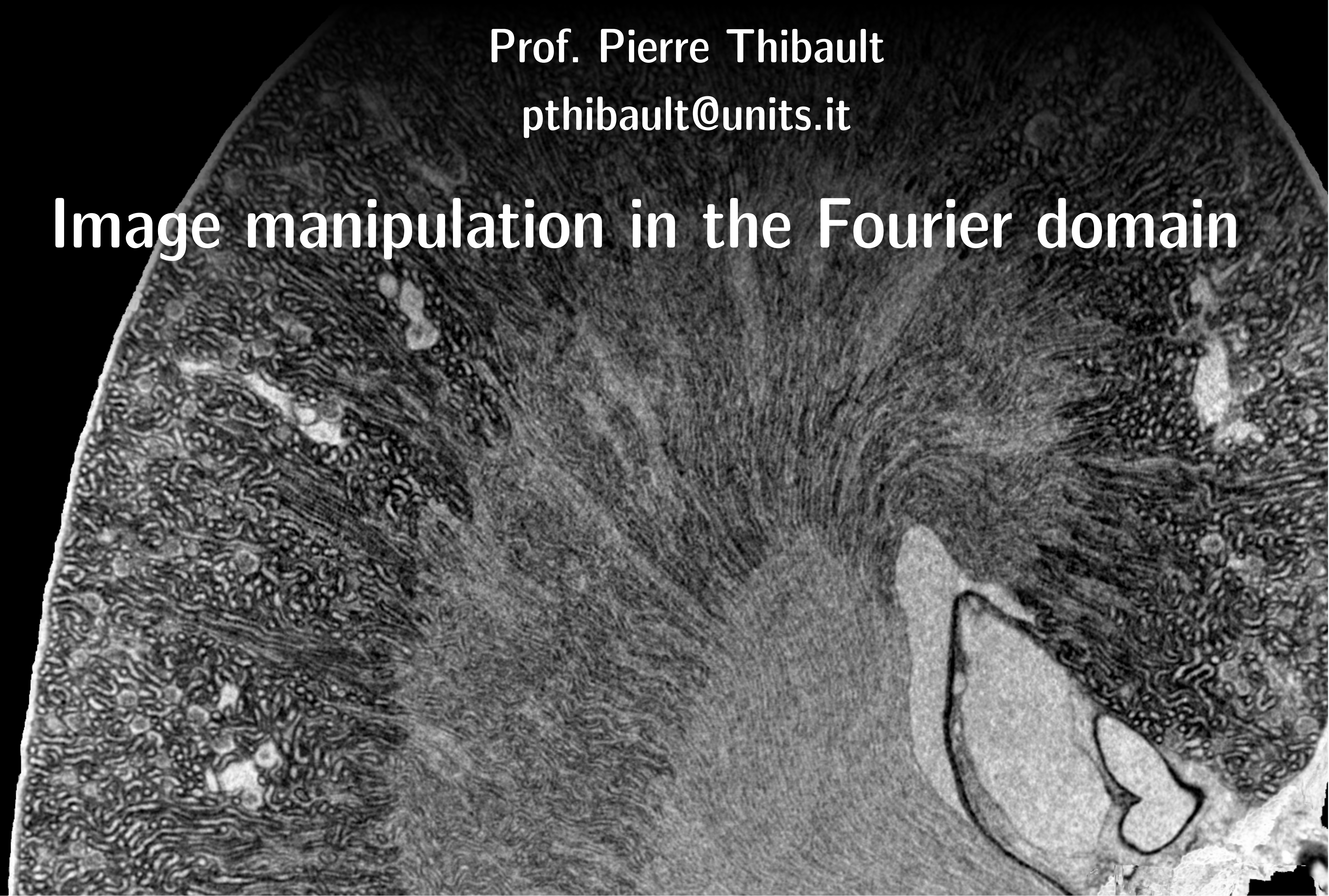
Image Processing for Physicists

Prof. Pierre Thibault

pthibault@units.it

Image manipulation in the Fourier domain



Overview

- The Fourier transform (FT)
  - introduction, properties
  - Fourier series, convolution, Dirac comb
  - Discrete Fourier transform (DFT), sampling, aliasing
- Linear filters
  - smoothing, sharpening, edge detection

Literature

- Rafael C. Gonzalez, “Digital Image Processing”, Prentice Hall International; (2008)
- E. Oran Brigham, “Fast Fourier Transform and Its Application”, Prentice Hall International; (1988)
- J.D. Gaskill, “Linear Systems, Fourier Transforms, and Optics”, John Wiley and Sons, (1978)

The Fourier transform

- First introduced by Joseph Fourier (1768-1830) to describe heat transfer
- today extremely important
- widely used in many fields
- fast computational implementation (FFT)
- original motivation: representation by easier-to-handle functions
- basis functions: oscillations (sine and cosine)
- describe signal by its frequency spectrum

What's a spatial frequency?

Analogy with time domain:

temporal frequency: $\dfrac{\#\text{cycles}}{\text{unit of time}}$

For images:

spatial frequency: $\dfrac{\#\text{cycles}}{\text{unit of length}}$

e.g.

printer resolution: 300 dpi "dots per inches"

What's a spatial frequency?

High spatial frequencies:

- “fast” changes in image content, small details, edges, ...

Low spatial frequencies:

- “slow” changes in image content, large areas, plane regions, ...

Single frequencies are not localized in an image!

Definitions

- Continuous Fourier transform

convention most common in imaging

$$\mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i u x}\, dx$$

$$f(x) = \mathcal{F}^{-1}\{F(u)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du \qquad \left[\text{physics } e^{-iq\cdot x} \quad u = \frac{q}{2\pi}\right]$$

- Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k x/p}$$

period of $f(x)$

$f(x)$: periodic function

$$c_k = \frac{1}{p}\int_{-p/2}^{p/2} f(x)\, e^{-2\pi i k x/p}\, dx$$

- Discrete Fourier transform

N: total number of samples

$$F_k = \sum_{n=0}^{N-1} f_n\, e^{-2\pi i k n/N}$$

f_n: sample points of a periodic function

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k\, e^{2\pi i n k/N}$$

Properties

- linearity

$$a\,f(x) + b\,g(x) \xrightarrow{\mathcal{F}} a\,F(u) + b\,G(u)$$

- scaling

$$f(a\cdot x) \xrightarrow{\mathcal{F}} \frac{1}{a} F\left(\frac{u}{a}\right)$$

reciprocal relationship

- shifting/modulation

$(i = \sqrt{-1})$

$$f(x - x_0) \xrightarrow{\mathcal{F}} F(u)\,\underbrace{e^{2\pi i u x_0}}_{\text{"phase ramp"}}$$

phase is linear in u

- Parseval's identity

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(u)|^2\,du$$

- 0-frequency term

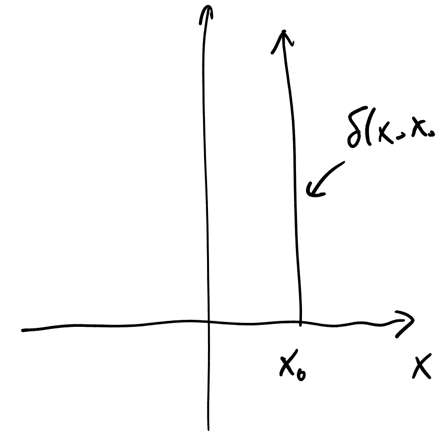
"direct current"

