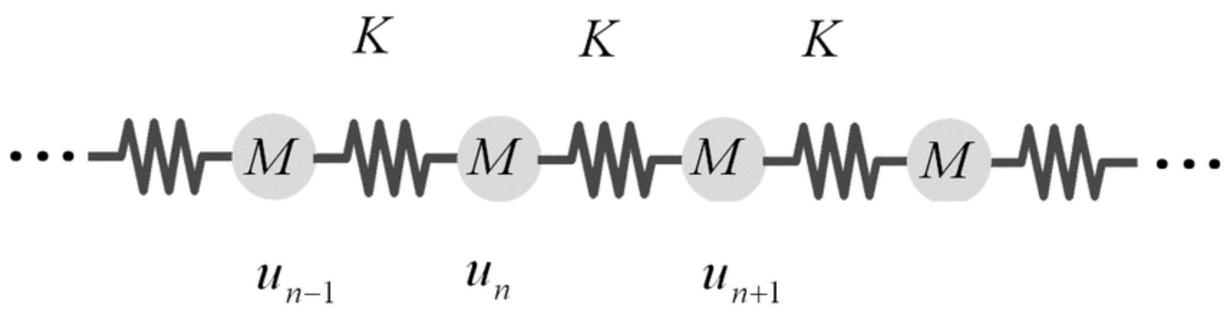
Coupled Oscillators

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**, **superpose** and **stand**

Normal modes of the system



What is a **wave**?

Small perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

Coupling of **harmonic oscillators** → the disturbances can **propagate**, **superpose** and **stand**

General form of LWE

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = v^2 \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

WAVE: organized **propagating imbalance**, satisfying differential equations of motion

Speed of waves

A general property of waves is that the speed of a wave depends on the properties of the medium, but is independent of the motion of the source of the waves.

Consider a wave moving along a rope experimentally we find

(i) the greater the tension in a rope the faster the waves propagate

(ii) waves propagate faster in a light rope than a heavy rope

ie $v \propto$ tension (F) and $v \propto$ 1/mass

known as **Mersenne's law**

Mersenne's law

L'Harmonie Universelle (1637)

