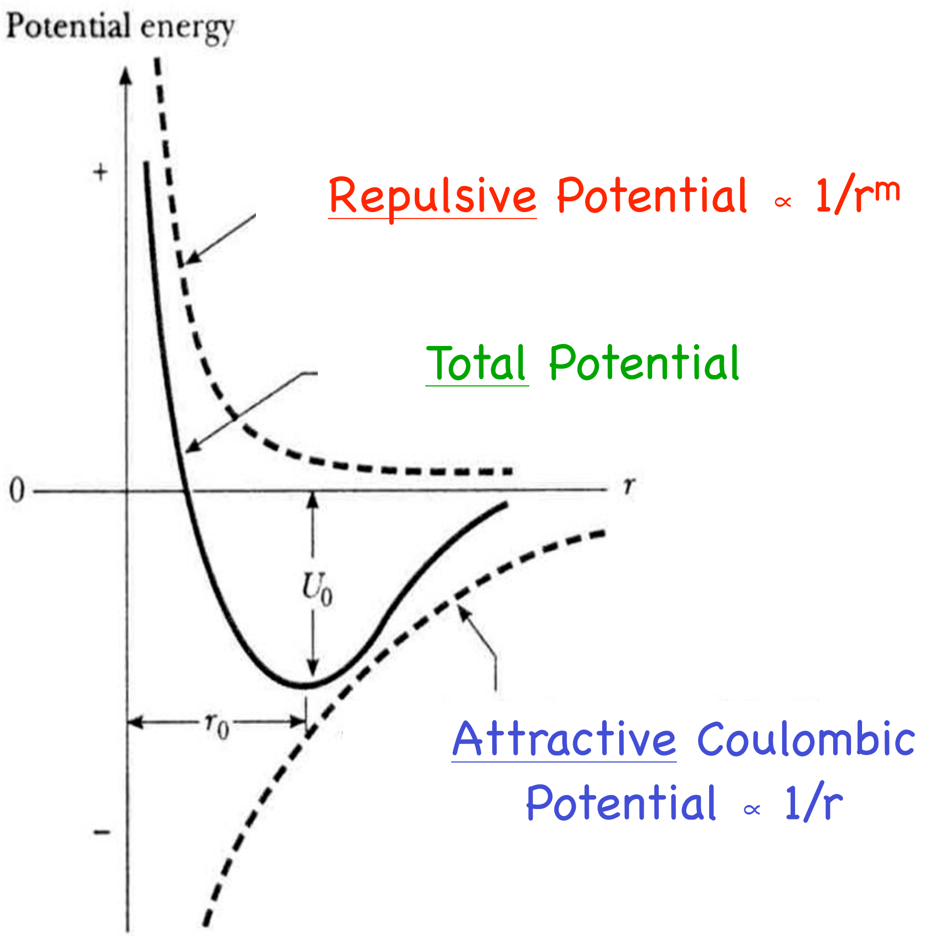
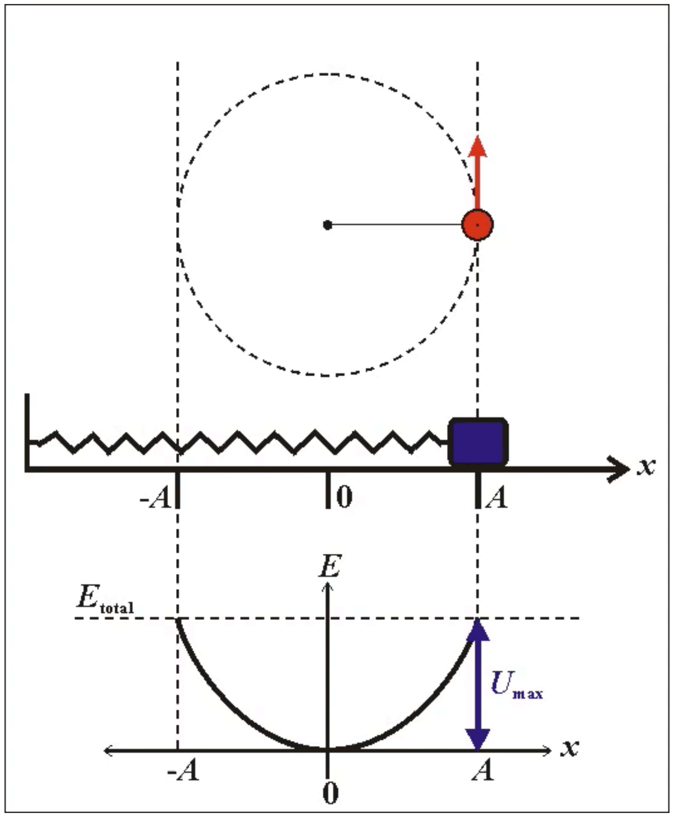
Harmonic oscillations

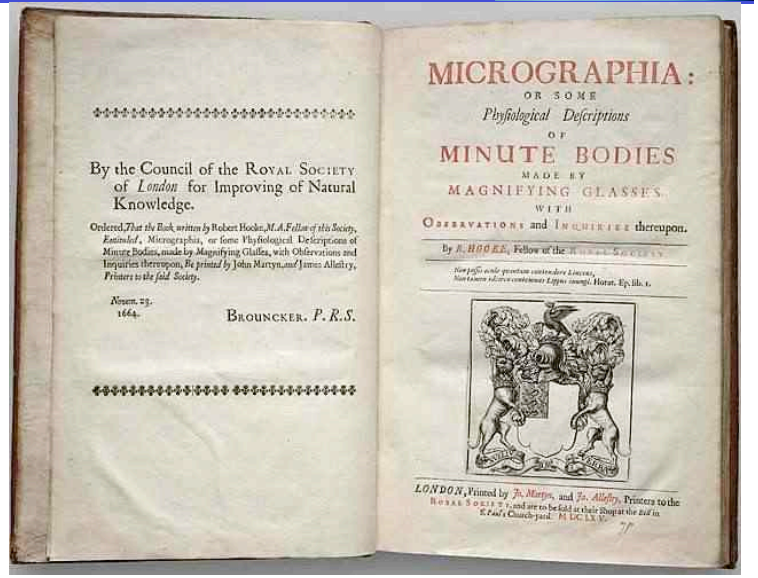
Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**





Hooke's law

Although **Robert Hooke**'s name is now usually associated with elasticity and springs, he was interested in many aspects of science and technology. His most famous written work is probably the Micrographia, a compendium of drawings he made of objects viewed under a magnifying glass.


By the Council of the ROYAL SOCIETY of *London* for Improving of Natural Knowledge.

Ordered, *That the Book written by* Robert Hooke, *M.A. Fellow of this Society, Entituled*, Micrographia, or some Physiological Descriptions of Minute Bodies, made by Magnifying Glasses, with Observations and Inquiries thereupon, *Be printed by* John Martyn, *and* James Allestry, *Printers to the said Society.*

Novem. 23. 1664.

BROUNCKER. *P. R. S.*

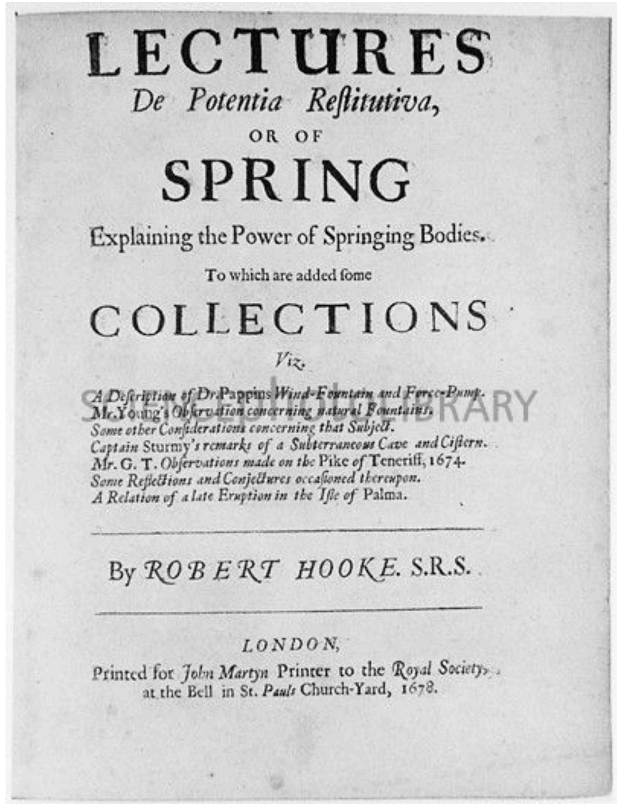
MICROGRAPHIA:
OR SOME
Physiological Descriptions
OF
MINUTE BODIES
MADE BY
MAGNIFYING GLASSES
WITH
OBSERVATIONS and INQUIRIES thereupon.

