$$\lim_{x\to 0} \frac{\log(e+tg^2x) - e^{x^2}}{x^2}$$

$$\frac{\log(e+tg^2x) - 1}{x^2} + \frac{1-e^{x^2}}{x^2}$$

$$-\frac{e^{x^2}-1}{x^2} \to -1$$

$$\lim_{y\to 0} \frac{\log(y+1)}{y}$$

$$\lim_{y\to 0} \frac{e^y-1}{y}$$

$$\frac{\log(e+tg^2x) - \log(e)}{x^2} = \frac{\log\left(\frac{e+tg^2x}{e}\right)}{x^2} = \frac{\log\left(1+\frac{tg^2x}{e}\right)}{\frac{tg^2x}{e}} \cdot \frac{tg^2x}{x^2} \cdot \frac{1}{e}$$

$$1 \qquad 1 \qquad \frac{1}{e}$$

$$= \frac{1}{e} - 1$$

$$\lim_{x \to 0} f(x) \qquad |f(x)| \leq \text{sen}^2 x \qquad \forall x \in]-1,1[$$

$$-\text{sen}^2 x \leq f(x) \leq \text{sen}^2 x \qquad \lim_{x \to 0} \text{sen}^2 x = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$$\lim_{y \to 0} \frac{1-\cos y}{y^2}$$

2 Carabinieri $\Rightarrow$ $f(x) \to 0$

$$\lim_{x \to +\infty} \exp\Big[\overbrace{\underbrace{x \log(\cos(x^4 e^{-x}))}_{1}}^{\to 0}\Big] = 1$$

$$\lim_{y \to 0} \frac{\log_a(1+y)}{y}$$

$1+y = \cos(x^4 e^{-x}) \qquad y = \cos(x^4 e^{-x}) - 1$

$$\lim_{x \to +\infty} \frac{\log(\cos(x^4 e^{-x}))}{\frac{1}{x}} =$$

li

$$= \lim_{x \to +\infty} \frac{\log(1 + \overbrace{[\cos(x^4 e^{-x}) - 1]}^{y})}{\cos(x^4 e^{-x}) - 1} \cdot \frac{\cos(x^4 e^{-x}) - 1}{(x^4 e^{-x})^2} \cdot x \cdot (x^4 e^{-x})^2$$

$$\searrow 1 \qquad \searrow -\frac{1}{2} \qquad x^9 \cdot e^{-2x} \to 0$$

Teorema (linearità della derivazione)

$f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ derivabili in x_0, $\alpha, \beta \in \mathbb{R}$ $\Rightarrow$ $\alpha f + \beta g$ è derivabile in x_0 e si ha

$$\boxed{(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0)}$$

$$\lim_{x \to x_0} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0)}{x - x_0} = \alpha \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \beta \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

sia $f : I \to \mathbb{R}$ derivabile su I; $f' : I \to \mathbb{R}$ $\quad f' = Df$

Possiamo considerare una funzione $D : \{\underbrace{\text{funzioni derivabili su } I}_{f : I \to \mathbb{R}}\} \to \{\text{funzioni su } I\}$ $\quad f : I \to \mathbb{R}$

$Df = f'$ $\quad$ D viene detto operatore

Definiamo $C(I) = C^0(I) = \{ f : I \to \mathbb{R}$ continue su $I \}$

$C^1(I) = \{ f : I \to \mathbb{R}$ tali che f è derivabile e $f' \in C(I) \}$

Esistono funzioni derivabili su I con derivate non continue.

$D : C^1(I) \to C(I)$

OSS: $C(I)$ e $C^1(I)$ sono spazi vettoriali.