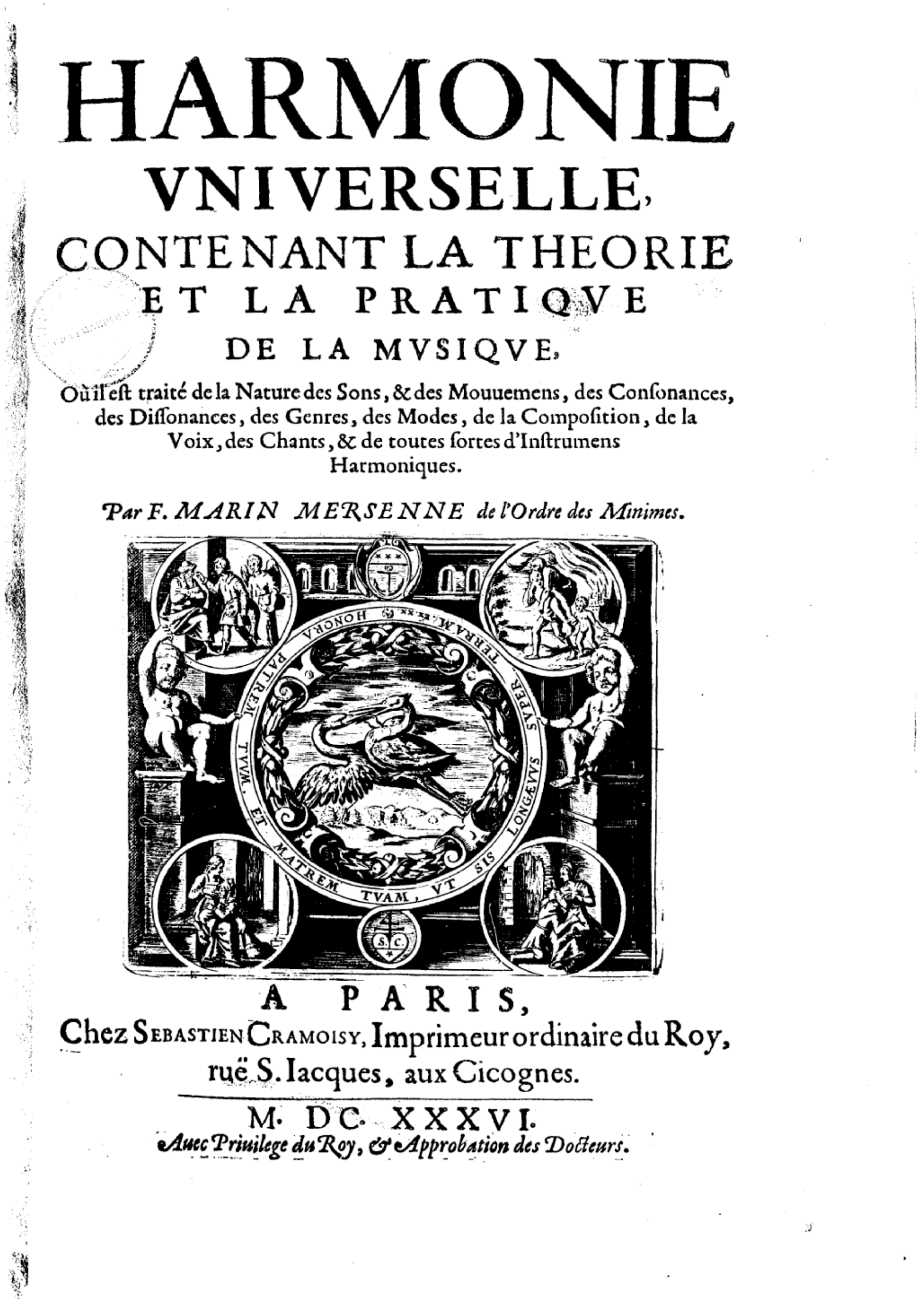
This book contains (Marine) Mersenne's laws which describe the frequency of oscillation of a **stretched string**.

This frequency is:

a) Inverse proportional to the length of the string (this was actually known to the ancients, and is usually credited to Pythagoras himself).

b) Proportional to the square root of the stretching force, and

c) Inverse proportional to the square root of the mass per unit length.



HARMONIE
VNIVERSELLE,
CONTENANT LA THEORIE
ET LA PRATIQVE
DE LA MVSIQVE,

Où il est traité de la Nature des Sons, & des Mouuemens, des Consonances, des Dissonances, des Genres, des Modes, de la Composition, de la Voix, des Chants, & de toutes sortes d'Instrumens Harmoniques.

Par F. MARIN MERSENNE de l'Ordre des Minimes.

A PARIS,
Chez SEBASTIEN CRAMOISY, Imprimeur ordinaire du Roy, ruë S. Iacques, aux Cicognes.

M. DC. XXXVI.
Auec Priuilege du Roy, & Approbation des Docteurs.

D'Alembert's solution

D'Alembert (1747) "Recherches sur la courbe que forme une corde tendue mise en vibration" (Researches on the curve that a tense cord forms [when] set into vibration), Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 3, pages 214-219.

D'Alembert (1750) "Addition au mémoire sur la courbe que forme une corde tenduë mise en vibration," Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 6, pages 355-360.

$$y(x,t) \to y(\xi,\eta) \text{ with } \xi=x-vt,\ \eta=x+vt$$

$$y_x = \frac{\partial y}{\partial x} = y_\xi \xi_x + y_\eta \eta_x = y_\xi + y_\eta;\ y_{xx} = \frac{\partial}{\partial x}\left(y_x\right) = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta},\ y_{tt} = v^2\left(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}\right)$$

$$\Rightarrow y_{\xi\eta} = \frac{\partial^2 y}{\partial\xi\,\partial\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial y}{\partial\eta}\right) = 0$$