By R. HOOKE, Fellow of the ROYAL SOCIETY.

LONDON, Printed by *Jo. Martyn*, and *Ja. Allestry*, Printers to the ROYAL SOCIETY, and are to be sold at their Shop at the Bell in S. Paul's Church-yard. M DC LX V.

ceiiinosssttuu

It's an anagram. In the time before patents and other intellectual property rights, publishing an anagram was a way to announce a discovery, establish priority, and still keep the details secret long enough to develop it fully. Hooke was hoping to apply his new theory to the design of timekeeping devices and didn't want the competition profiting off his discovery.

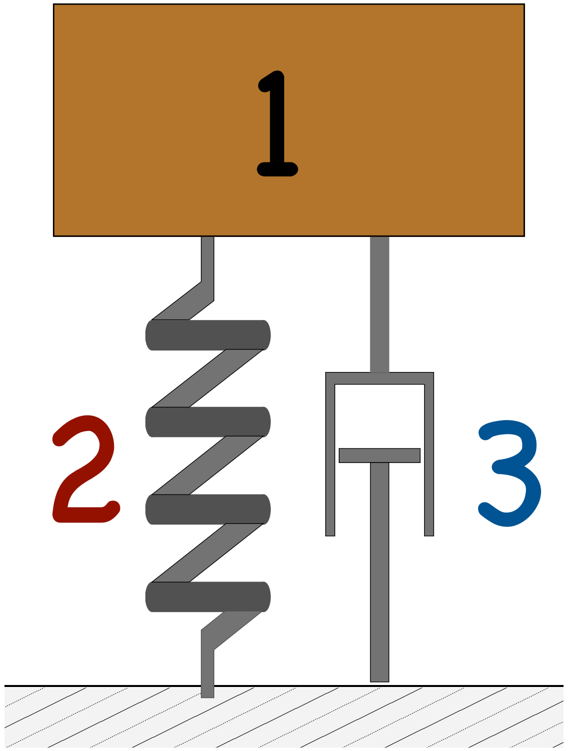

LECTURES
De Potentia Restitutiva,
OR OF
SPRING
Explaining the Power of Springing Bodies.
To which are added some
COLLECTIONS
Viz.
A Description of Dr. Pappins *Wind-Fountain and Force-Pump.*
Mr. Young's *Observation concerning natural Fountains.*
Some other Considerations concerning that Subject.
Captain Sturmy's *remarks of a Subterraneous Cave and Cistern.*
Mr. G. T. *Observations made on the* Pike of Teneriff, 1674.
Some Reflections and Conjectures occasioned thereupon.
A Relation of a late Eruption in the Isle of Palma.

By *ROBERT HOOKE.* S.R.S.

LONDON,
Printed for *John Martyn* Printer to the *Royal Society*, at the Bell in St. *Pauls* Church-Yard, 1678.

1678: "About two years since I printed this Theory in an Anagram at the end of my Book of the Descriptions of Helioscopes, viz. ceiinosssttuu, that is **Ut tensio sic vis**."

Modeling Vibration



The Ingredients:

- **Inertia** (stores kinetic energy)
- **Elasticity** (stores potential energy)

Realistic Addition:

- **Dissipation**

- **mass**
- **stiffness**
- **damping**

- to model lots of physical systems: engines, water towers, building etc...

A mass under a restoring force

From Newton's 2nd Law $F = ma$

where $a = \frac{d^2x}{dt^2}$

therefore $-kx = m\frac{d^2x}{dt^2}$

or
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This is the condition for simple harmonic motion

SHM

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

An object moves with simple harmonic motion (SHM) when the acceleration of the object is proportional to its displacement and in the opposite direction.

Some definitions:

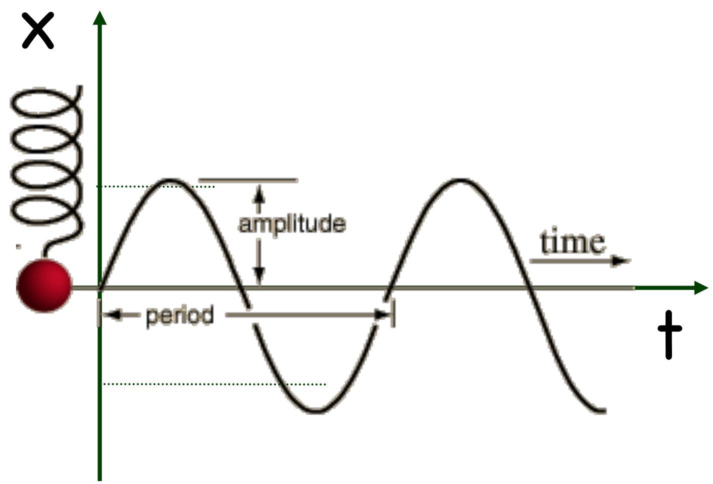
The time taken to make one complete oscillation is the **period**, **T**.

The **frequency** of oscillation, **f** = 1/T in s^{-1} or Hertz

The distance from equilibrium to maximum displacement is the **amplitude** of oscillation, **A**.

SHM

Consider the following:



The general equation for the curve traced out by the pen is $x = A\cos(\omega t + \delta)$

where $(\omega t + \delta)$ is the **phase** of the motion

and δ is the **phase constant**

SHM - natural frequency

We can show that the expression x = A cos (ωt + δ) is a solution of $\frac{d^2x}{dt^2} = -\frac{k}{m}x$ by differentiating wrt time

$$x = A\cos(\omega t + \delta)$$

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t + \delta)$$

$$a = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \delta)$$

or $$a = -\omega^2 x$$

Compare this to a = -(k/m)x

x = A cos (ωt + δ) is a solution if $\omega = \sqrt{\frac{k}{m}}$

SHM - IC

We can determine the amplitude of the oscillation (A) and the phase constant (δ) from the initial position x_o and the initial velocity v_o

$$\text{if} \quad x = A\cos(\omega t + \delta) \text{ then } \quad x_o = A\cos(\delta)$$

$$\text{if } v = -A\omega\sin(\omega t + \delta) \quad \text{then} \quad v_o = -A\omega\sin(\delta)$$

The system repeats the oscillation every T seconds

therefore $x(t) = x(t+T)$

and

$$A\cos(\omega t + \delta) = A\cos(\omega(t + T) + \delta)$$
$$= A\cos(\omega t + \delta + \omega T)$$