"DC term"

$$F(u=0) = \int_{-\infty}^{\infty} f(x)\,dx$$

constant term that doesn't oscillate

Dirac distribution



- "sifting" property

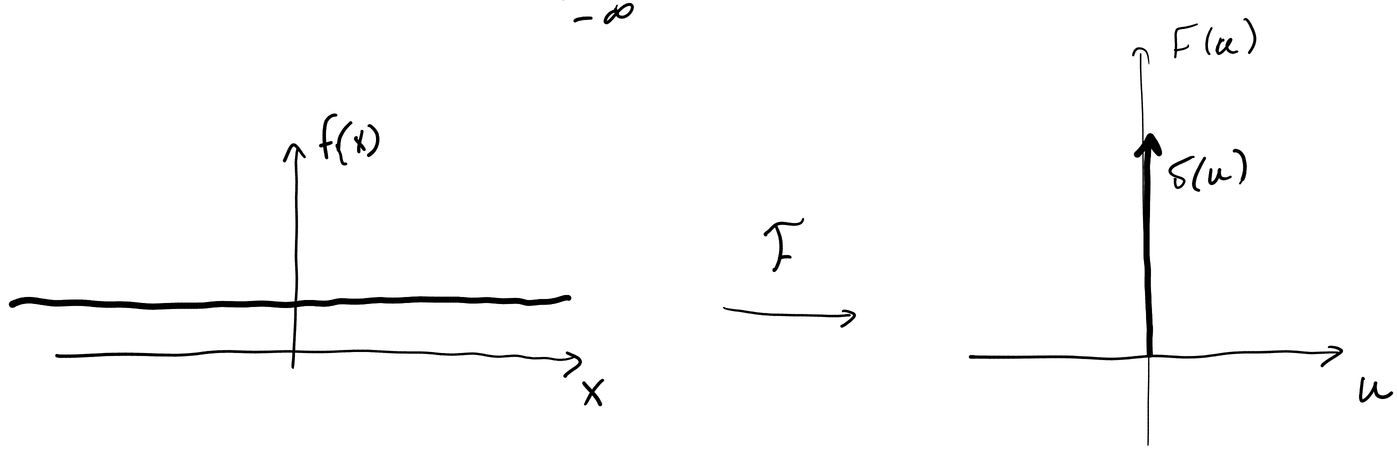
$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

- normalization

$$\int_{\infty}^{\infty} \delta(x)\,dx = 1$$

- relation to Fourier transforms

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-2\pi i u x}\,dx = \delta(u)$$



Convolution

$[f*g](x)$

- definition

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(s)\, g(x-s)\, ds$$

- commutativity, associativity, distributivity

$$f * g = g * f$$

$$(f*g)*h = f*(g*h)$$

$$f*(g+h) = f*g + f*h$$

- Dirac distribution: indentity/translation

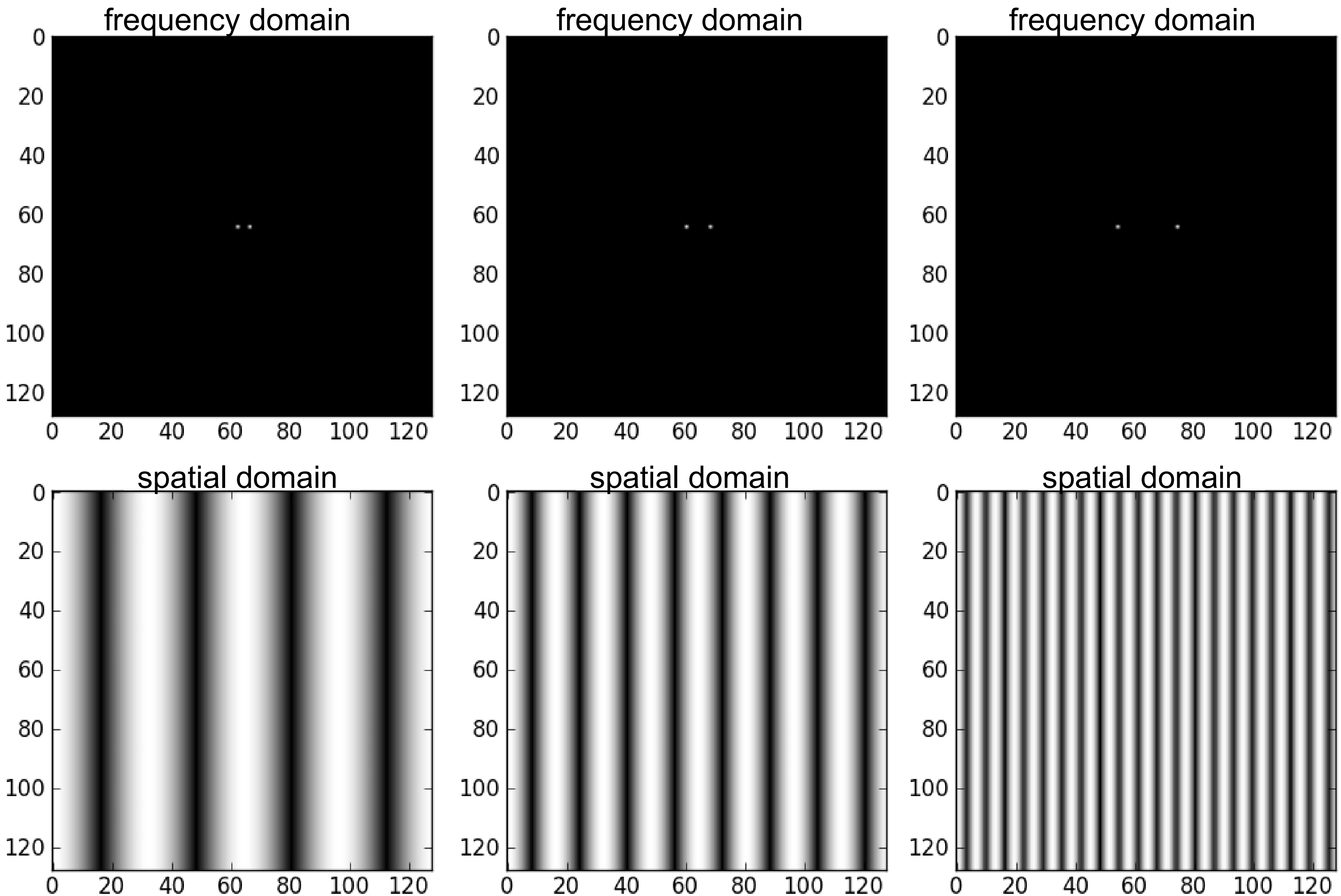
$$\left[f(x') * \delta(x'-x_0)\right](x) = f(x-x_0)$$

