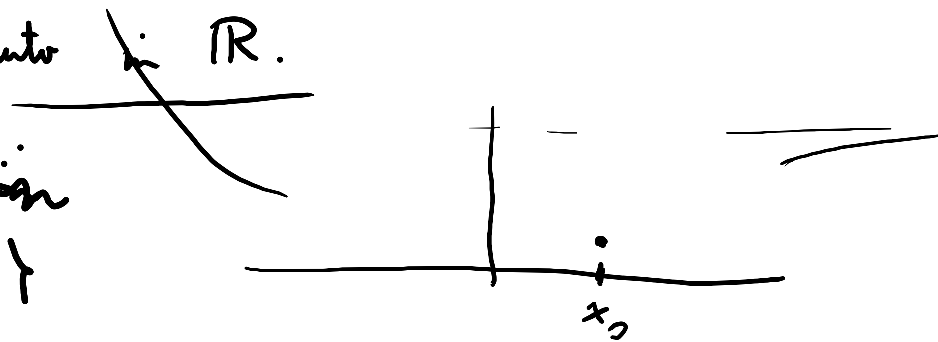
28 ottobre

Esercizio Se $f \in C^0(\mathbb{R})$ con $\lim_{x \to +\infty} f(x) = b$, $\lim_{x \to -\infty} f(x) = a$, $a, b \in \overline{\mathbb{R}}$ e per un $x_0 \in \mathbb{R}$ si ha

$f(x_0) < a$, $f(x_0) < b$. Allora f ha punti di minimo assoluto in $\mathbb{R}$.



Si considera $\{y_n\}$ in

$f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$

$\lim_{n \to \infty} y_n = \inf f(\mathbb{R})$

Si considera $\{y_n\}$ in $f(\mathbb{R})$ t.c. $\lim_{n\to+\infty} y_n = \inf f(\mathbb{R})$

Corrispondentemente sia $\{x_n\}$ una successione in $\mathbb{R}$ con $y_n = f(x_n)$. Resta definita $\{x_n\}$ e per un esercizio già discusso so che $\exists$ una $\{x_{n_k}\}$ e un $\bar{x} \in \overline{\mathbb{R}}$ t.c. $\lim_{k\to+\infty} x_{n_k} = \bar{x}$.

Verifichiamo ora che $\bar{x} \in \mathbb{R}$. Mi limito ad escludere il caso $\bar{x} = +\infty$. Se per assurdo $\bar{x} = +\infty$

seguirebbe che $\lim\limits_{k\to+\infty} f(\underbrace{x_{n_k}}_{y_{n_k}}) = \lim\limits_{x\to+\infty} f(x) = b > f(x_0)$

$\lim\limits_{n\to+\infty} y_n = \inf f(\mathbb{R}) \Rightarrow \lim\limits_{k\to+\infty} y_{n_k} = \inf f(\mathbb{R})$

Dall'unicità del $f(x_0) < b = \inf f(\mathbb{R}) \leq f(x_0)$

$f(x_0) < f(x_0)$ assurdo. Pertanto $\bar{x} \neq +\infty$

Analogamente $\bar{x} \neq -\infty$. Concludiamo che $\bar{x} \in \mathbb{R}$

$\lim\limits_{k\to+\infty} f(x_{n_k}) = \inf f(\mathbb{R})$. $\lim\limits_{k\to+\infty} x_{n_k} = \bar{x} \Rightarrow \lim\limits_{k\to+\infty} f(x_{n_k}) = f(\bar{x})$

<u>Teor</u> Sia f una funzione continua e strettamente monotona in un intervallo I aperto e sia $J = f(I)$ e sia $g: J \to I$ l'inversa di f.
Se in un punto $x_0 \in I$ esiste $f'(x_0) \neq 0$ e se $y_0 = f(x_0)$ allora ho

$$g'(y_0) = \frac{1}{f'(x_0)}$$

$g = \arcsin$

$f = \sin$

Dim

$$\frac{g(y)-g(y_0)}{y-y_0} = \frac{x - x_0}{f(x)-f(x_0)}$$

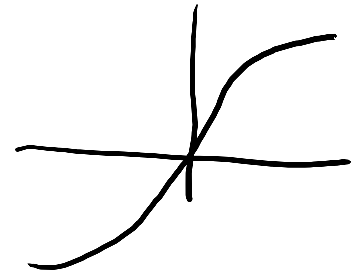
$y = f(x)$

$x = g(y)$

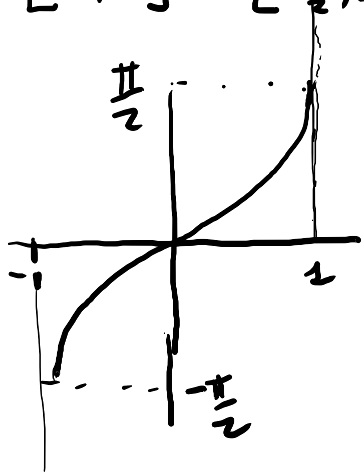
$$\lim_{y\to y_0} \frac{g(y)-g(y_0)}{y-y_0} = \lim_{x\to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$$