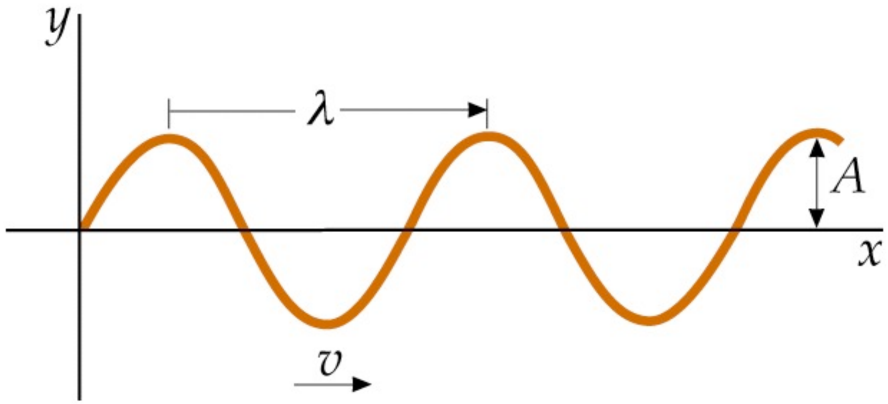
$$y = h(\xi) + g(\eta) \ \Rightarrow\ y(x,t) = h(x-vt) + g(x+vt)$$

and if the initial conditions are y(x,0)=f(x) and initial velocity=0

$$y(x,t) = \frac{1}{2}\left[f(x-vt) + f(x+vt)\right]$$

Harmonic Waves

A harmonic wave is sinusoidal in shape, and has a displacement y at time t=0



$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$

A is the **amplitude** of the wave and λ is the **wavelength** (the distance between two crests);
if the wave is moving to the right with speed v, the wavefunction at some t is given by:

$$y = A\sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Harmonic Waves

Time taken to travel one wavelength is the **period** T

Velocity, wavelength and period are related by

$$v = \frac{\lambda}{T} \quad \text{or} \quad \lambda = vT$$

$$\therefore \quad y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

The wavefunction shows the periodic nature of y:

at any time t y has the same value at x, x+λ, x+2λ………

and at any x y has the same value at times t, t+T, t+2T……

Harmonic Waves

It is convenient to express the harmonic wavefunction by defining the **wavenumber k**, and the **angular frequency** ω

$$\text{where} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

$$\therefore \quad y = A\sin(kx - \omega t)$$

This assumes that the displacement is zero at x=0 and t=0. If this is not the case we can use a more general form

$$y = A\sin(kx - \omega t - \phi)$$

where φ is the **phase constant** and is determined from initial conditions

Harmonic Waves

The wavefunction can be used to describe the motion of any point P.

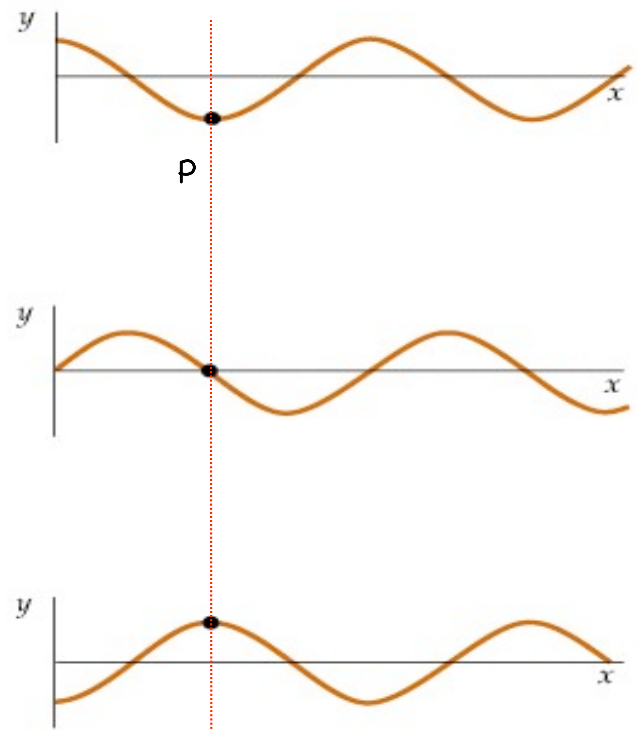
If $y = A\sin(kx - \omega t)$

Transverse velocity v_y

$$v_y = \left.\frac{dy}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial y}{\partial t}$$