- relation to Fourier transforms

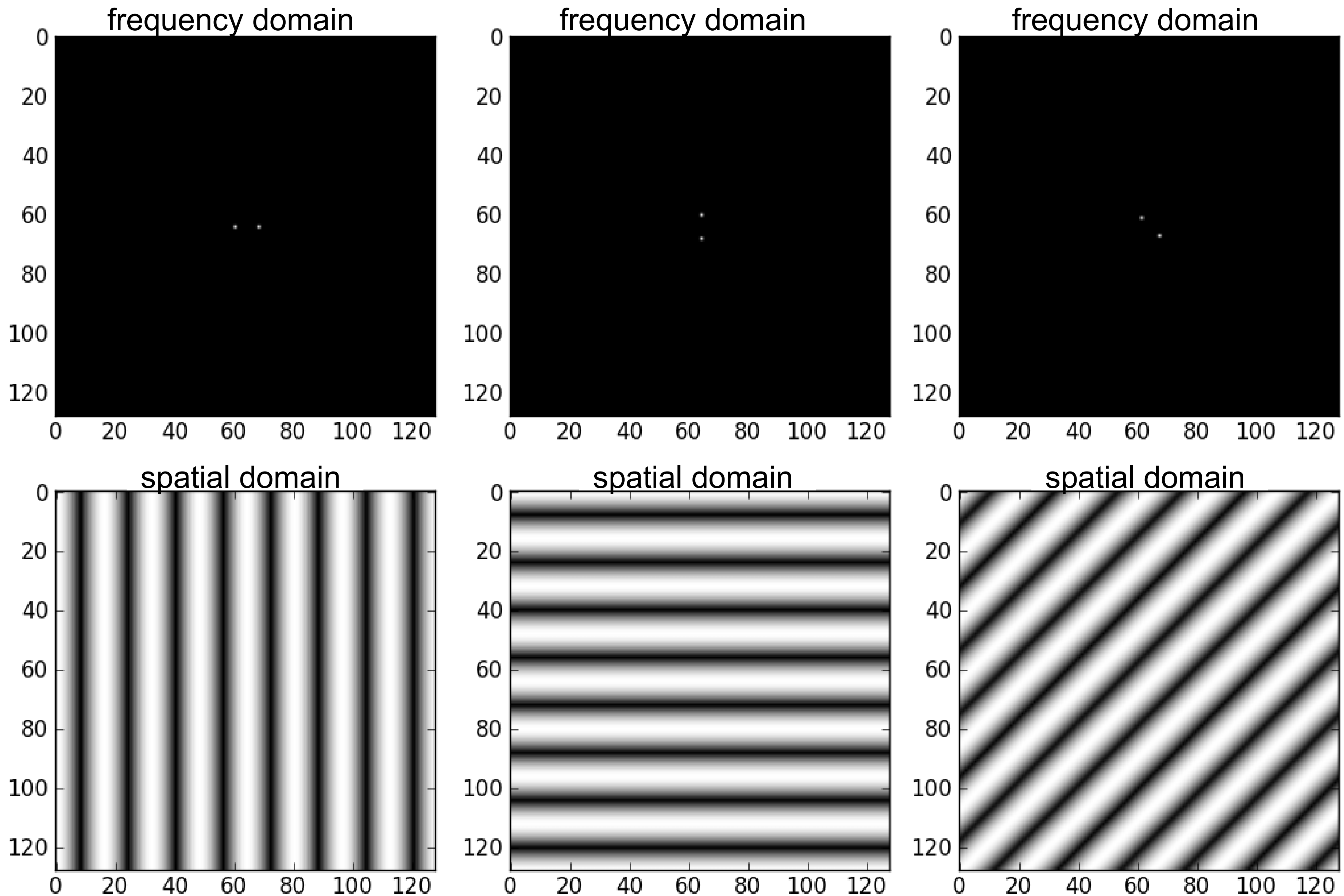
$$\mathcal{F}\{f*g\} = F(u) \cdot G(u)$$

important!

Basic functions and their spectra



Basic functions and their spectra



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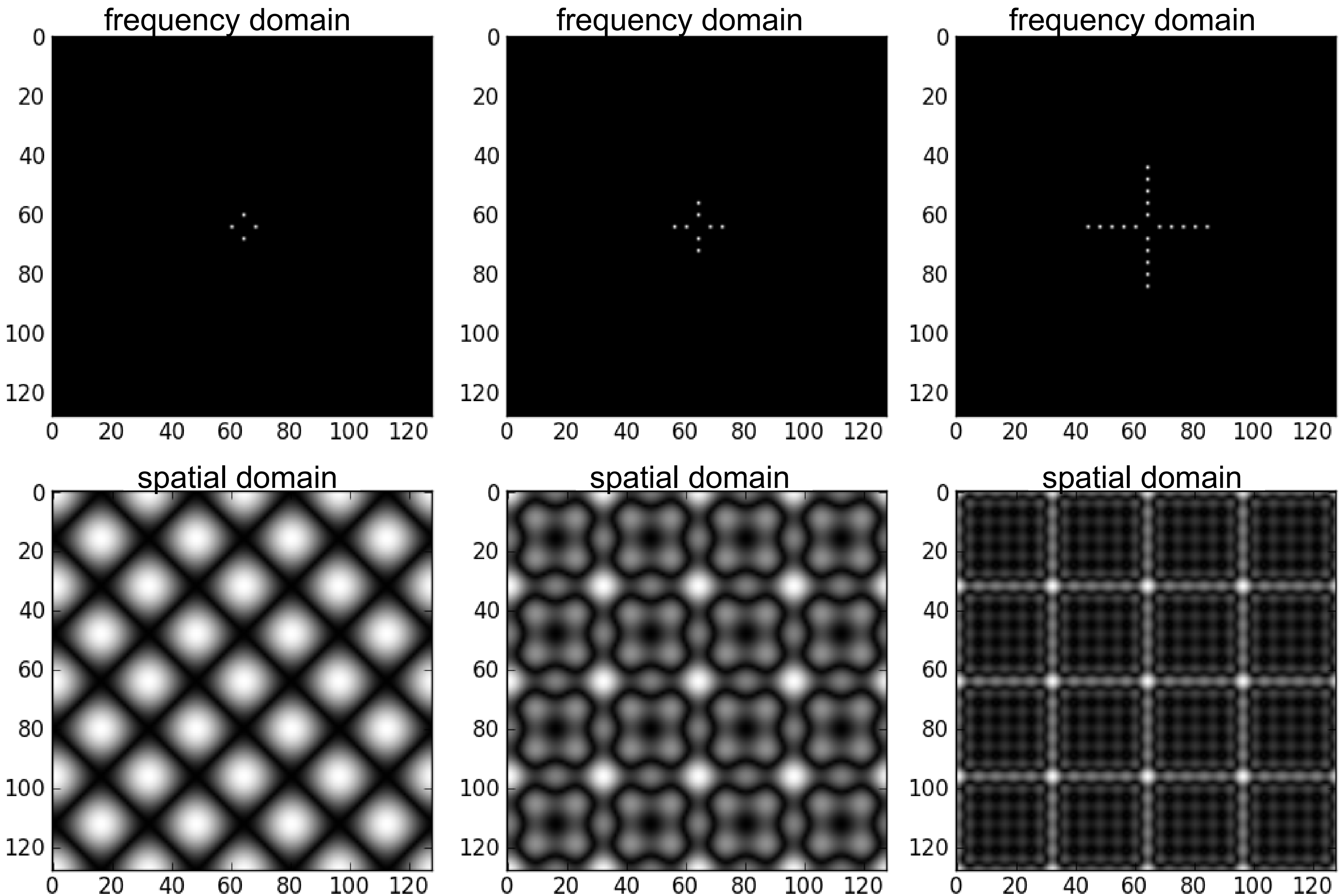

