2)\lambda^2 = \boxed{(1+c^2)\lambda^2 = 5}$$

$$\gamma(1) = c_0 c_1 \lambda^2 = \boxed{c\lambda^2 = 2}$$

$$\begin{cases} (1+c^2)\lambda^2 = 5 \\ c\lambda^2 = 2 \longrightarrow \lambda^2 = \frac{2}{c} \end{cases}$$

$$2c^2 - 5c + 2 = 0$$

$$c = \begin{cases} +2 \quad \text{NO} \\ +1/2 \end{cases}$$

$$C(z) = (1 + c z^{-1}) \Rightarrow W(z) = \frac{z+c}{z}$$

$|p_i| < 1$ ✓ $\qquad N(z)\ D(z)$ monici ✓

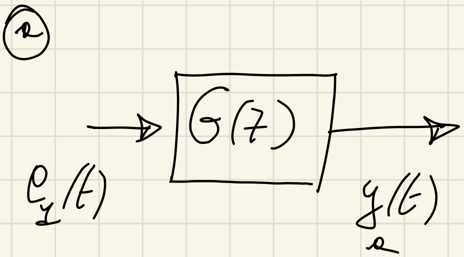
$|z_i| \leq 1$ $\qquad$ stesso grado ✓

$c = +\frac{1}{2}$ $\qquad W(z) = \frac{z + 1/2}{z}$ $\qquad$ fattore spettrale canonico

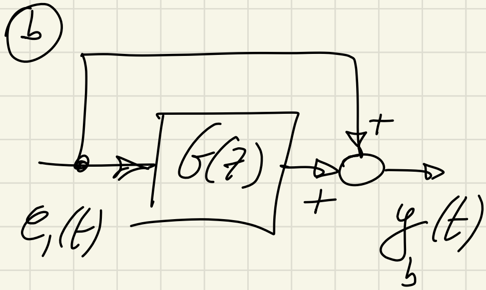
$\lambda^2 = 2/c = 4$

$\Gamma(\omega) = ?$ $\qquad \Gamma(\omega) = |W(e^{j\omega})|^2 \lambda^2$ ✓

(a)



(b)



$e_1(\cdot) \sim WN(0,1)$

$$G(z) = \frac{z}{z - 1/4}$$

$E[y(t)]$ (a) (b) ?

(a) (b) MA AR ARMA ?

$\Gamma_y(\omega)$ (a) (b)

$\boxed{a}$

$$y_a(t) = \frac{z}{z-\frac{1}{4}} e_1(t) = \frac{1}{1-\frac{1}{4}z^{-1}} e_1(t)$$

$$\left[1 - \frac{1}{4} z^{-1}\right] y_a(t) = e_1(t)$$

$$y_a(t) = \frac{1}{4} y_a(t-1) + e_1(t) \quad AR(1)$$

$$E[y_a(t)] = 0$$

perché $G(z)$ è AS. STABILE
e perché a regime stazionario
$AR(1) \simeq MA(\infty)$ che è proc. stat

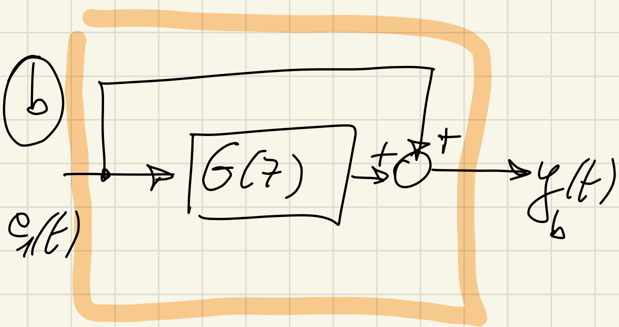
$$G(z) = \frac{z}{z-\frac{1}{4}}$$

$$\Gamma_{y_a}(\omega) = \left|G(e^{j\omega})\right|^2 \cdot \underbrace{\lambda^2}_{\to 1}$$

$$= \frac{\left|e^{j\omega}\right|^2}{\left|e^{j\omega}-\frac{1}{4}\right|^2} \cdot 1 = \frac{1}{\left|\cos\omega + j\operatorname{sen}\omega - \frac{1}{4}\right|^2}$$

$$= \frac{16}{17-8\cos\omega} \qquad \omega \in [-\pi \quad \pi]$$

$\omega = 0 \qquad \omega = \pi/2 \qquad \omega = \pi$



$$G(z) = \frac{z}{z-\frac{1}{4}}$$

$$e_1(\cdot) \sim WN(0,1)$$

$$W(z) = 1 + G(z) = 1 + \frac{z}{z-\frac{1}{4}} = \frac{2z - 1/4}{z-\frac{1}{4}}$$