$$= -\omega A\cos(kx - \omega t)$$



which has a maximum value, $(v_y)_{max} = \omega A$, when $y = 0$

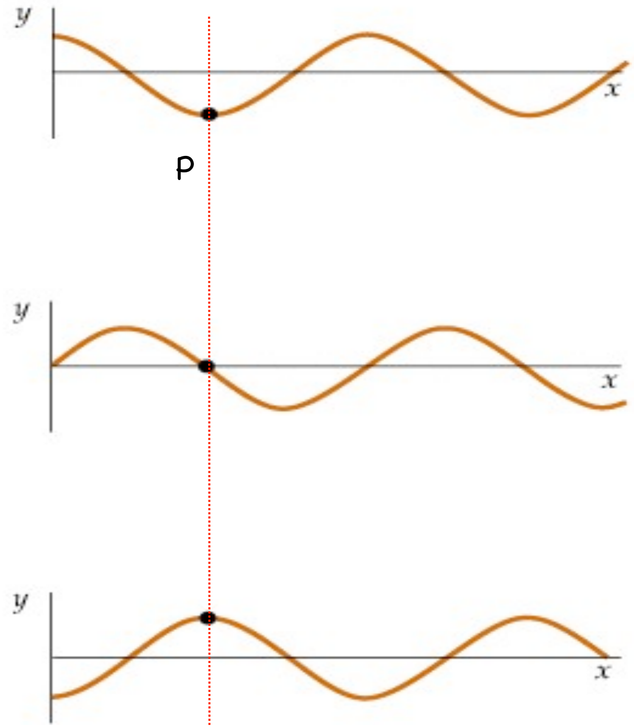
Harmonic Waves

Transverse acceleration a_y

$$a_y = \left.\frac{dv_y}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial v_y}{\partial t}$$

$$= -\omega^2 A \sin(kx - \omega t)$$

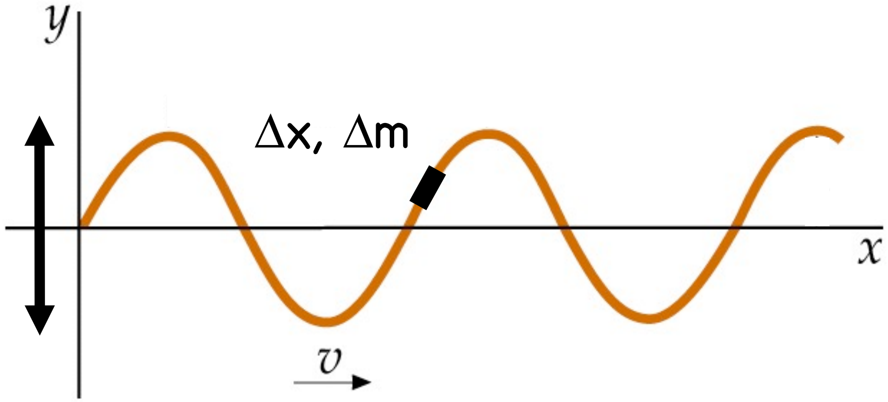


which has a maximum absolute value, $(a_y)_{max} = \omega^2 A$, when t=0

NB: x-coordinates of P are constant

Energy of waves on a string

Consider a harmonic wave travelling on a string.



Source of energy is an external agent on the left of the wave which does work in producing oscillations.

Consider a small segment, length Δx and mass Δm.

The segment moves vertically with SHM, frequency ω and amplitude A.