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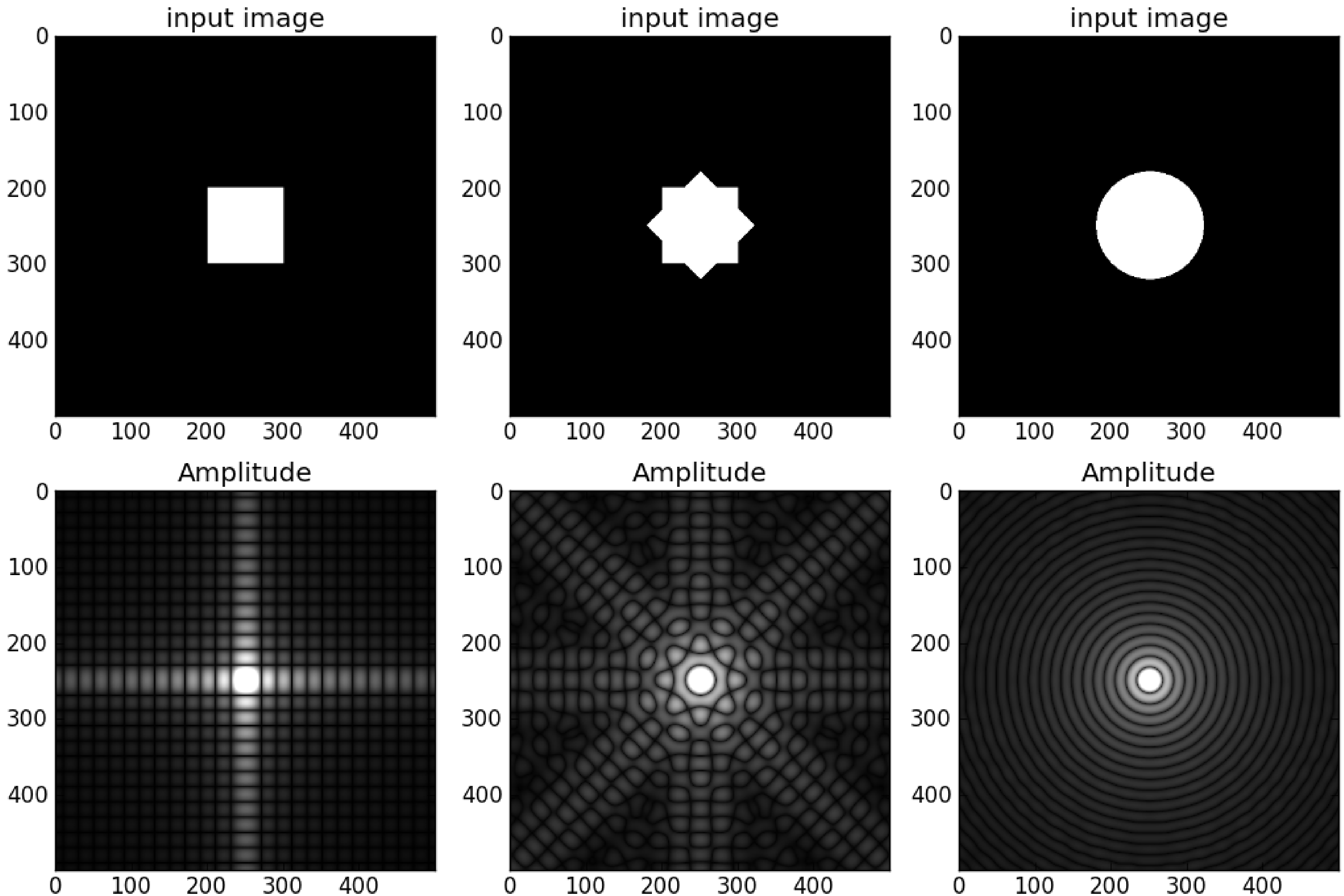

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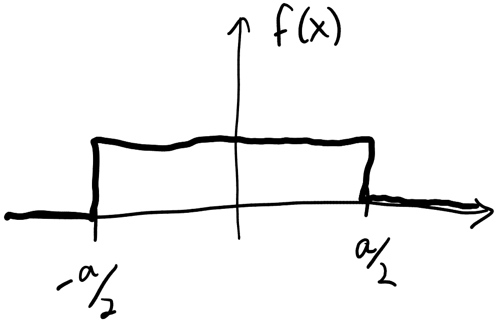
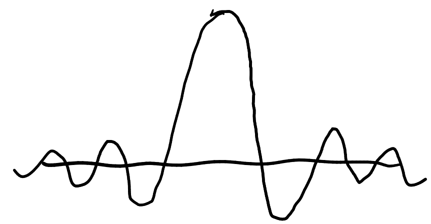
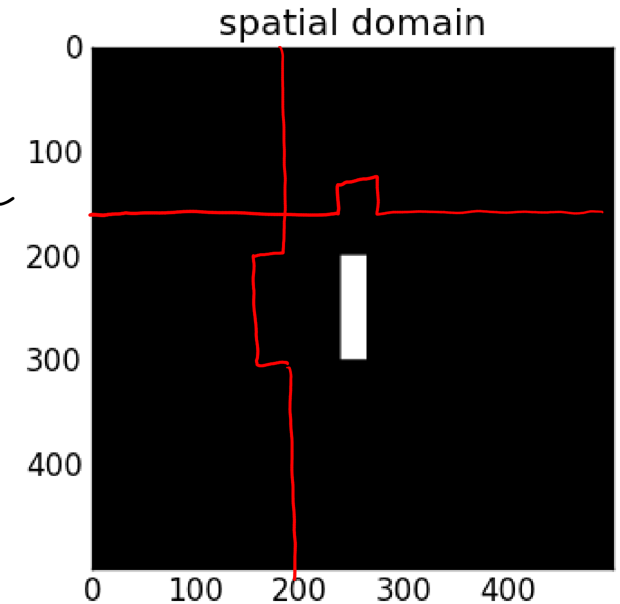
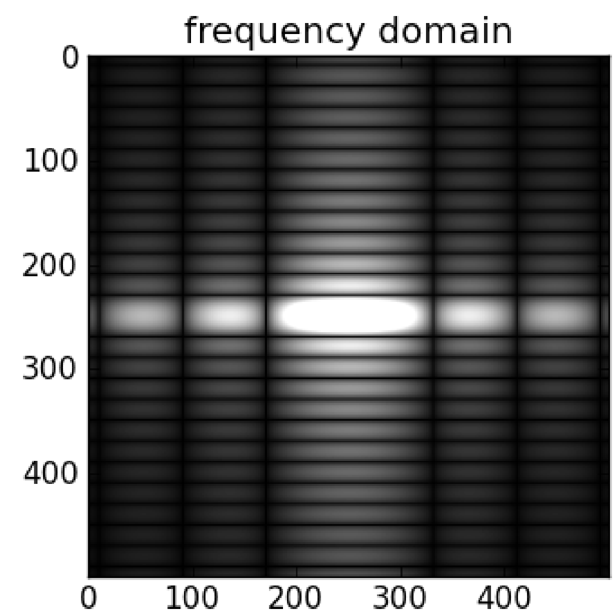








$$\mathcal{F}\{f(x)\} = \int_{-a/2}^{a/2} e^{-2\pi i u x}\, dx$$

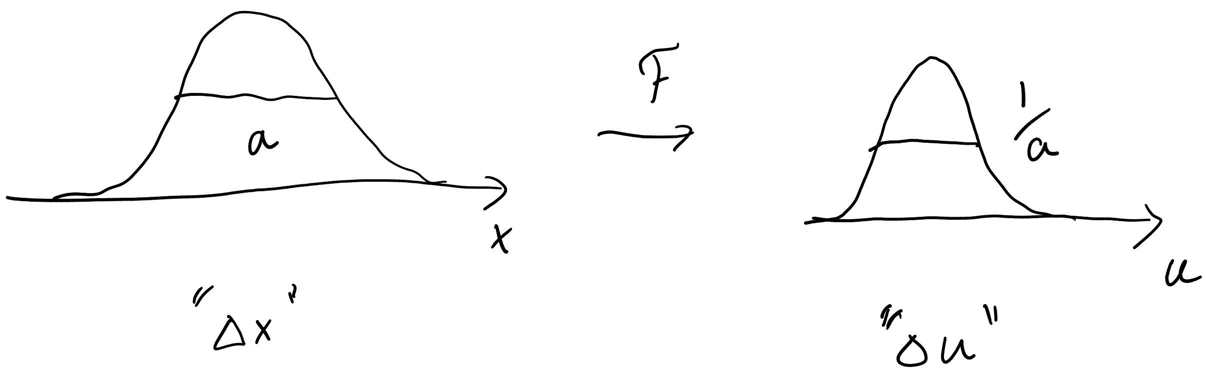
$$= \frac{1}{-2\pi i u} e^{-2\pi i u x}\Big|_{-a/2}^{a/2}$$