D è un'applicazione lineare tra gli spazi $C^1(I)$ e $C(I)$;

$\forall f, g \in C^1(I)$, $\alpha, \beta \in \mathbb{R}$ $\boxed{D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)}$

$C^{n+1}(I) = \{ f : I \to \mathbb{R}$ tali che f è $n+1$ volte derivabile su I

e $f' \in C^{n}(I) \} = \{ f : I \to \mathbb{R} : f^{(n)} \in C^{1}(I) \}$

$D : C^{n+1}(I) \to C^{n}(I) \qquad Df = f'$

$D^{k} : C^{n}(I) \to C^{n-k}(I) \qquad D^{k} f = f^{(k)} \qquad k \leq n$

D^{k} è un applicazione lineare

Es: $D[\operatorname{sen}(x) - 3x^{2} + x - 1] = \cos x - 3 \cdot 2x + 1$

Teorema (derivata del prodotto)

⚠ È falso $\cancel{D[f \cdot g] = (Df) \cdot (Dg)}$

$$D[x^2] = 2x \qquad D[x] \cdot D[x] = 1$$

$f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ $\quad x_0 \in I$ $\quad$ f e g derivabili in x_0. Allora la funzione prodotto $f \cdot g$ è derivabile in x_0 e si ha

$$\boxed{(f \cdot g)'(x_0) = f'(x_0)\, g(x_0) + f(x_0) \cdot g'(x_0)}$$

$$(f\cdot g)'(x_0) = \lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\cdot g(x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0} \frac{f(x)\,g(x) - f(x_0)\,g(x) + f(x_0)\,g(x) - f(x_0)\,g(x_0)}{x-x_0} =$$

$$= \lim_{x\to x_0}\left[g(x)\cdot \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{f'(x_0)} + f(x_0)\,\underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{g'(x_0)}\right] = g(x_0)\,f'(x_0) + f(x_0)\cdot g'(x_0)$$

vero perché g è continua

Es:

$$D\left[x^2 \cdot \operatorname{sen} x\right] = 2x \cdot \operatorname{sen} x + x^2 \cos x$$

$f(x)$ $g(x)$

$f'(x) = 2x$ $g'(x) = \cos x$

$$D x^2 = 2x$$

$$D\left[x \cdot x\right] = x \cdot 1 + 1 \cdot x = 2x$$

$$D\left[x^3\right] = D\left[x^2 \cdot x\right] = 2x \cdot x + x^2 \cdot 1 = 3x^2$$

$$D\left[x^4\right] = 4x^3$$

$$D\left[x^n\right] = n x^{n-1}$$

$n = 1$ vero $D\left[x^1\right] = 1 = 1 \cdot x^0$

$n \rightsquigarrow n+1$ $D\left[x^{n+1}\right] = D\left[x^n \cdot x\right] =$

$$= \underbrace{n x^{n-1} \cdot x}_{\parallel \\ x^n} + x^n \cdot 1 = (n+1)\, x^n$$

$\alpha \in \mathbb{R}$

$x > 0$

$$D\, x^{\alpha} = \alpha\, x^{\alpha - 1}$$

$$\lim_{h \to 0} \frac{(x+h)^{\alpha} - x^{\alpha}}{h} = \lim_{h \to 0} x^{\alpha} \frac{\left(1 + \frac{h}{x}\right)^{\alpha} - 1}{\frac{h}{x}} \frac{1}{x}$$

$\to \alpha$

$$= \alpha\, x^{\alpha - 1}$$

Teorema di Heine - Cantor

$(a_n)_n$ $\quad$ $(b_n)_n$ $\quad$ $|a_n - b_n| < \frac{1}{n}$ $\quad$ $|f(a_n) - f(b_n)| \geq \varepsilon$

$(a_{n_k})_k \to \alpha$ $\quad$ $(b_{n_k})_k$ $\quad$ $(b_{n_{k_j}})_j \to \beta$ $\quad$ $a_{n_{k_1}}$ $\quad$ $b_{n_{k_{j_0}}}$

$n_k = 2k$

(n pari)

$(b_{n_i})_i \to \beta$

$n_i = 2i+1$

$|a_n - b_n| < \frac{1}{n} \to 0$

$a_{n_k} - b_{n_i}$

1		b_1
2	a_2	
3		b_3
4	a_4	
5		b_5
6	a_6	
7		b_7
8	a_8	
9		b_9
	$\downarrow$	$\downarrow$
	α	β

$\alpha = \beta$?