Generally

$$E = \frac{1}{2} m\omega^2 A^2$$

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If we apply this to our small segment, the total energy of the element is

$$\Delta E = \frac{1}{2} (\Delta m)\,\omega^2 A^2$$

If μ is the mass per unit length, then the element Δx has mass $\Delta m = \mu\ \Delta x$

$$\Delta E = \frac{1}{2} (\mu\,\Delta x)\,\omega^2 A^2$$

If the wave is travelling from left to right, the energy ΔE arises from the work done on element Δm_i by the element Δm_{i-1} (to the left).

Similarly Δm_i does work on element Δm_{i+1} (to the right) ie. energy is transmitted to the right.

The rate at which energy is transmitted along the string is the power and is given by dE/dt.

If Δx -> 0 then

$$\text{Power} = \frac{dE}{dt} = \frac{1}{2}(\mu\frac{dx}{dt})\omega^2 A^2$$

but dx/dt = speed

$$\therefore \quad \text{Power} = \frac{1}{2}\mu\,\omega^2 A^2 v$$

$$\text{Power} = \frac{1}{2}\mu\,\omega^2 A^2 v$$

Power transmitted on a harmonic wave is proportional to

(a) the wave speed v

(b) the square of the angular frequency ω

(c) the square of the amplitude A

All harmonic waves have the following general properties:

The power transmitted by any harmonic wave is proportional to the square of the frequency and to the square of the amplitude.

Wavefunction for a standing wave

Consider two sinusoidal waves in the same medium with the same amplitude, frequency and wavelength but travelling in opposite directions

$$y_1 = A_o \sin(kx - \omega t) \qquad\qquad y_2 = A_o \sin(kx + \omega t)$$

$$y = A_o \left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$$

Using the identity $\sin A + \sin B = 2\cos\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$

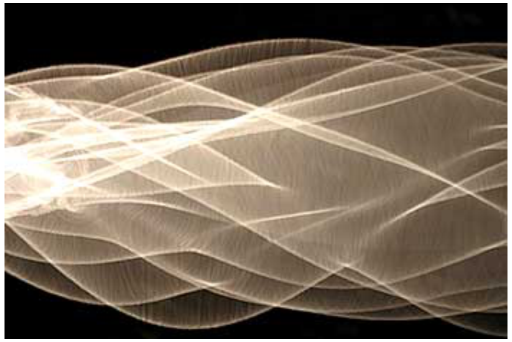
$$y = 2A_o \sin(kx)\cos(\omega t)$$

This is the wavefunction of a **standing** wave

Separation of variables

- A starting point to study differential equations is to guess solutions of a certain form (ansatz). Dealing with linear PDEs, the superposition principle principle guarantees that linear combinations of separated solutions will also satisfy both the equation and the homogeneous boundary conditions.
    - Separation of variables: a PDE of n variables ⇒ n ODEs
    - Solving the ODEs by BCs to get **normal modes** (solutions satisfying PDE and BCs).
- The proper choice of linear combination will allow for the initial conditions to be satisfied
- Determining exact solution (expansion coefficients of modes) by ICs

Separation of variables: string



$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

and if it has separable solutions:

$$y(x,t) = X(x)T(t)$$

$$\frac{d^2X(x)}{dx^2} + k^2X(x) = 0 \qquad T''(t) + c^2k^2T(t) = 0$$

$$X(x) = A\cos(kx) + B\sin(kx) \qquad T(t) = C\cos(\omega t) + D\sin(\omega t)$$

$\omega = ck$

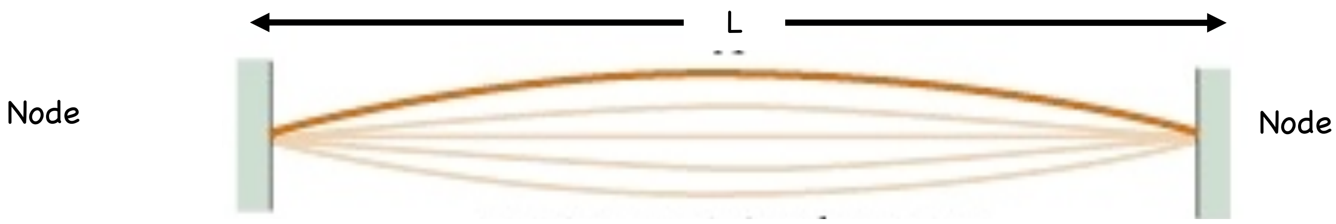
To be determined by **initial** and **boundary** conditions