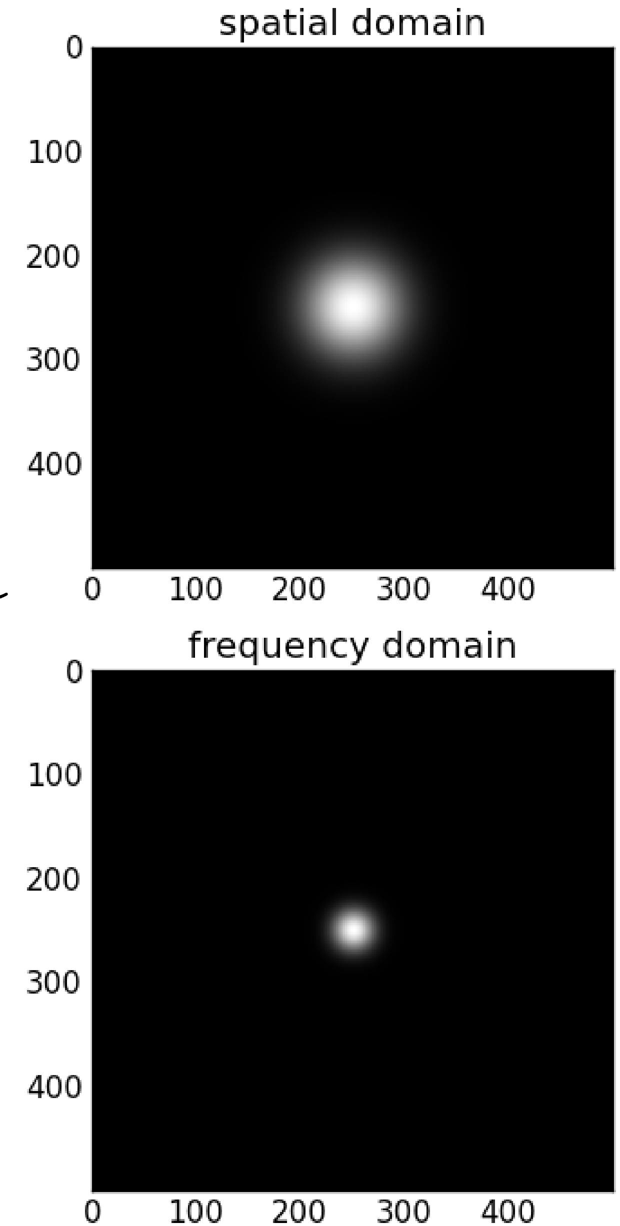
$$= \frac{1}{\pi u}\left(\frac{e^{\pi i u a} - e^{-\pi i u a}}{2i}\right)$$

$$= \frac{\sin(\pi u a)}{\pi u}$$ "sinc" $\frac{\sin x}{x}$

$$= a\,\mathrm{sinc}(ua)$$

Basic functions and their spectra





Additional properties

- uncertainty principle

$$\Delta x \, \Delta u \geqslant \frac{1}{4\pi}$$

- power spectrum

$$P(u) = |F(u)|^2$$

- derivatives

$$\mathcal{F}\left\{ \frac{\partial}{\partial x} f(x) \right\} = 2\pi i u \, F(u)$$

$$\frac{\partial^n}{\partial x^n} f \xrightarrow{\mathcal{F}} (2\pi i u)^n F(u)$$

- "Friedel" (crystallography terminology) symmetry:

if $f(x) \in \mathbb{R}$ then $F(u) = F^*(-u)$

Periodic signals

$f(x)$: Periodic function with period p

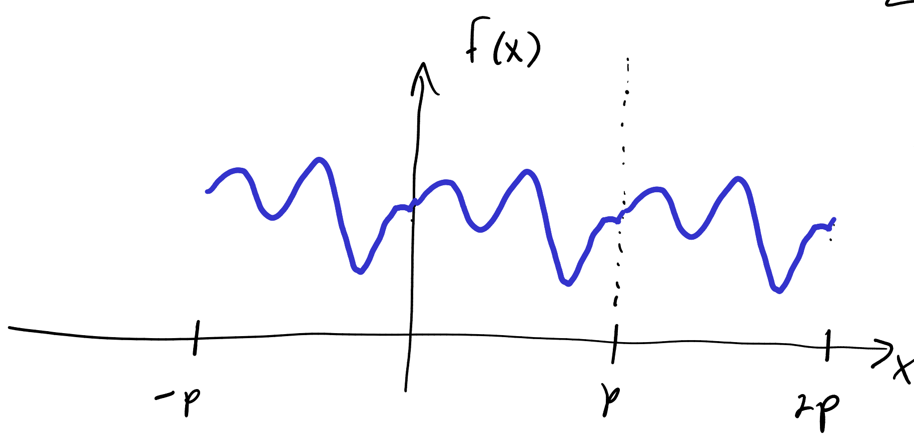
$$f(x) = \int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, dx$$ ("Fourier synthesis" or inverse Fourier transform)

but also:

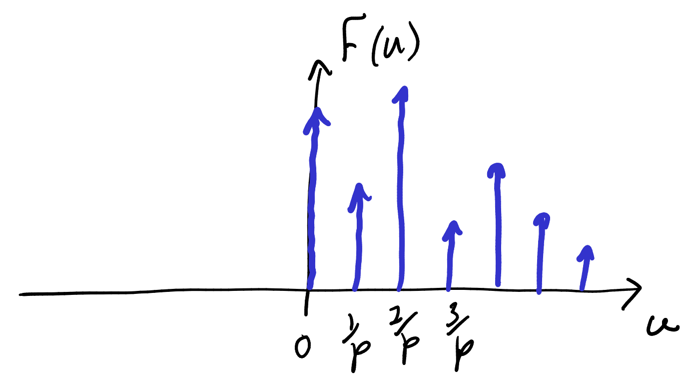
$$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k x/p}$$ (Fourier series)