The function will repeat when $\omega T = 2\pi$

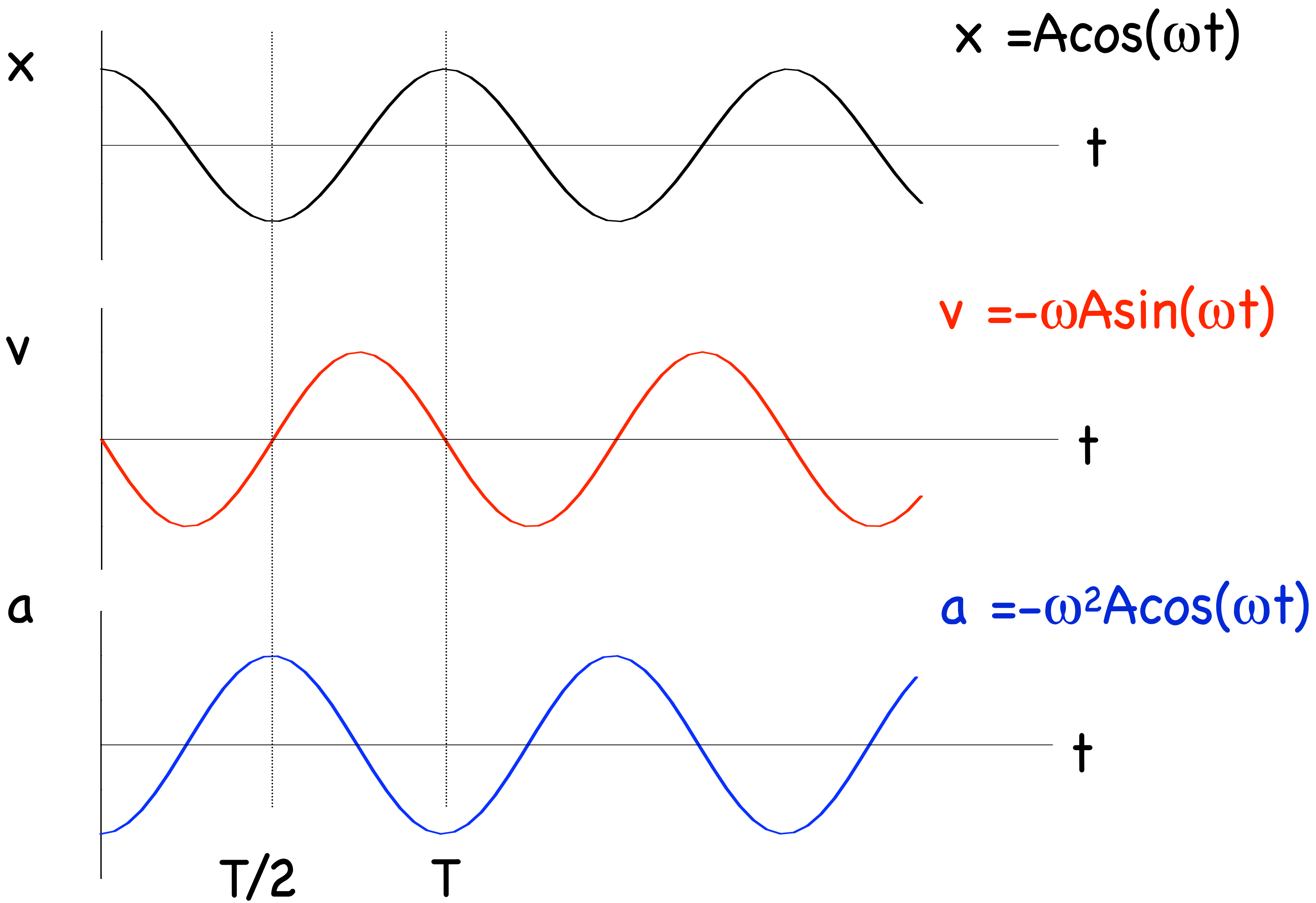
We can relate ω, f and the spring constant k using the following expressions.

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

ω is known as the **angular frequency** and has units of rad·s^{-1}

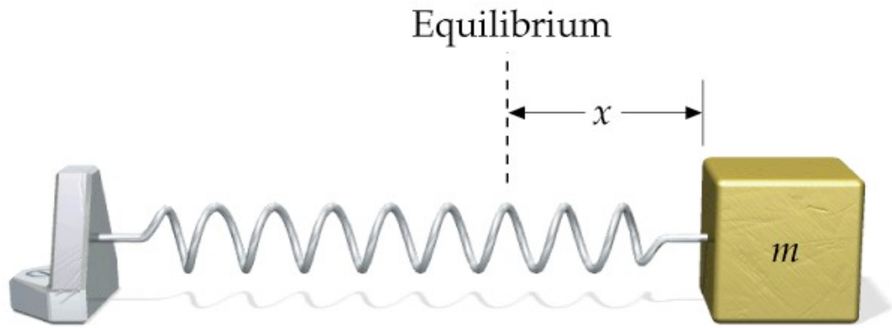
x, v, a time dependence in SHM



Energy of SHM

In SHM the total energy (E) of a system is **constant** but the kinetic energy (K) and the potential energy (U) vary wrt.

Consider a mass a distance x from equilibrium and acted upon by a restoring force



Kinetic Energy

$$K = \tfrac{1}{2} m v^2$$

$$v = -A\omega \sin(\omega t + \delta)$$

$$K = \tfrac{1}{2} m A^2 \omega^2 \sin^2(\omega t + \delta)$$

Substitute $\omega^2 = k/m$

$$K = \tfrac{1}{2} k A^2 \sin^2(\omega t + \delta)$$

Potential Energy

$$U = \tfrac{1}{2} k x^2$$

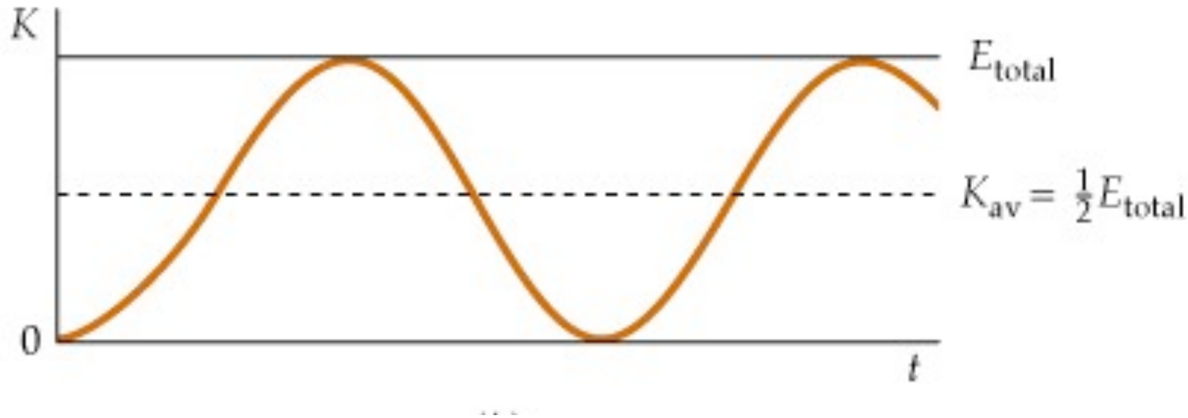
$$x = A\cos(\omega t + \delta)$$

$$U = \tfrac{1}{2} k A^2 \cos^2(\omega t + \delta)$$

Graphical representation

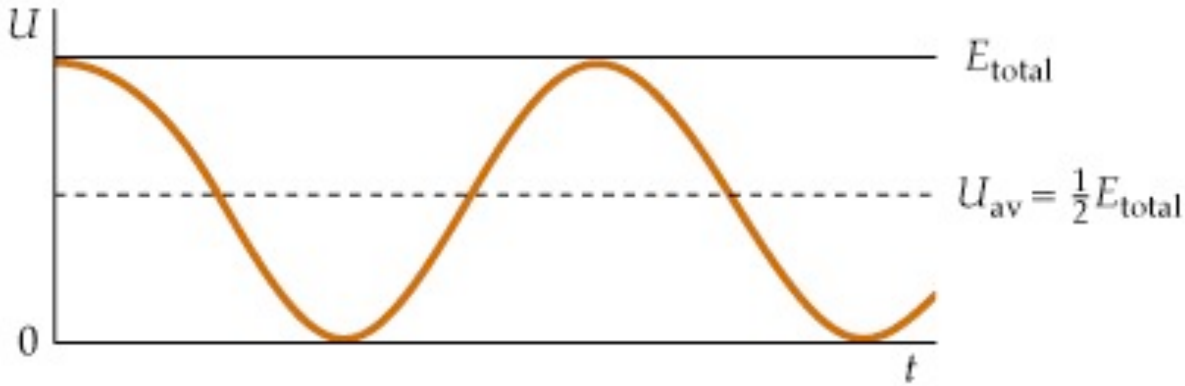
Kinetic

Energy



Potential

Energy



Total Energy

Total energy E = K + U

$$= \tfrac{1}{2}kA^2\sin^2(\omega t+\delta) + \tfrac{1}{2}kA^2\cos^2(\omega t+\delta)$$

$$= \tfrac{1}{2}kA^2\left(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right)$$