Derivata della funzione reciproca

$f: I \to \mathbb{R}$ $\quad f$ derivabile in $x_0 \in I$ $\quad f(x_0) \neq 0$ $\quad [\, f(x) \neq 0$ in un intorno di $x_0 \,]$

$\frac{1}{f} : \{x \in I : f(x) \neq 0\} \to \mathbb{R}$ $\quad$ Allora $\frac{1}{f}$ è derivabile in x_0 e si ha

$$\left(\frac{1}{f}\right)'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$$

$$= \lim_{x \to x_0} \frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x - x_0} = \lim_{x \to x_0} \underbrace{\frac{f(x_0) - f(x)}{x - x_0}}_{\to -f'(x_0)} \frac{1}{\underbrace{f(x)}_{\to f(x_0)} \cdot f(x_0)} = -\frac{f'(x_0)}{f(x_0)^2}$$

Derivata del quoziente

$$D\,\frac{f}{g}(x) = D\left(f\cdot\frac{1}{g}\right) = f'\cdot\frac{1}{g} + f\cdot\left(\frac{1}{g}\right)' = f'\cdot\frac{1}{g} + f\cdot\left(-\frac{g'}{g^2}\right)$$

$$= \frac{f'\cdot g - f\,g'}{g^2}$$

$$\left[\frac{f}{g}(x)\right]' = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$$

Esempi:

$f(x) = \dfrac{1}{x}$

$D\,\dfrac{1}{x} = -\dfrac{g'(x)}{g(x)^2} = -\dfrac{1}{x^2}$

$g(x) = x$

$g'(x) = 1$

$\underbrace{D\left[x^{-1}\right]}_{} = \underbrace{-1 \cdot x^{-2}}_{}$

$D\left[x^n\right] = n\,x^{n-1} \quad \forall n \in \mathbb{Z}$

$x^\alpha \qquad \alpha \in \mathbb{R}$

$x > 0$

$$D\,\frac{1}{\operatorname{sen} x} = -\frac{\cos x}{\operatorname{sen}^2 x}$$

$$D\,\frac{1}{\cos x} = -\frac{-\operatorname{sen} x}{\cos^2 x} = \frac{\operatorname{sen} x}{\cos^2 x}$$

$$D\,\operatorname{tg} x = D\,\frac{\operatorname{sen} x}{\cos x} = \frac{\cos x \cdot \cos x - \operatorname{sen} x \cdot (-\operatorname{sen} x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \operatorname{tg}^2 x + 1$$

$$D\,\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$\frac{\operatorname{sen}^2 x + \cos^2 x}{\cos^2 x}$$

$$\boxed{D\,\operatorname{tg} x = \operatorname{tg}^2 x + 1} \leftarrow \boxed{f' = 1 + f^2}$$

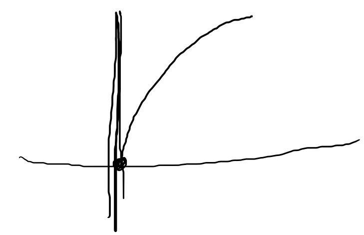
$$\underline{D\,e^x = e^x} \qquad f' = f$$

Derivata della funzione inversa

$f: I \to f(I)$ invertibile $\quad x_0 \in I \quad y_0 \in f(I) \quad y_0 = f(x_0)$

f derivabile in x_0 $f'(x_0) \neq 0$ $\Longrightarrow$ f^{-1} è derivabile in y_0 ?

FALSO

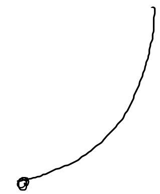


Teorema

$I \subseteq \mathbb{R}$ intervallo. $f: I \to f(I)$ continua strettamente monotona su I e derivabile in $x_0 \in I$. Sia $f'(x_0) \neq 0$. Allora $f^{-1}: f(I) \to I$ è derivabile in $y_0 = f(x_0)$ e si ha