$$F(u) = \sum_{k=-\infty}^{\infty} c_k\, \delta(u - k/p)$$

$\boxed{\text{periodic} \longrightarrow \text{discrete}}$

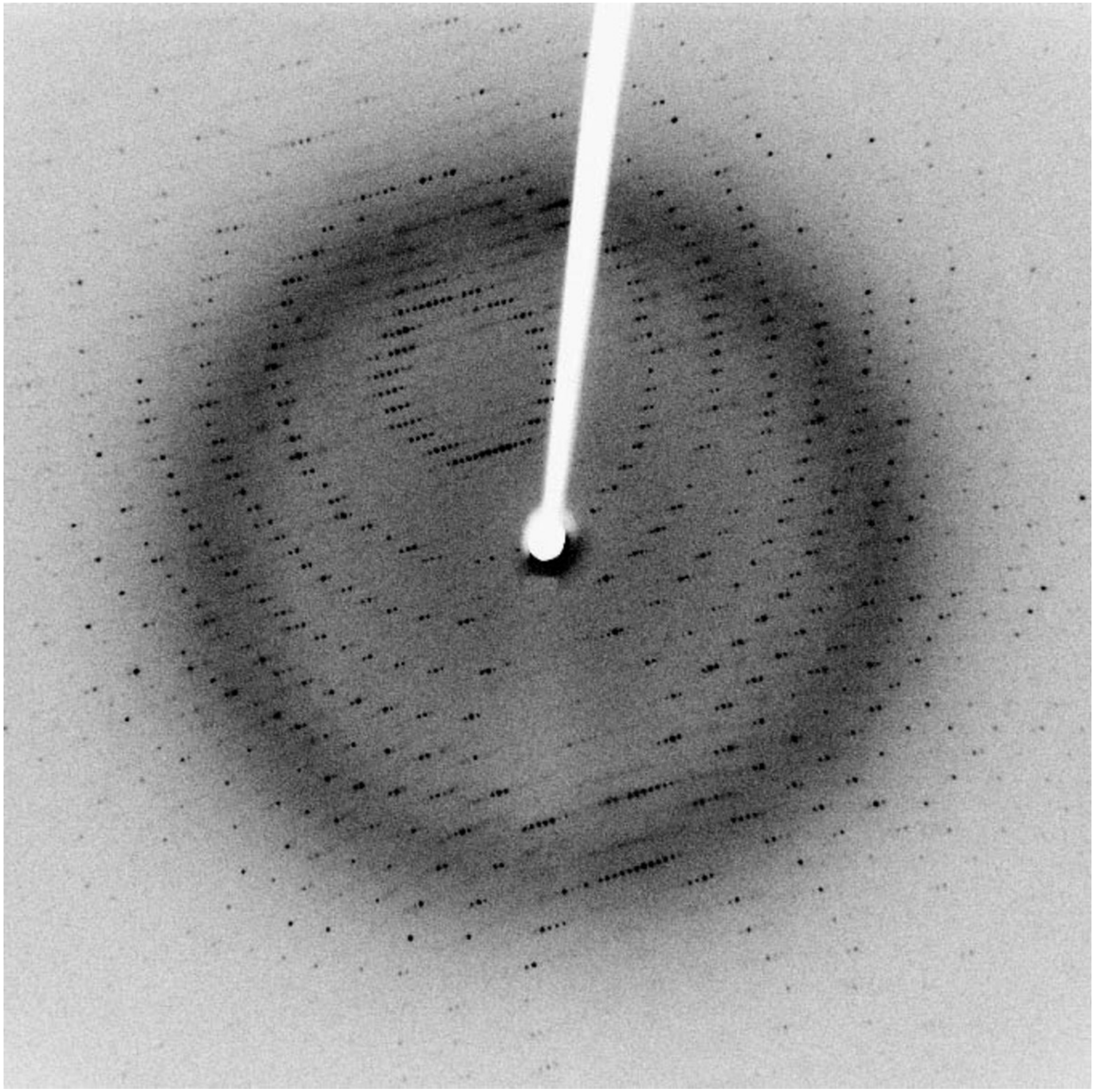


$\xrightarrow{\mathcal{F}}$



Periodic signals

X-ray diffraction by a crystal



"Bragg peaks"

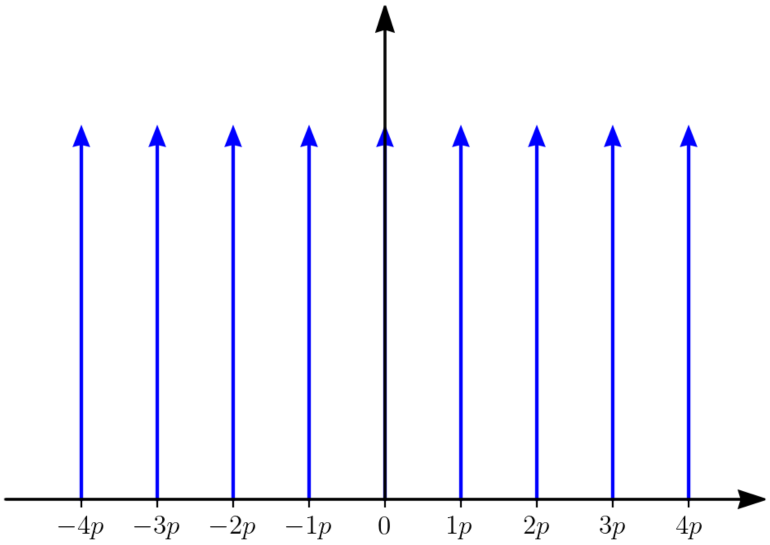
$\Updownarrow$

Dirac deltas caused by periodicity

The Dirac comb

A periodic function made of Dirac functions

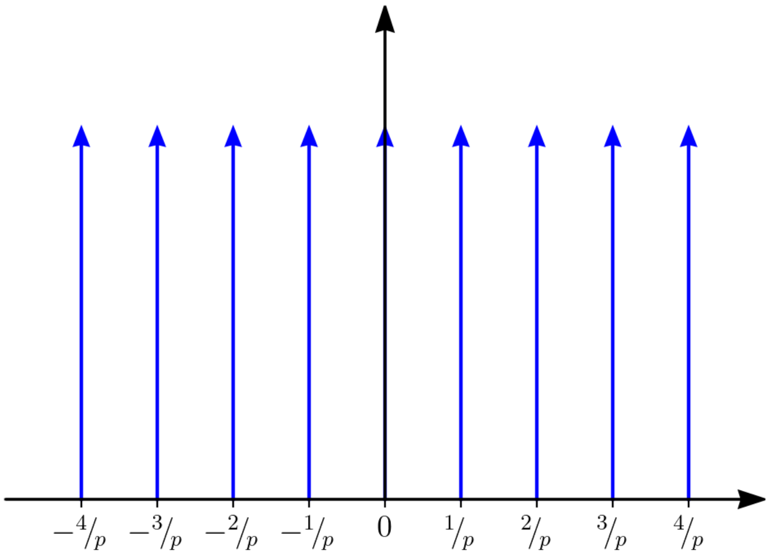
$$\Delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np)$$



$$\Delta_{\frac{1}{p}}(u) = \sum_{k=-\infty}^{\infty} \delta(u - k/p)$$

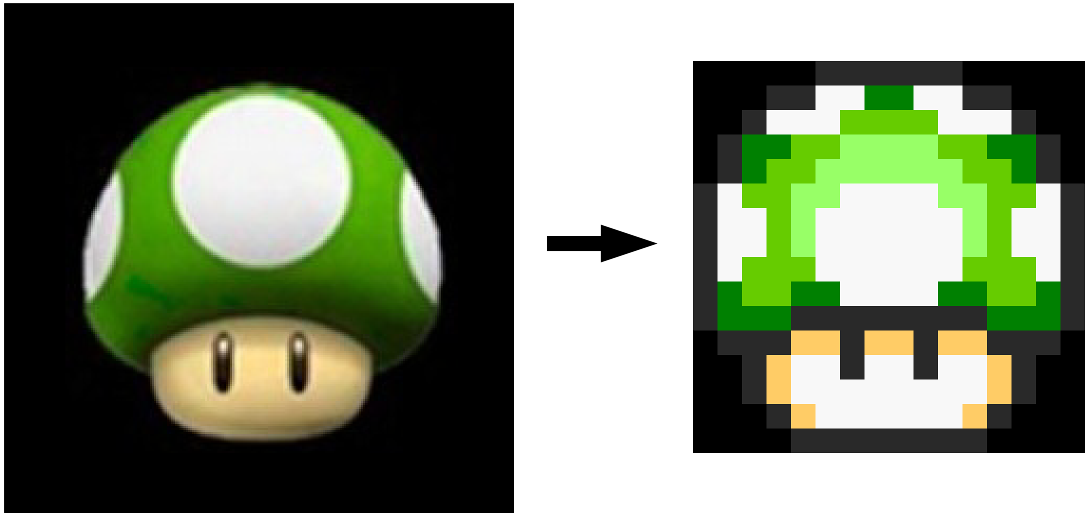
$$\mathcal{F}\{\Delta_p(x)\} = \frac{1}{p}\Delta_{\frac{1}{p}}(u)$$