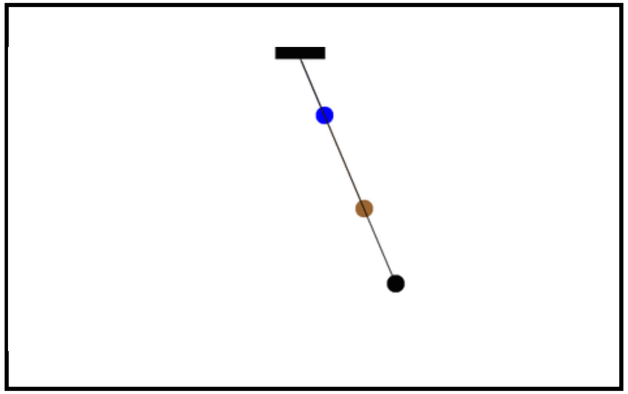
but $\left(\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)\right) = 1$

$$\therefore \quad E = \tfrac{1}{2}kA^2$$

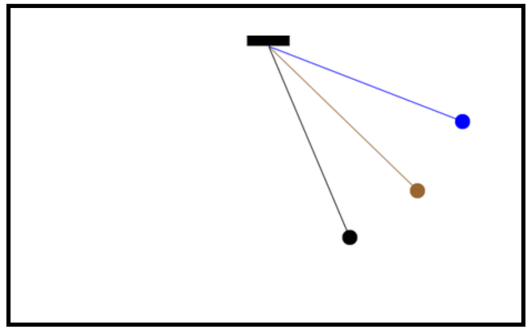
In SHM the total energy of the system is proportional to the square of the amplitude of the motion

The Simple Pendulum

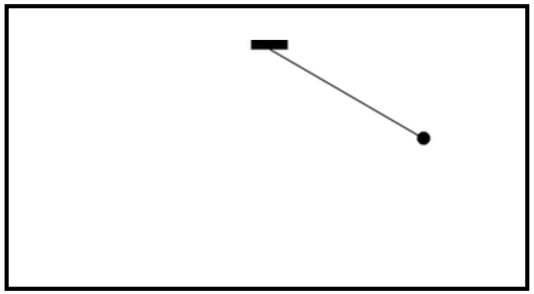
A simple pendulum consists of a string of length L and a bob of mass m.



When the mass is displaced and released from an initial angle ϕ with the vertical it will swing back and forth with a period T.



We are going to derive an expression for T.



Forces on mass:

mg (downwards)

tension (upwards)

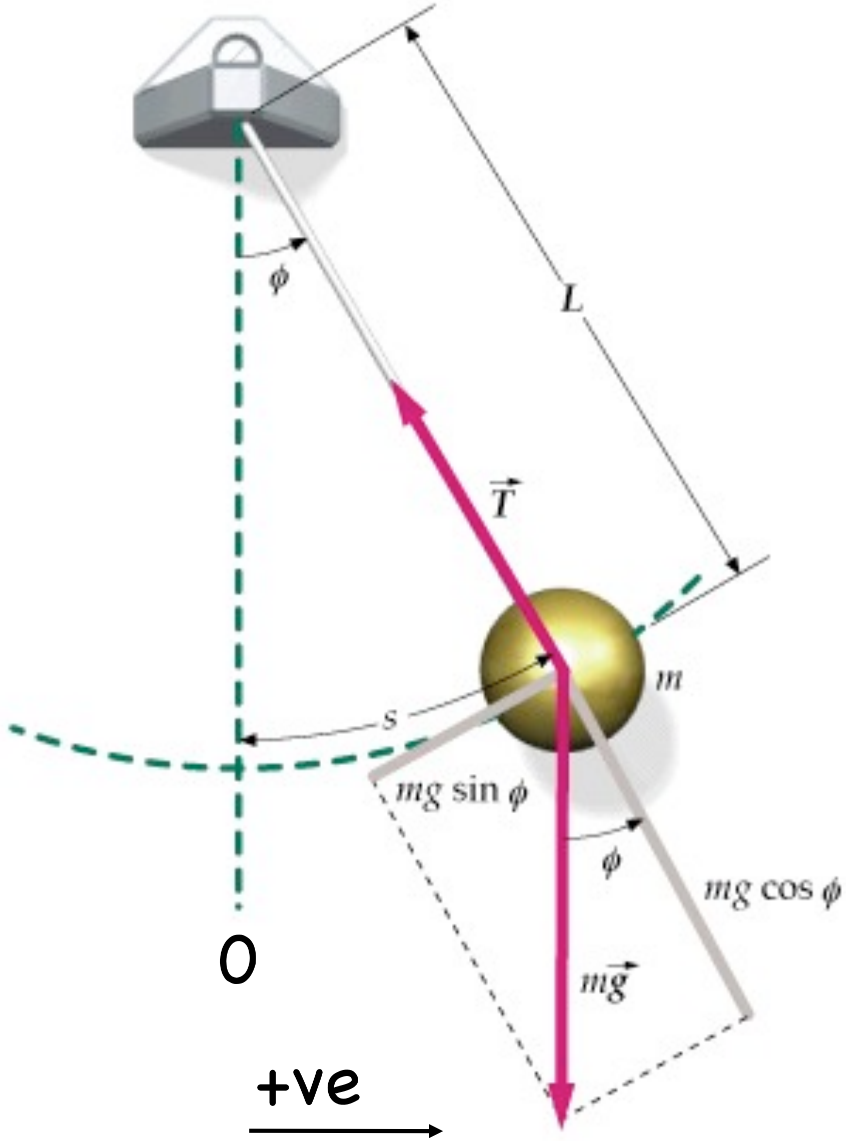
When mass is at an angle ϕ to the vertical these forces have t be resolved.

Tangentially:

weight = mg sin ϕ (towards 0)