$n_c = 1$

$n_a = 1$

ARMA(1,1)

$$W(z) = \frac{2z - \frac{1}{4}}{z - \frac{1}{4}}$$

$N(z)$ $D(z)$ monici

$W(z)$ non è fattore spettrale canonico

$$F(z) = 2\alpha \frac{z - 1/8}{z - \frac{1}{4}} \qquad \alpha = \frac{1}{2}$$

1

$$\frac{z - \frac{1}{8}}{z - \frac{1}{4}}$$

$$\phi(z) = W(z)\, W(z^{-1}) \cdot \lambda^2 = \alpha W(z) \cdot \alpha W(z^{-1})\, \hat{\lambda}^2$$

$$\hat{\lambda}^2 = \frac{\lambda^2}{\alpha^2}$$

$$\eta(t) \longrightarrow \boxed{F(z)} \longrightarrow y_b(t)$$

$\eta(t) \sim WN(0, 4)$

$$\Gamma_{y_b}(\omega) = \left| F(e^{j\omega}) \right|^2 \cdot \hat{\lambda}^2$$

$$= \frac{\left| e^{j\omega} - \frac{1}{8} \right|^2}{\left| e^{j\omega} - \frac{1}{4} \right|^2} \cdot 4$$

$$\Gamma_{y_b}(\omega) = 4 \cdot \frac{\left|\cos\omega - \frac{1}{8} + j\,\mathrm{sen}\,\omega\right|^2}{\left|\cos\omega - \frac{1}{4} + j\,\mathrm{sen}\,\omega\right|^2} =$$

$$= 4 \cdot \frac{\left[\cos^2\omega + \frac{1}{64} - \frac{1}{4}\cos\omega + \mathrm{sen}^2\omega\right]}{\left[\cos^2\omega + \frac{1}{16} - \frac{1}{2}\cos\omega + \mathrm{sen}^2\omega\right]}$$

$$\cos^2\omega + \mathrm{sen}^2\omega = 1$$

$$= 4 \cdot \frac{\frac{65}{64} - \frac{1}{4}\cos\omega}{\frac{17}{16} - \frac{1}{2}\cos\omega} = \cancel{4} \cdot \frac{65 - 16\cos\omega}{\cancel{64}_{\cancel{4}\,1}} \cdot \frac{\cancel{16}^{\,1}}{17 - 8\cos\omega}$$

$$\Gamma_{y_b}(\omega) = \frac{65 - 16\cos\omega}{17 - 8\cos\omega} \qquad \omega \in [-\pi; \pi]$$

E se avessi utilizzato la forma non canonica?

Si DEVE ottenere il medesimo spettro!

Verifica: $\Gamma(\omega) = |W(e^{j\omega})|^2 \cdot \lambda^2$

$\Updownarrow$

$\lambda^2 = 1$

$$W(z) = \frac{2z - 1/4}{z - 1/4}$$

$$\Gamma(\omega) = \frac{\left|2e^{j\omega} - \frac{1}{4}\right|^2}{\left|e^{j\omega} - \frac{1}{4}\right|^2} \cdot 1 =$$

$$= \frac{\left|2\cos\omega - \frac{1}{4} + 2j\,\text{sen}\,\omega\right|^2}{\left|\cos\omega - \frac{1}{4} + j\,\text{sen}\,\omega\right|^2}$$

$$= \frac{4\cos^2\omega + \frac{1}{16} - \cos\omega + 4\,\text{sen}^2\omega}{\cos^2\omega + \frac{1}{16} - \frac{1}{2}\cos\omega + \text{sen}^2\omega} =$$

$$= \frac{\frac{65}{16} - \cos\omega}{17/16 - \frac{1}{2}\cos\omega} = \frac{65 - 16\cos\omega}{\cancelto{1}{16}} \cdot \frac{\cancelto{1}{16}}{17 - 8\cos\omega}$$

$$\Gamma(\omega) = \frac{65 - 16\cos\omega}{17 - 8\cos\omega} \qquad \omega \in [-\pi; \pi]$$