F.T. of a Dirac comb is a Dirac comb

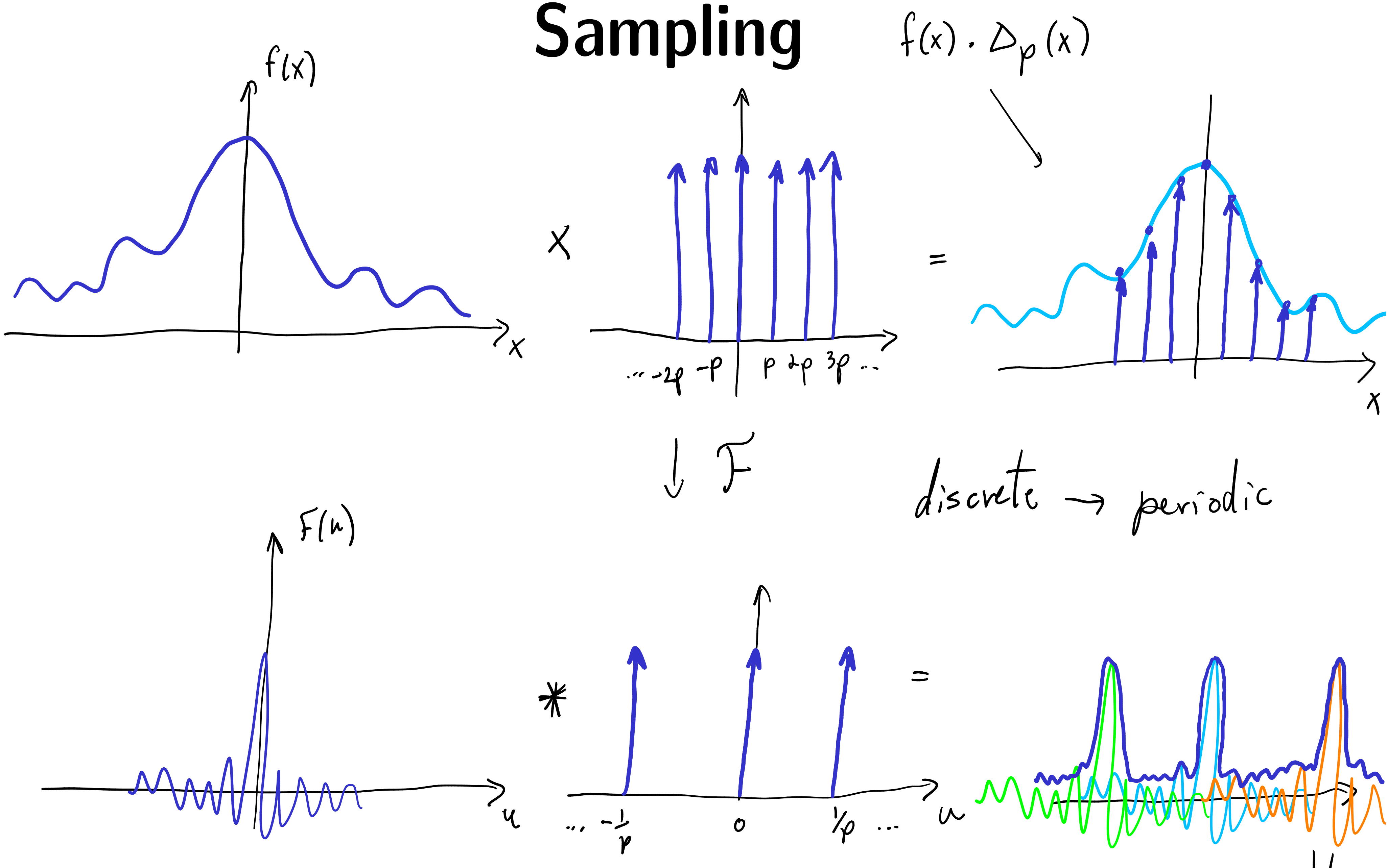


The discrete Fourier transform

- additional ingredients needed:
  - sampling in space
  - finite field of view in space
  - sampling in frequency domain
  - finite frequency band
- discrete approximation of some continuous function



Sampling



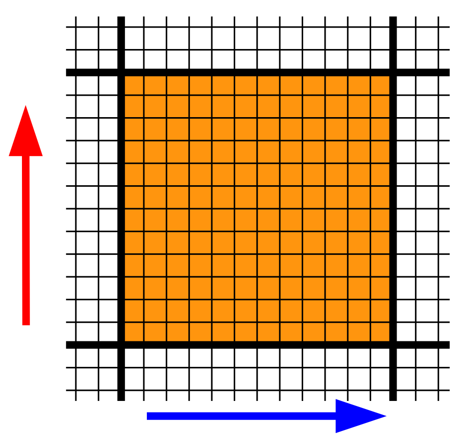
discrete → periodic

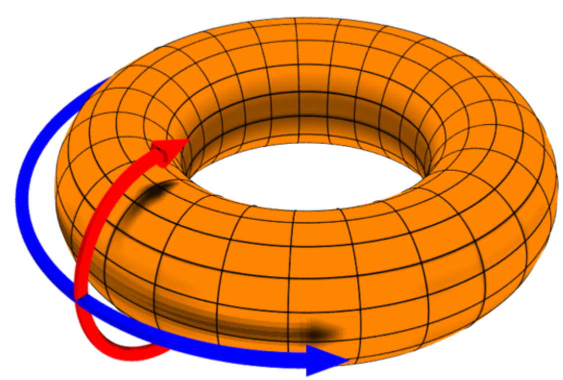
overlap causes problems

"aliasing"