tension = T cos 90 = 0

$$\sum F_{tang} = -mg\sin\phi$$



Using $$\frac{\phi(\text{rads})}{2\pi} = \frac{s}{2\pi L}$$

we find $$s = L\phi$$

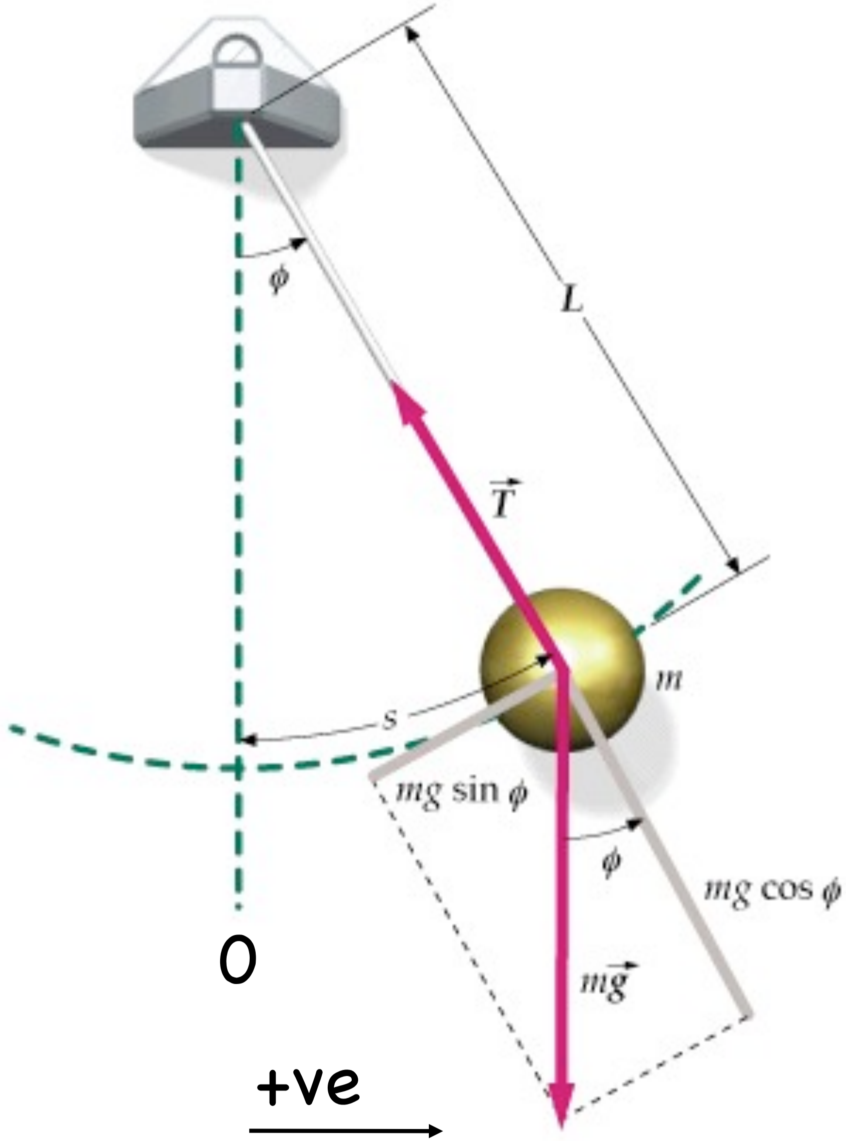
From Newton's 2nd Law (N2)

$$\sum F_{tang} = -mg\sin\phi$$

$$= ma$$

$$= m\frac{d^2s}{dt^2}$$

$$= mL\frac{d^2\phi}{dt^2}$$



$$-mg\sin\phi = mL\frac{d^2\phi}{dt^2}$$

or $$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\sin\phi$$

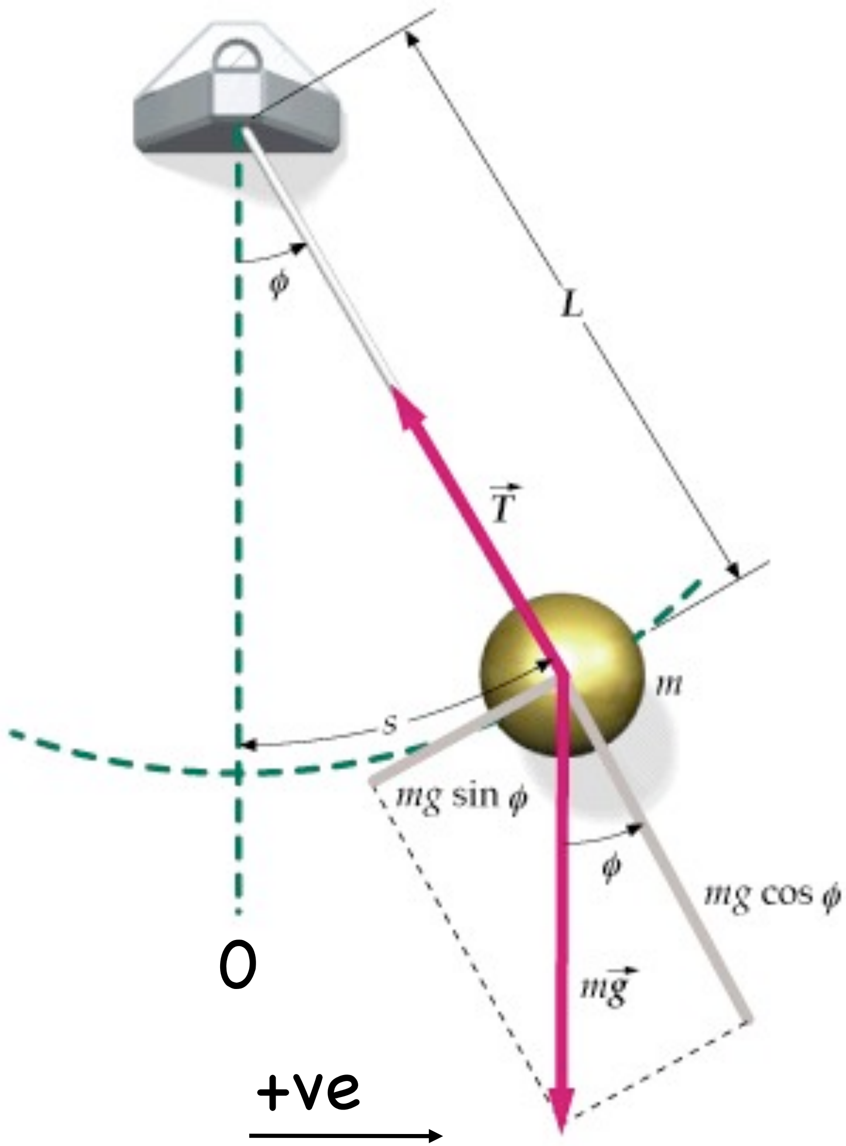
For small ϕ $\quad \sin\phi \sim \phi$

$$\frac{d^2\phi}{dt^2} = -\frac{g}{L}\phi$$

ie SHM with $\omega^2 = \frac{g}{L}$

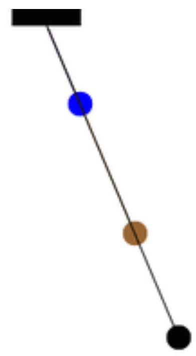
This has the solution

$$\phi = \phi_o \cos(\omega t + \delta)$$



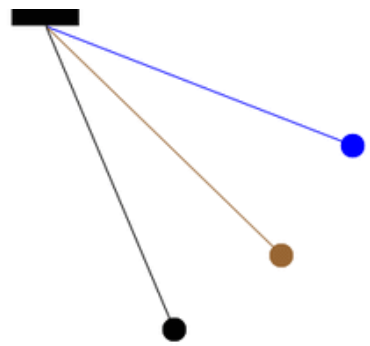
Period of the motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



ie the longer the pendulum the greater the period

Note: T does not depend upon amplitude of oscillation



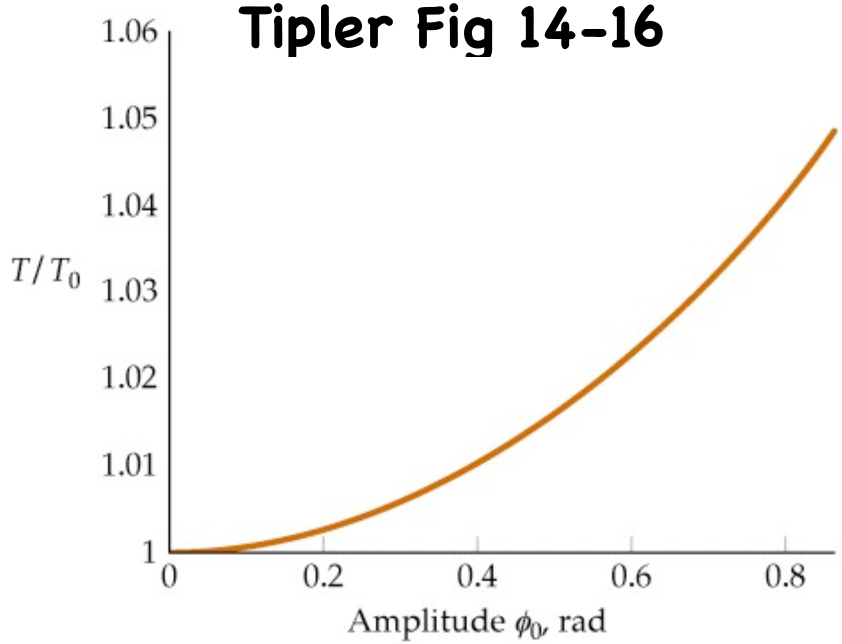
even if a clock pendulum changes amplitude it will still keep time