Standing waves in a string fixed at both ends

Consider a string of length L and fixed at both ends

The string has a number of natural patterns of vibration called **NORMAL MODES**

Each normal mode has a characteristic frequency which we can easily calculate



When the string is displaced at its mid point the centre of the string becomes an antinode.

Standing waves in a string fixed at both ends

String is fixed at both ends $\therefore$ $y(x,t) = 0$ at $x = 0$ and L

$y(0,t)=0$ when $x = 0$ as $\sin(kx) = 0$ at $x = 0$

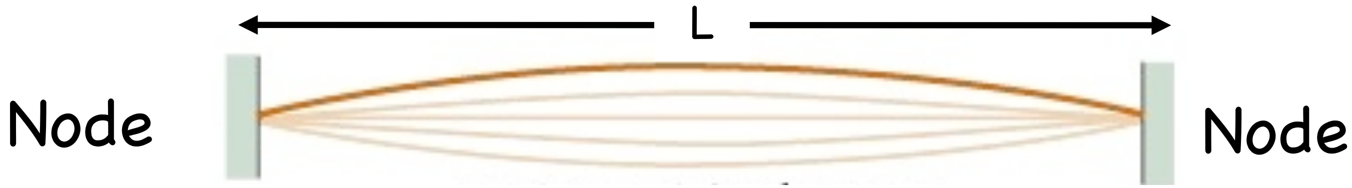
$$y(x,t) = 2A_0 \sin(kx)\cos(\omega t)$$

$y(L,t) = 0$ when $\sin(kL) = 0$ ie $k_n L = n\pi$ $n=1,2,3....$

but $k_n = 2\pi / \lambda$ $\therefore (2\pi / \lambda_n)L = n\pi$ or $\boxed{\lambda_n = 2L/n}$

Standing waves in a string fixed at both ends

For first normal mode $L = \lambda_1 / 2$



The next normal mode occurs when the length of the string L = one wavelength, i.e. $L = \lambda_2$

The third normal mode occurs when $L = 3\lambda_3 /2$

Generally normal modes occur when $L = n\lambda_n /2$

ie $\lambda_n = \frac{2L}{n}$ where $n = 1,2,3.......$

Standing waves in a string fixed at both ends

The natural frequencies associated with these modes can be derived from $f = v/\lambda$