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

$x_0 = f^{-1}(y_0)$



Se $f'(x_0) = 0$ e f è crescente, allora $\exists\, (f^{-1})'(y_0) = +\infty$

Se $f'(x_0) = 0$ e f è decrescente, allora $\exists\, (f^{-1})'(y_0) = -\infty$

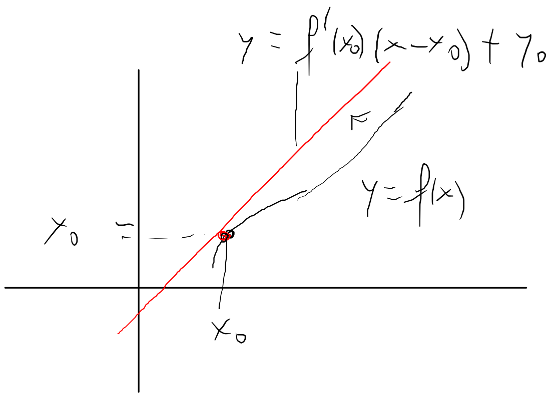
Se esiste $f'(x_0) = \pm\infty$, allora f^{-1} è derivabile in y_0, $(f^{-1})'(y_0) = 0$

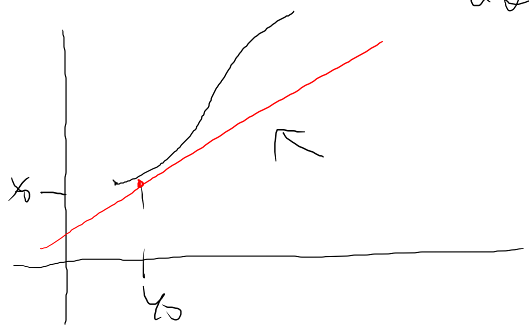
Dim

$$\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \lim_{x\to x_0}\frac{x-x_0}{f(x)-f(x_0)} = \begin{cases} \dfrac{1}{f'(x_0)} & \text{se } f'(x_0)\neq 0 \\ \pm\infty & \text{se } f'(x)=0 \quad f \text{ crescente o decrescente} \\ 0 & \text{se } f'(x_0)=\pm\infty \end{cases}$$

$y=f(x) \qquad x=f^{-1}(y)$

se $y\to y_0$ allora per la continuità di f^{-1} si ha $f^{-1}(y)\to f^{-1}(y_0)$
cioè $x\to x_0$





$$y = f'(x_0)(x-x_0)+y_0$$

$$y = m(x-x_0)+y_0$$

$$mx = y-y_0+mx_0$$

$$\boxed{\begin{aligned} x &= \frac{1}{m}(y-y_0)+x_0 \\ x &= (f^{-1})'(x_0)(y-y_0)+f^{-1}(y_0) \end{aligned}}$$

$$(f \circ f^{-1})(y) = y \qquad (f^{-1} \circ f)(x) = x$$

$$\underbrace{D\left[(f^{-1} \circ f)(x)\right]}_{\parallel} = 1$$

$$(f^{-1})'\big(\underbrace{f(x_0)}_{\parallel\, y_0}\big) \cdot f'(x_0)$$

$$\rightsquigarrow \quad (f^{-1})'(y_0) \cdot f'(x_0) = 1$$

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Esempi

$$D\sqrt{x} = \frac{1}{2\sqrt{x}} \qquad D\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x}$$

$\sqrt{y}$ è l'inverso di $f(x) = x^2\Big|_{[0,+\infty[}$ $\qquad f'(x) = 2x$

$\Downarrow$

$f^{-1}(y)$

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{2x} = \frac{1}{2\sqrt{y}} \quad //$$

$\uparrow$

$y \neq 0$ $\qquad y = x^2 \qquad x = \sqrt{y}$

$D \log_a y$

$\|$

$f^{-1}(y)$

$a^x = y$

$f(x) = a^x \qquad f'(x) = a^x \cdot \log a$

$$D f^{-1}(y) = \frac{1}{f'(x)} = \frac{1}{a^x \cdot \log a} = \frac{1}{y \log a}$$

$\uparrow$

$D \, \operatorname{arctg} y$

$f^{-1}(y)$

$f(x) = \operatorname{tg} x$ $\quad f'(x) = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$

$D \, \operatorname{arctg} y = \frac{1}{f'(x)} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x$

$= \frac{1}{1 + \operatorname{tg}^2 x} = \frac{1}{1 + y^2}$

$y = \operatorname{tg} x$