Period of the motion

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

This is only true for $\phi < 10^{o}$

Tipler Fig 14-16



Generally

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\varphi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\varphi}{2}\right) + \ldots\right)$$

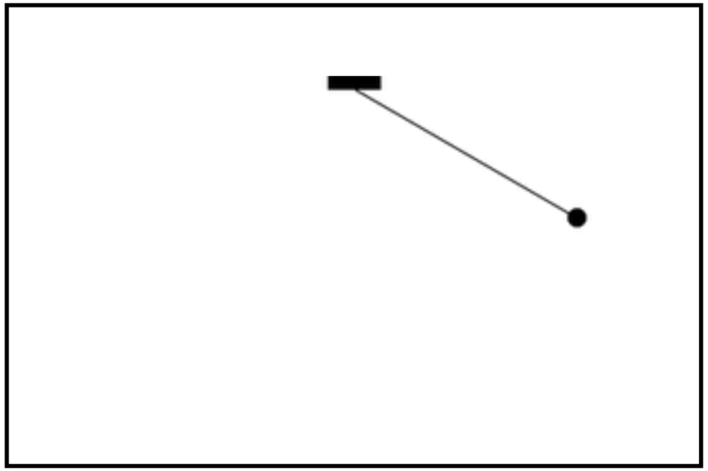
$$T = T_o\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\varphi}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\varphi}{2}\right) + \ldots\right)$$

If the initial angular displacement is significantly large the small angle approximation is no longer valid

The error between the simple harmonic solution and the actual solution becomes apparent almost immediately, and grows as time progresses.

Dark blue pendulum is the simple approximation,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$



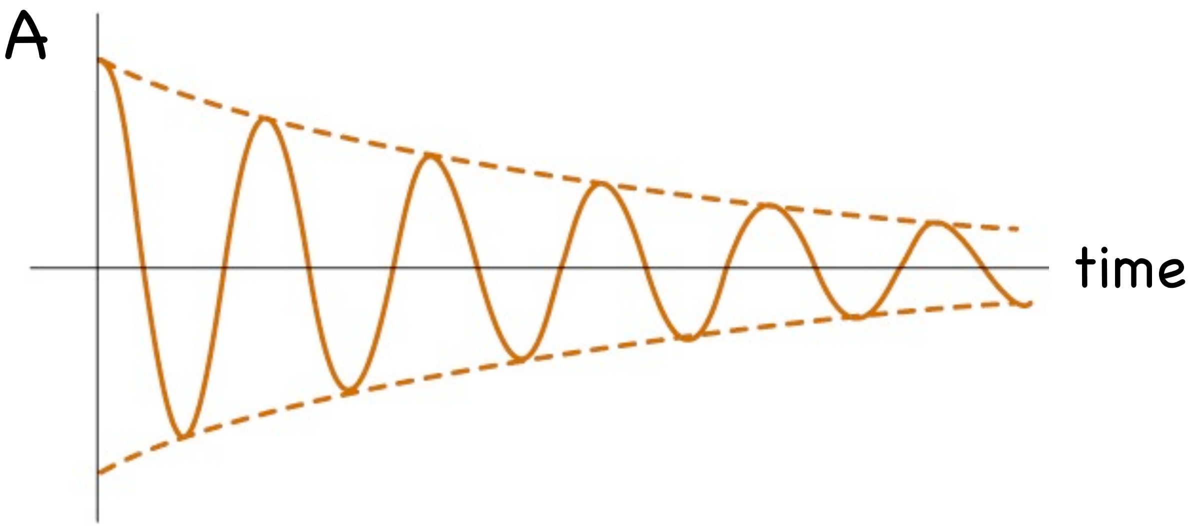
light blue pendulum shows the numerical solution of the nonlinear differential equation of motion.

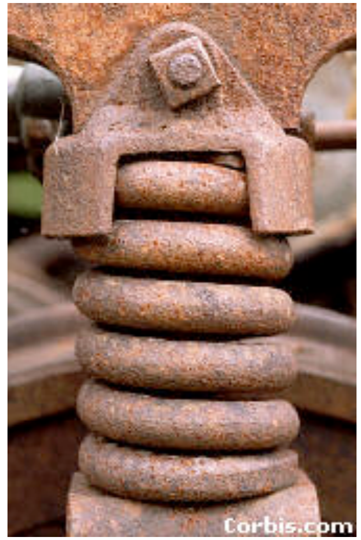
$$T = T_o \left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\varphi}{2}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^2 \sin^4\left(\frac{\varphi}{2}\right) + \dots\right)$$

Damped Oscillations

All real oscillations are subject to frictional or dissipative forces.

These forces remove energy from the oscillating system and reduce A.





Damped Oscillations

Consider mass m on the end of a spring with a spring constant k

Restoring force = kx when mass is a distance x from equilibrium

Drag force ∝ dx/dt

$$F = ma$$

$$-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

where $\gamma = b/m$ and $\omega^2 = k/m$

Auxiliary equation

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = 0$$ where $\gamma = b/m$ and $\omega_0 = (k/m)^{1/2}$

In order to find the auxiliary eq. one tries: $x(t) = e^{-\beta t}$

$$\beta^2 - \gamma\beta + \omega_o^2 = 0 \qquad \beta_{1/2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

1) $\gamma > 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+\frac{\sqrt{\gamma^2-4\omega_0^2}}{2}t}$

2) $\gamma = 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t} + Bte^{-\frac{\gamma}{2}t}$

3) $\gamma < 2\omega_0$ $x(t) = Ae^{-\frac{\gamma}{2}t}e^{-i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t} + Be^{-\frac{\gamma}{2}t}e^{+i\frac{\sqrt{4\omega_0^2-\gamma^2}}{2}t}$

Initial conditions

and the constants can be determined applying the initial conditions, e.g. $x(0)=x_0$ and $v(0)=0$.

1) $x(t) = e^{-\frac{\gamma}{2}t}\left[\left(\frac{x_0}{2} - \frac{\gamma x_0}{4\omega}\right)e^{-\omega t} + \left(\frac{x_0}{2} + \frac{\gamma x_0}{4\omega}\right)e^{+\omega t}\right]$ **overdamped**

2) $x(t) = e^{-\frac{\gamma}{2}t}\left[x_0 + \frac{\gamma x_0}{2}t\right]$ **critically damped**

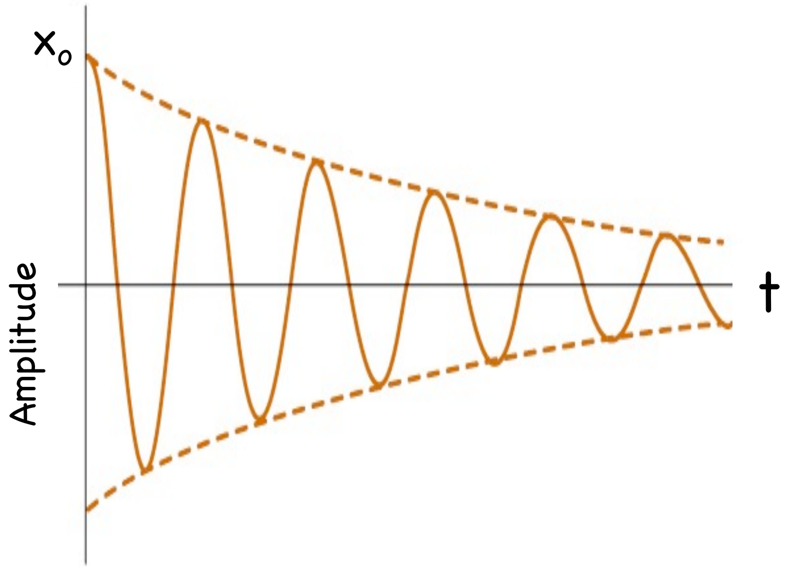
3) $x(t) = e^{-\frac{\gamma}{2}t}\left[\left(x_0\right)\cos\omega t + \left(\frac{\gamma x_0}{2\omega}\right)\sin\omega t\right]$ **underdamped**