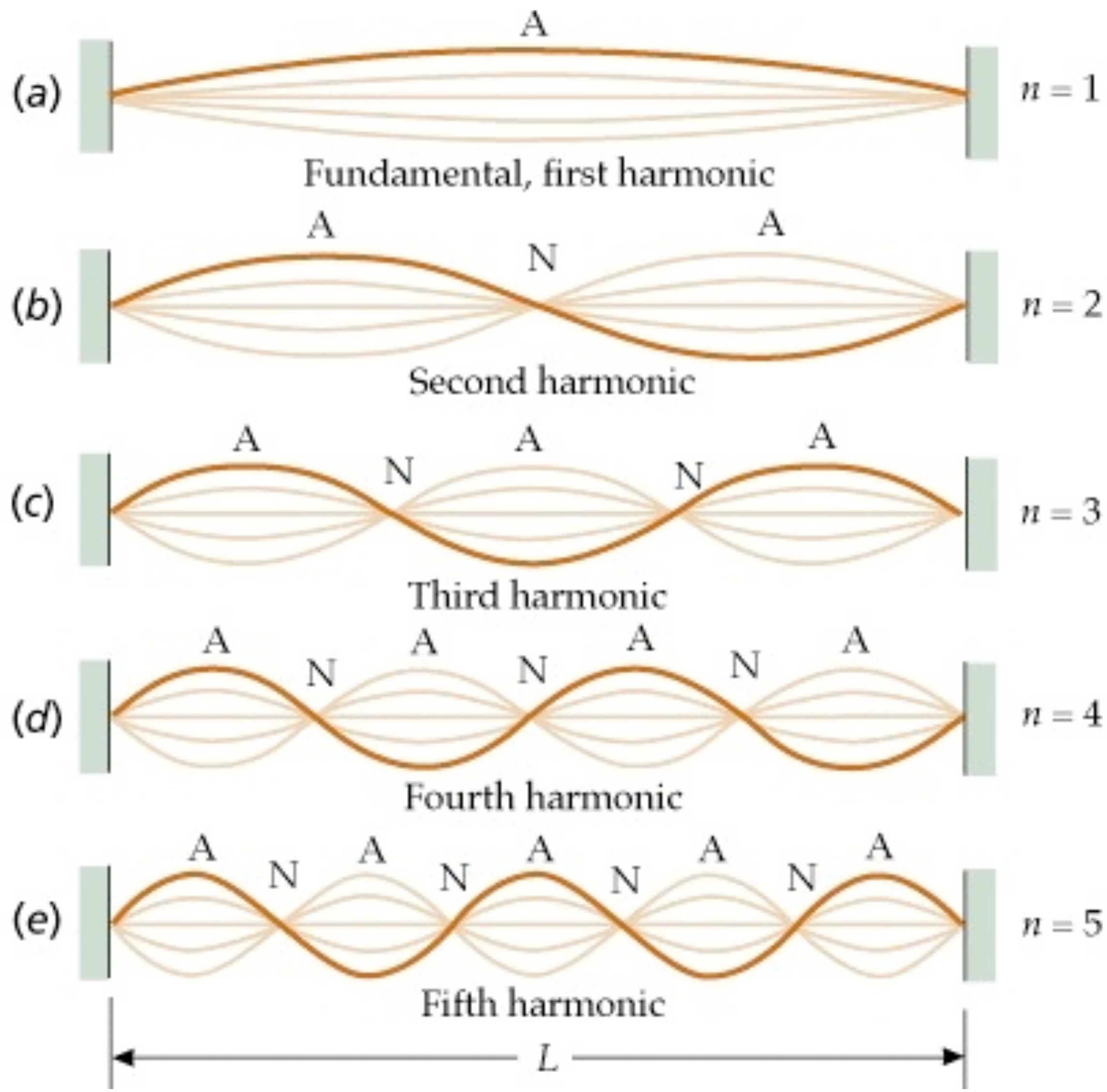
$$f = \frac{v}{\lambda} = \frac{n}{2L}v \quad \text{with } n = 1,2,3.....$$

For a string of mass/unit length μ, under tension F we can replace v by $(F/\mu)^{1/2}$

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad \text{with } n = 1,2,3.....$$

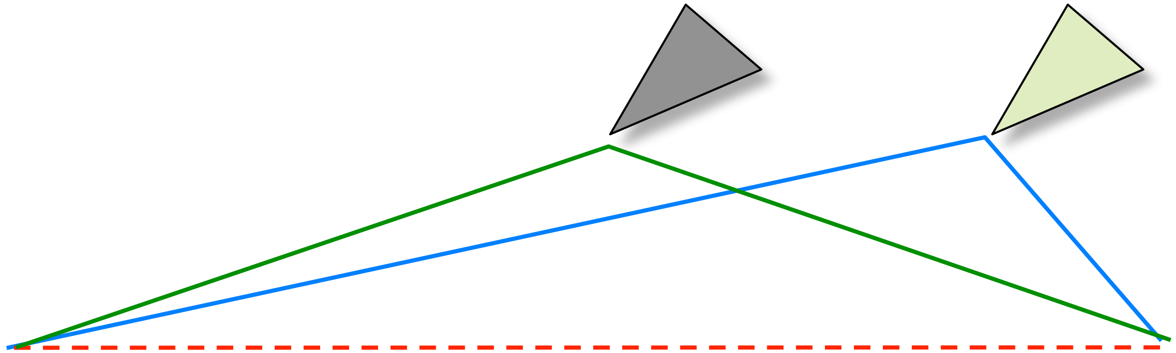
The lowest frequency (**fundamental**) corresponds to n = 1

ie $f = \frac{1}{2L}v$ or $f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$