Per def. $g'(y_0) = \dfrac{1}{f'(x_0)}$

$\sin(x) : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$



$\arcsin x : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$\lim_{x \to 1^-} \frac{1}{\sqrt{1-x^2}} = +\infty$$

$$\lim_{x \to -1^+} \frac{1}{\sqrt{1-x^2}} = +\infty$$

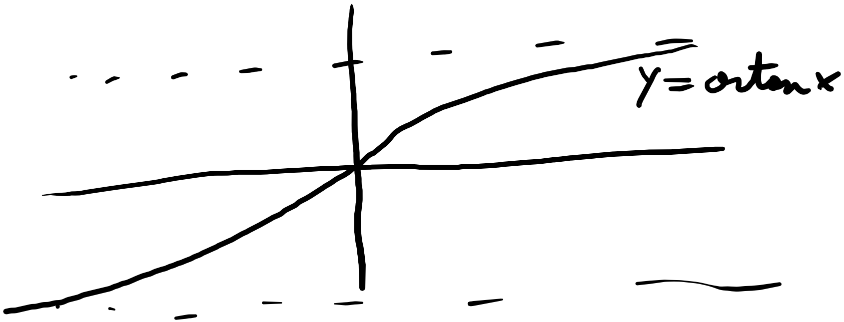
$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

$y = \sin x$

$$-1 < y < 1 \iff -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$\tan x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

$\arctan : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$$(\arctan x)' = \frac{1}{1+x^2}$$

$$g'(y) = \frac{1}{f'(x)}$$

$g(y) = \arctan y$

$f(x) = \tan x$

$$(\arctan y)' = \frac{1}{(\tan x)'} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

$$y = \tan x$$

Funzioni iperboliche

$\cosh(x)$, $\sinh(x)$ $\qquad$ $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$ch(x)$ $\qquad$ $sh(x)$ $\qquad$ th

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

$$ch(x) = \frac{e^x + e^{-x}}{2} \qquad sh(x) = \frac{e^x - e^{-x}}{2}$$

$$ch^2(x) - sh^2(x) = 1$$

$$\left(\frac{e^x+e^{-x}}{2}\right)^2 - \left(\frac{e^x-e^{-x}}{2}\right)^2 = \frac{\cancel{e^{2x}} + 2\cancel{e^x e^{-x}} + \cancel{e^{-2x}}}{4} - \frac{\cancel{e^{2x}} - 2\cancel{e^x e^{-x}} + \cancel{e^{-2x}}}{4}$$

$$= \frac{2}{4} - \frac{-2}{4} = \frac{2}{4} + \frac{2}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

$$(ch\, x)' = sh\, x \qquad (sh\, x)' = ch\, x$$

$$(\cos x)' = -\sin \qquad (\sin x)' = \cos x$$

$$(\mathrm{ch}\,x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{1}{2}\left(e^x + e^{-x}\right)' =$$

$$= \frac{1}{2}\left(\left(e^x\right)' + \left(e^{-x}\right)'\right) = \frac{1}{2}\left(e^x + e^{-x}\underbrace{(-x)'}_{-1}\right) =$$