with $\omega = \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$

Energy of a damped oscillator

Generally $$E = \frac{1}{2} m\omega^2 A^2$$ energy $E \propto$ amplitude A^2

if amplitude is decreasing exponentially then energy will also decrease exponentially



$$x(t) = x_o\, e^{\frac{-\gamma t}{2}} \cos(\omega t)$$

max displacement when cos=1

$$x(t) = x_o\, e^{\frac{-\gamma t}{2}}$$

$$\therefore \quad E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

Quality factor - Q

A damped oscillator is often described by its quality-factor or Q-factor

$$Q = \frac{\omega_o m}{b} = \frac{\omega_o}{\gamma}$$

this can be related to the fractional energy lost per cycle

$$E = \frac{1}{2} m\omega^2 (x_o e^{-\frac{\gamma t}{2}})^2$$

$$= E_o \, e^{-\gamma t}$$

$$dE = -\gamma \, E_o \, e^{-\gamma t} dt$$

$$= -\gamma \, E \, dt$$

In a weakly damped system the energy lost / cycle is small

$$dE = \Delta E \qquad \text{and} \qquad dt = T$$

$$\Delta E = -\gamma E T$$

$$\frac{|\Delta E|}{E} = \gamma T$$

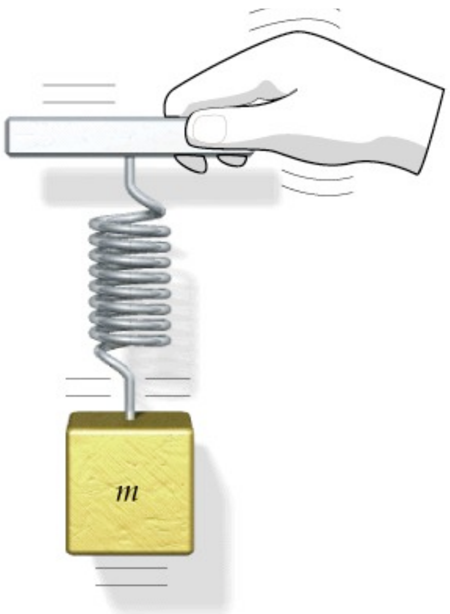
$$\frac{|\Delta E|}{E} = \frac{\gamma 2\pi}{\omega_o}$$

but $Q = \dfrac{\omega_o}{\gamma}$ ie $\gamma = \dfrac{\omega_o}{Q}$

$$\boxed{\frac{|\Delta E|}{E} = \frac{2\pi}{Q}}$$

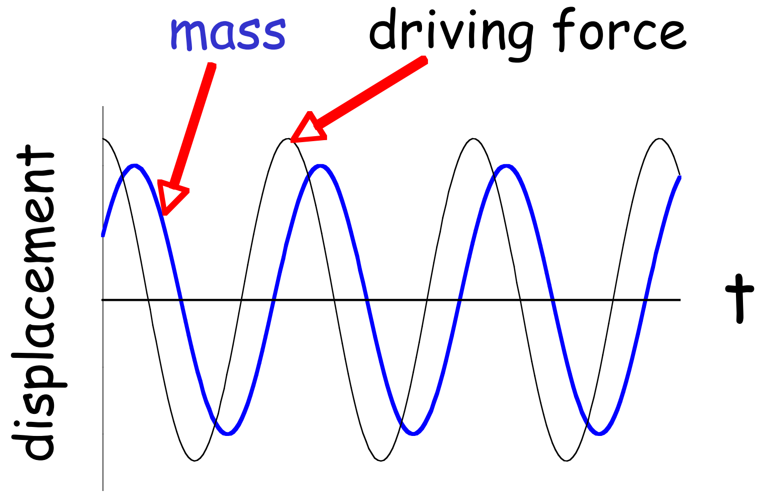
Driven oscillations

Consider the steady state behaviour of a mass oscillating on a spring under the influence of a driving force.



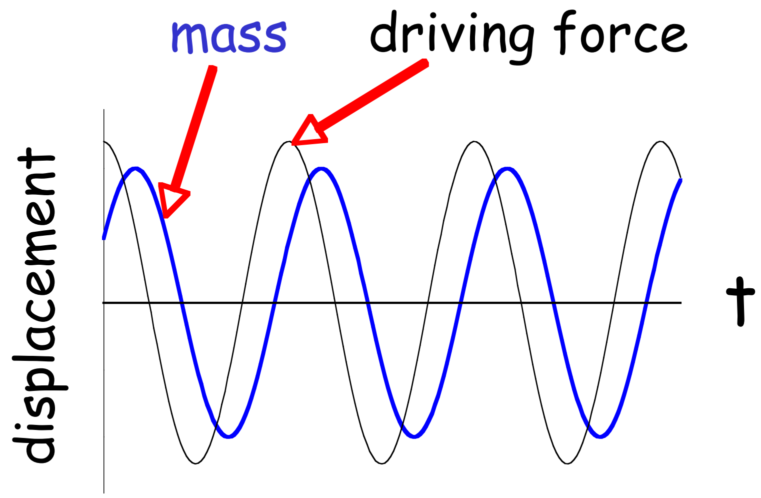
The mass oscillates at the same frequency of the driving force with a constant amplitude x_o.

The oscillations are out of phase, ie the displacement lags behind the driving force.



Force = $F_o\cos(\omega t)$ has +ve peaks at $t = 0,\ 2\pi/\omega,\ 4\pi/\omega$…………

+ve peaks of the displacement occur at $t = \Delta t,\ (2\pi/\omega)+\Delta t,\ (4\pi/\omega)+\Delta t$ …………



$$\therefore \text{the displacement } x = x_o\cos(\omega t - \phi) \quad \text{where} \quad \phi = \omega\Delta t = \frac{2\pi\Delta t}{T}$$

This describes a displacement with the same frequency as the driving force, has constant amplitude and a phase lag ϕ with respect to the driving force.

Equation of motion for a driven oscillator is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t) \quad \text{where } \gamma = b/m \quad \text{and } \omega^2 = k/m$$

Solution of this equation is $x = x_o\cos(\omega t - \phi)$

To determine the x_o and ϕ we need to substitute the solution into the equation of motion.

We need
$$\frac{dx}{dt} = -\omega x_o \sin(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} = -\omega^2 x_o \cos(\omega t - \phi)$$

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_o^2 x = \frac{F_o}{m}\cos(\omega t)$$

$$-\omega^2 x_o\cos(\omega t - \phi) - \gamma\omega x_o\sin(\omega t - \phi) + \omega_o^2 x_o\cos(\omega t - \phi) = \frac{F_o}{m}\cos(\omega t)$$

$$(\omega_o^2 - \omega^2)x_o\cos(\omega t - \phi) - \gamma\omega x_o\sin(\omega t - \phi) = \frac{F_o}{m}\cos(\omega t)$$

This equation must be true at all times.

To solve for x_o and ϕ we need to consider two situations.