Summary

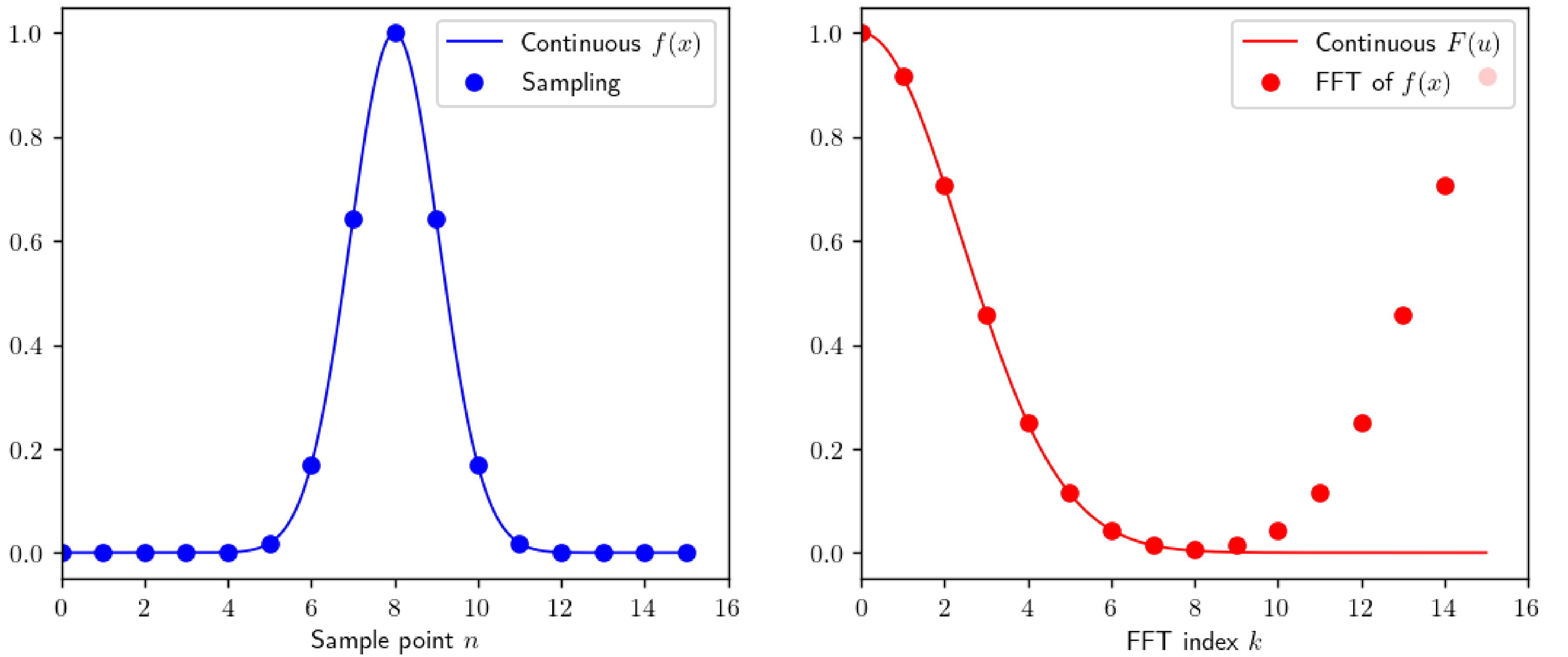
	real space	Fourier space
F.T.	continuous, infinite domain	continuous, infinite domain
F.S.	continuous, periodic	discrete, infinite domain
D.F.T.	discrete, periodic	discrete, periodic





DFT example

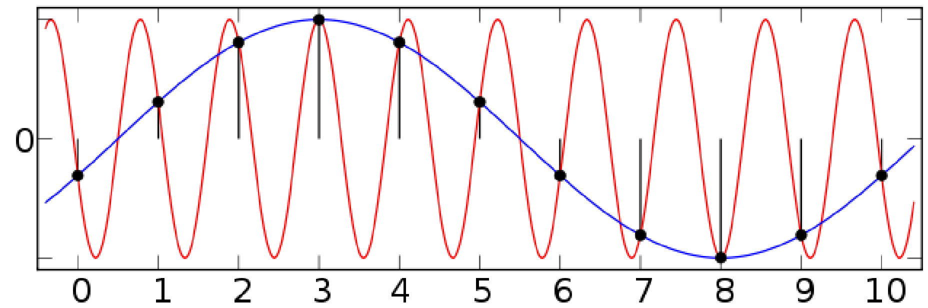
- Example: relation between space, sampling and frequency



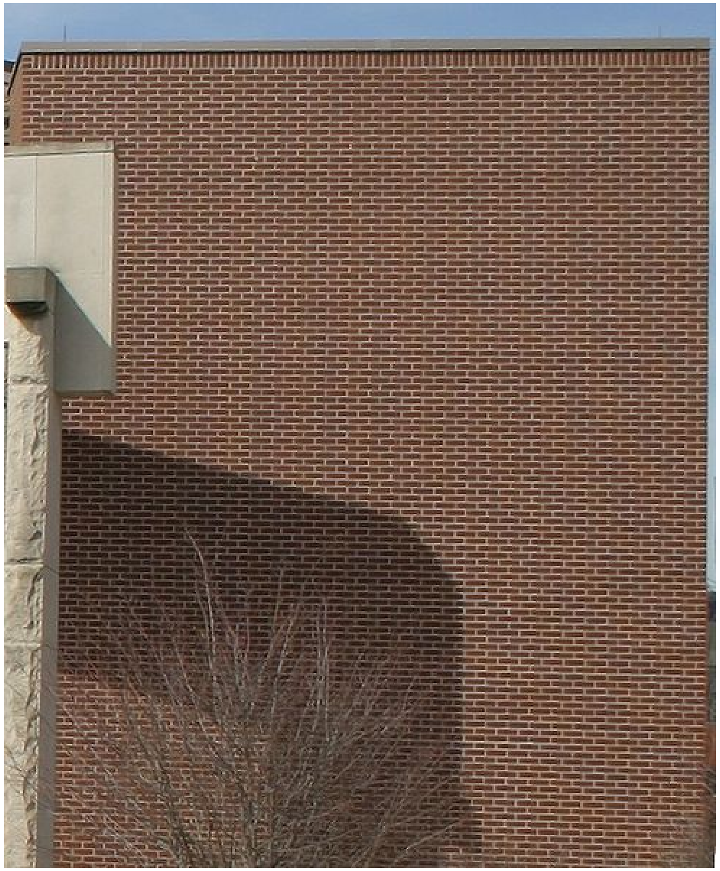
zero frequency component is in the top left corner output array.

Aliasing

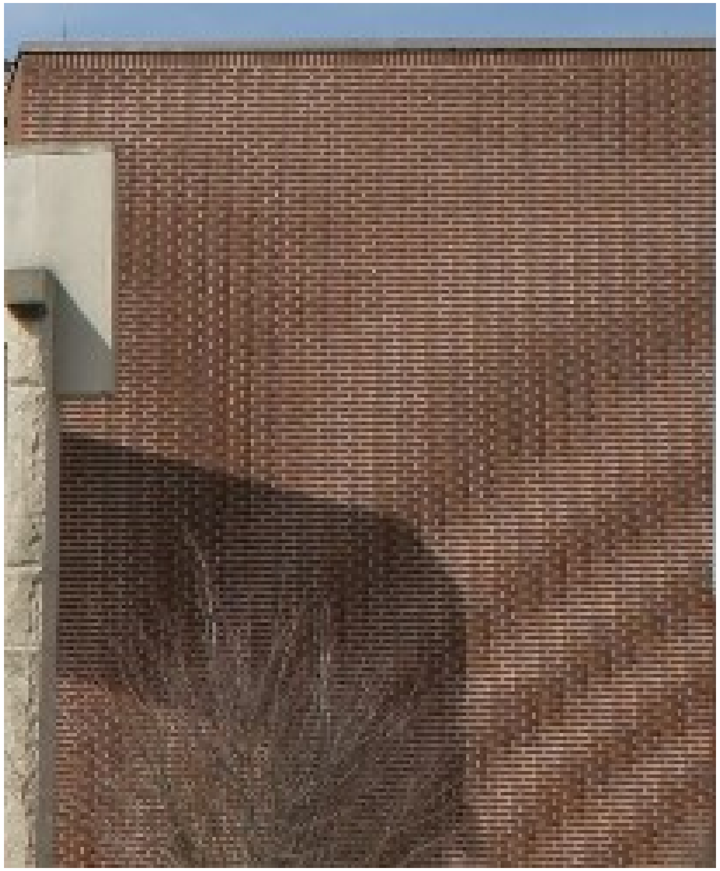
Moiré: after resampling, high spatial frequencies appear as low spatial frequencies



high resolution



low resolution



source: http://wikipedia.org