$$= \frac{1}{2}\left(e^x - e^{-x}\right) = \mathrm{sh}\,x$$

$$\left(e^{f(x)}\right)' = e^{f(x)} \quad f'(x)$$

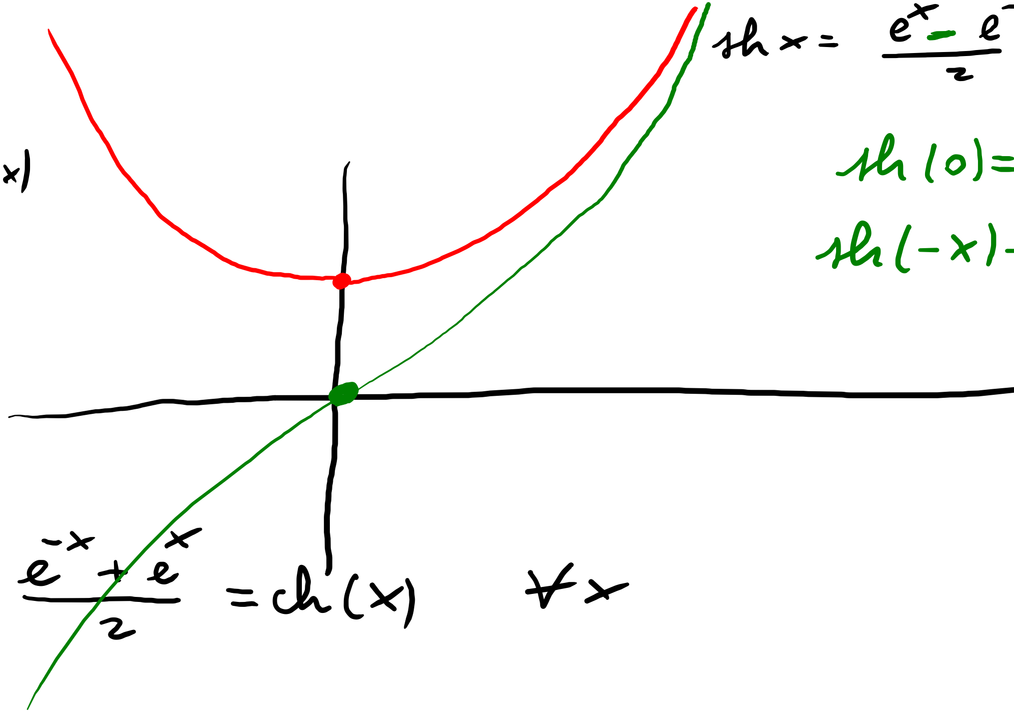
$ch(x) > sh(x) \quad \forall x$

$ch\, x = \dfrac{e^{x} + e^{-x}}{2}$

$sh\, x = \dfrac{e^{x} - e^{-x}}{2}$

$ch(0) = 1$

$ch(-x) = ch(x)$



$sh(0) = 0$

$sh(-x) = -sh(x)$

$ch(-x) = \dfrac{e^{-x} + e^{x}}{2} = ch(x) \quad \forall x$

$sh : \mathbb{R} \to \mathbb{R}$ è biettiva. Calcoliamo l'inversa

la inversa $g(x) = lg\left(x + \sqrt{x^2+1}\right)$

$g(y)$

$$y = sh(x) = \frac{e^x - e^{-x}}{2} \qquad y = \frac{e^x - e^{-x}}{2}$$

$$2y - e^x + e^{-x} = 0 \qquad \cdot e^x$$

$2ye^x - e^{2x} + 1 = 0$ è un polinomio quadratico in e^x

$$(e^x)^2 - 2y\, e^x - 1 = 0 \qquad (e^x)_\pm = y \pm \sqrt{y^2+1}$$

$$(e^x)_\pm = y \pm \sqrt{1+y^2}$$

e^x — $(e^x)_+$

$? \quad \cancel{(e^x)_-}$

$$(e^x)_- = y - \sqrt{1+y^2} < 0 \qquad \text{se } y \neq 0$$

$\Uparrow$

$$e^x = y + \sqrt{1+y^2}$$

$$y - \sqrt{1+y^2} < 0 \iff y < \sqrt{1+y^2} \qquad x = \lg\left(y + \sqrt{1+y^2}\right)$$

se $y \geq 0$ $\qquad y < \sqrt{1+y^2} \iff \cancel{y^2} < 1+y^2$

$$\left(e^{\arctan(\operatorname{sh} x)}\right)' = e^{\arctan(\operatorname{sh} x)} \left(\arctan(\operatorname{sh} x)\right)' =$$

$$= e^{\arctan(\operatorname{sh} x)} \arctan'(\operatorname{sh} x) \operatorname{sh}'(x)$$

$$= e^{\arctan(\operatorname{sh} x)} \frac{1}{1+\operatorname{sh}^2(x)} \operatorname{ch} x =$$

$$= e^{\arctan(\operatorname{sh} x)} \frac{1}{\operatorname{ch}^{\cancel{2}\,1} x} \cancel{\operatorname{ch} x} = \frac{e^{\arctan(\operatorname{sh} x)}}{\operatorname{ch} x}.$$

Def Sia $X \subseteq \mathbb{R}$ ed $f: X \to \mathbb{R}$. Un punto $x_0 \in X$ si dice un punto di massimo (minimo) locale o relativo se $\exists\ \delta > 0$ t.c. $|x - x_0| < \delta$ e $x \in X \Rightarrow f(x) \leq f(x_0)$

$f(x) \geq f(x_0)$