1. $(\omega t - \phi) = 0$ $\quad\therefore \sin(\omega t - \phi) = 0$ and $\cos(\omega t)=\cos\phi$

2. $(\omega t - \phi) = \pi/2$ $\quad\therefore \cos(\omega t - \phi) = 0$ and $\cos(\omega t)=\cos(\pi/2 + \phi)$

This leaves us with two simultaneous equations:

$$(\omega_o^2 - \omega^2)x_o = \frac{F_o}{m}\cos(\phi)$$

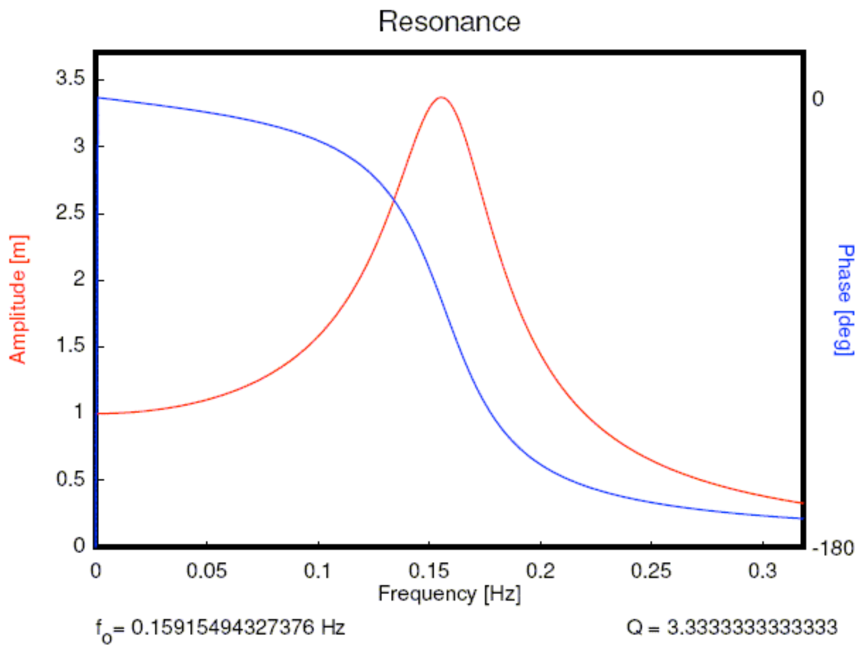
$$-\gamma\omega x_o = \frac{F_o}{m}\cos(\frac{\pi}{2} + \phi)$$

Remember $\cos(\frac{\pi}{2} + \phi) = -\sin\phi$ and $\cos^2 A + \sin^2 A = 1$

The solutions are

$$x_o = \frac{F_o / m}{\sqrt{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\tan\phi = \frac{\omega\gamma}{(\omega_o^2 - \omega^2)}$$



Absorbed power

The average rate at which power is absorbed equals the average power delivered by the driving force, to replace the energy dissipated by the drag force.

Over a period it is:

$$\langle P \rangle = \left\langle \frac{Fdx}{dt} \right\rangle = \left\langle -b\left(\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right) \right\rangle = b\left\langle \left(\frac{dx}{dt}\right)^2 \right\rangle$$

$$\langle P \rangle = \frac{F_0^2}{2m\gamma}\left[\frac{\gamma^2\omega^2}{\left(\omega_0^2 - \omega^2\right) + \gamma^2\omega^2}\right]$$

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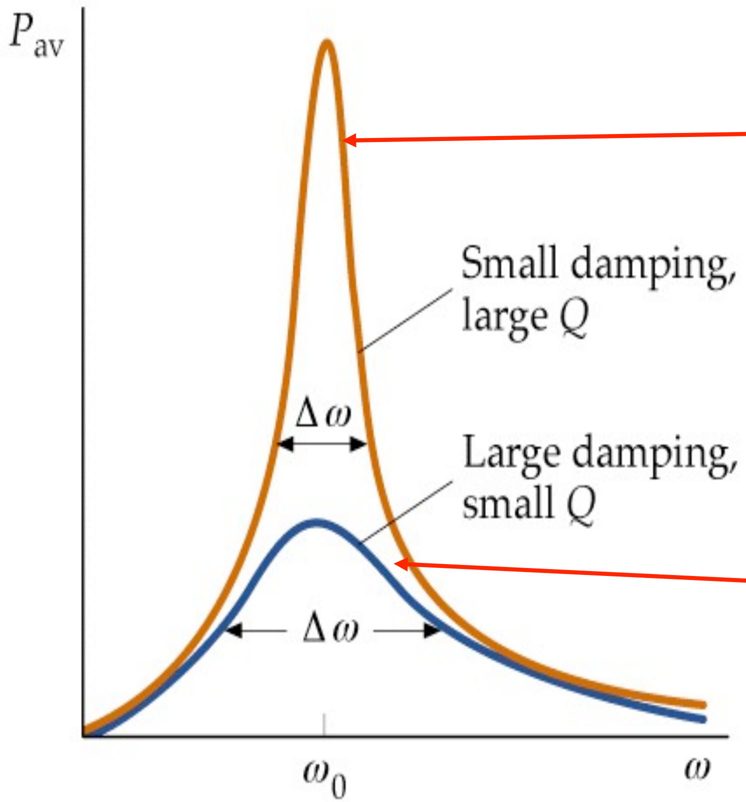
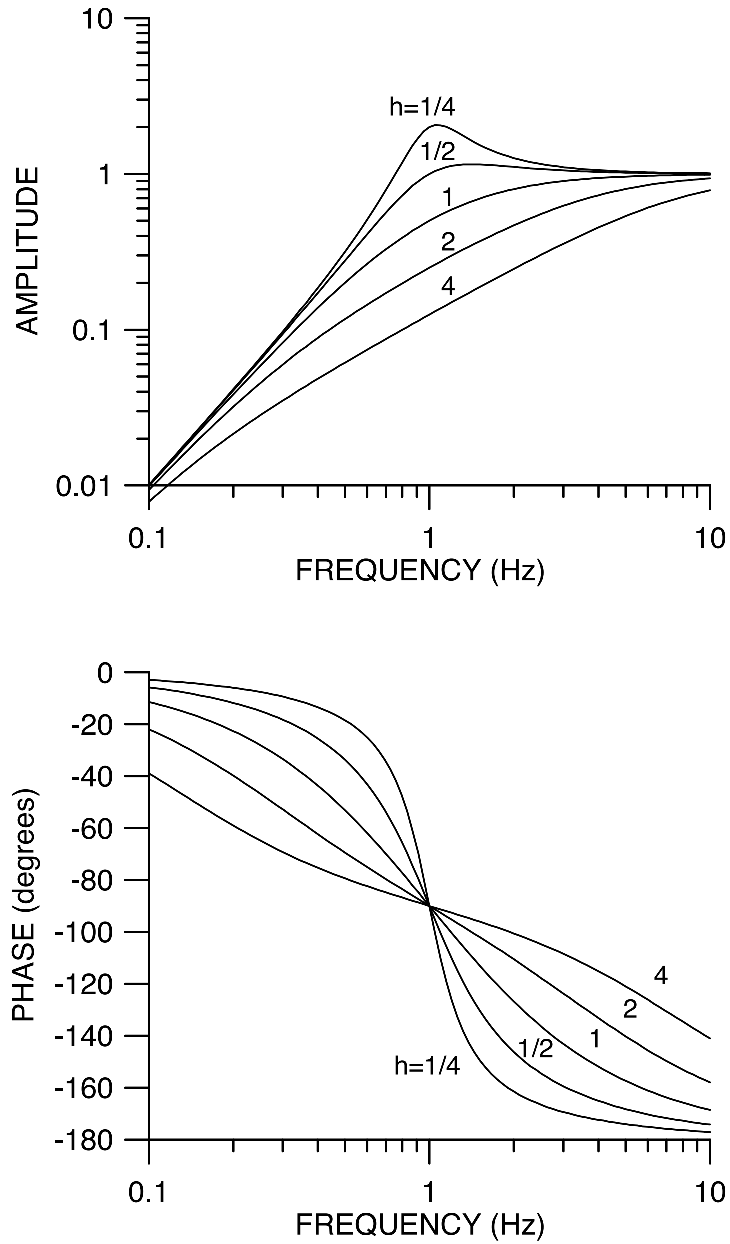


When damping is small oscillator absorbs much more energy from driving force.

Resonance peak is narrow

When damping is large oscillator resonance curve is broad

For small damping $\frac{\Delta\omega}{\omega_0} = \frac{\Delta f}{f_0} = \frac{1}{Q}$



- At very high frequencies ($\omega >> \omega_0$), $|A(\omega)| \approx 1$, and $\theta \approx \pi$, so the displacement from equilibrium is the negative of the forcing displacement, that is moving so rapidly that the mass cannot follow the motion at all.

- At very low frequencies ($\omega << \omega_0$) we have $|x_0(\omega)| \approx \omega^2/\omega_0^2$, so that the amplitude of the response falls off quadratically with frequency. From the time domain representation, we see that this response is proportional to the negative of the forcing acceleration