A
(a)
n = 1
Fundamental, first harmonic
A
N
A
(b)
n = 2
Second harmonic
A
N
A
N
A
(c)
n = 3
Third harmonic
A
N
A
N
A
N
A
(d)
n = 4
Fourth harmonic
A
N
A
N
A
N
A
N
A
(e)
n = 5
Fifth harmonic
L

Plucked string

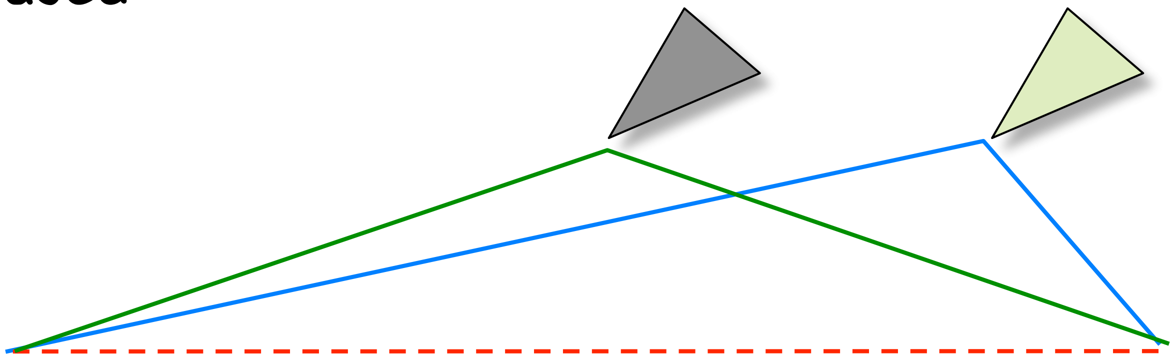
Can one predict the amplitude of each mode (overtone/harmonic?) following plucking?



Using the procedure to measure the Fourier coefficients it is possible to predict the amplitude of each harmonic tone.

Initial conditions

- You know the shape just before it is plucked.
- You know that each mode moves at its own frequency
- The shape when released
- We rewrite this as



$$\text{shape} = f(x, t = 0)$$

$$f(x, t = 0) = \sum_n A_n \sin(k_n x)$$

The plucked string (continued)

Each harmonic has its own frequency of oscillation, the m-th harmonic moves at a frequency $f_m = m f_0$ or m times that of the fundamental mode.

$$f(x, t = 0) = \sum_n A_n \sin(k_n x)$$

$$f(x, t) = \sum_n A_n \sin(k_n x) \cos(\omega_n t)$$

http://www.falstad.com/loadedstring/

Modal summation on a string

Recall modes on a string:

$$u(x, t) = \sum_{n=0}^{\infty} A_n U_n(x, \omega_n) \cos(\omega_n t)$$

This is the sum of standing waves or *eigenfunctions,* $U_n(x, \omega_n)$, each of which is weighted by the amplitude A_n and vibrates at its *eigenfrequency* ω_n.

The eigenfunctions and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

$$U_n(x, \omega_n) = \sin(n\pi x/L) = \sin(\omega_n x/\mathrm{v}) \qquad \omega_n = n\pi \mathrm{v}/L = 2\pi \mathrm{v}/\lambda$$

Source excitation

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\, F(\omega_n)\, \sin(n\pi x/L)\, \cos(\omega_n t)$$

The source, at $x_s = 8$, is described by

$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

with $\tau = 0.2$.