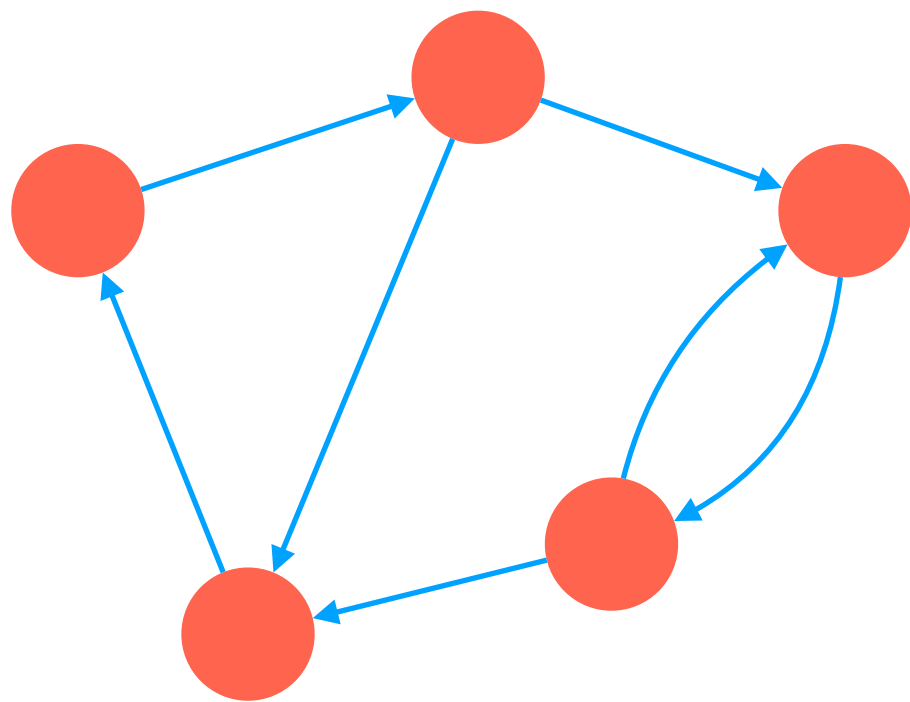


Computabilità, Complessità e Logica

Lezione 14

3-SAT \leq_P Ciclo hamiltoniano



Dato un grafo $G = (V, E)$ con $|V| = n$ vertici

Un ciclo hamiltoniano è una sequenza di $n + 1$ vertici $v_0, v_1, \dots, v_{n-1}, v_n = v_0$ tale per cui

1. Tutti i vertici sono nel ciclo e, escluso il primo, appaiono tutti una e una sola volta
2. $(v_i, v_{i+1}) \in E$ per ogni $0 \leq i < n$
i.e., due vertici consecutivi sono collegati da un arco

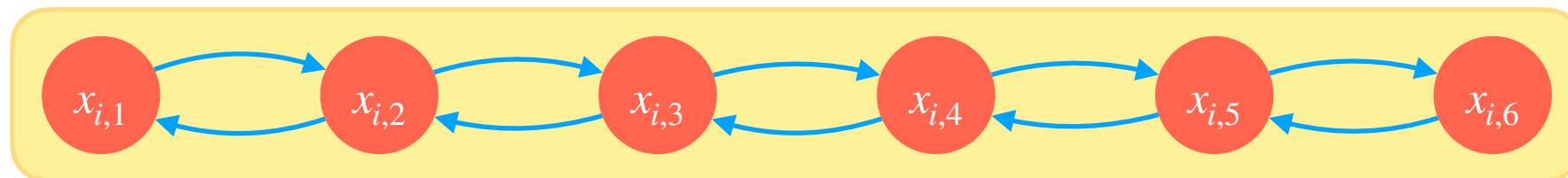
Il linguaggio **HAM** è dato da tutti i grafi per i quali esiste un ciclo hamiltoniano

3-SAT \leq_P Ciclo hamiltoniano

Data la seguente formula in 3-CNF:

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$$

Per ogni variabile x_i costruiamo un grafo con $2m$ nodi (dove m è il numero di clausole della formula) etichettati $x_{i,1}, x_{i,2}, \dots, x_{i,2m}$



In questo caso abbiamo **sei** vertici dato che la formula ha **tre** clausole.

Gli archi in questo grafo saranno:

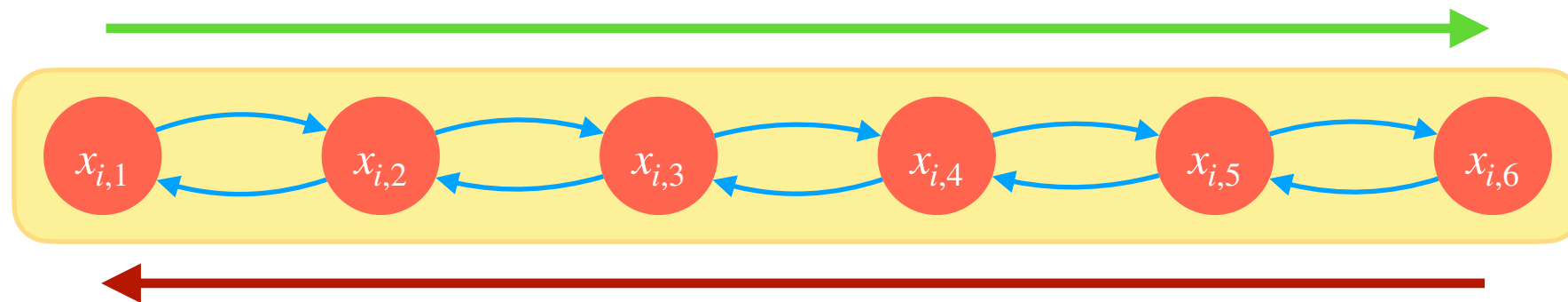
$$(x_{i,j}, x_{i,j+1}) \in E \text{ per } 1 \leq j \leq m - 1$$

$$\text{e } (x_{i,j}, x_{i,j-1}) \in E \text{ per } 2 \leq j \leq m$$

3-SAT \leq_P Ciclo hamiltoniano

Sfruttiamo la direzione di visita per codificare gli assegnamenti

Questa direzione di visita indica che
l'assegnamento di questa variabile è a **true**

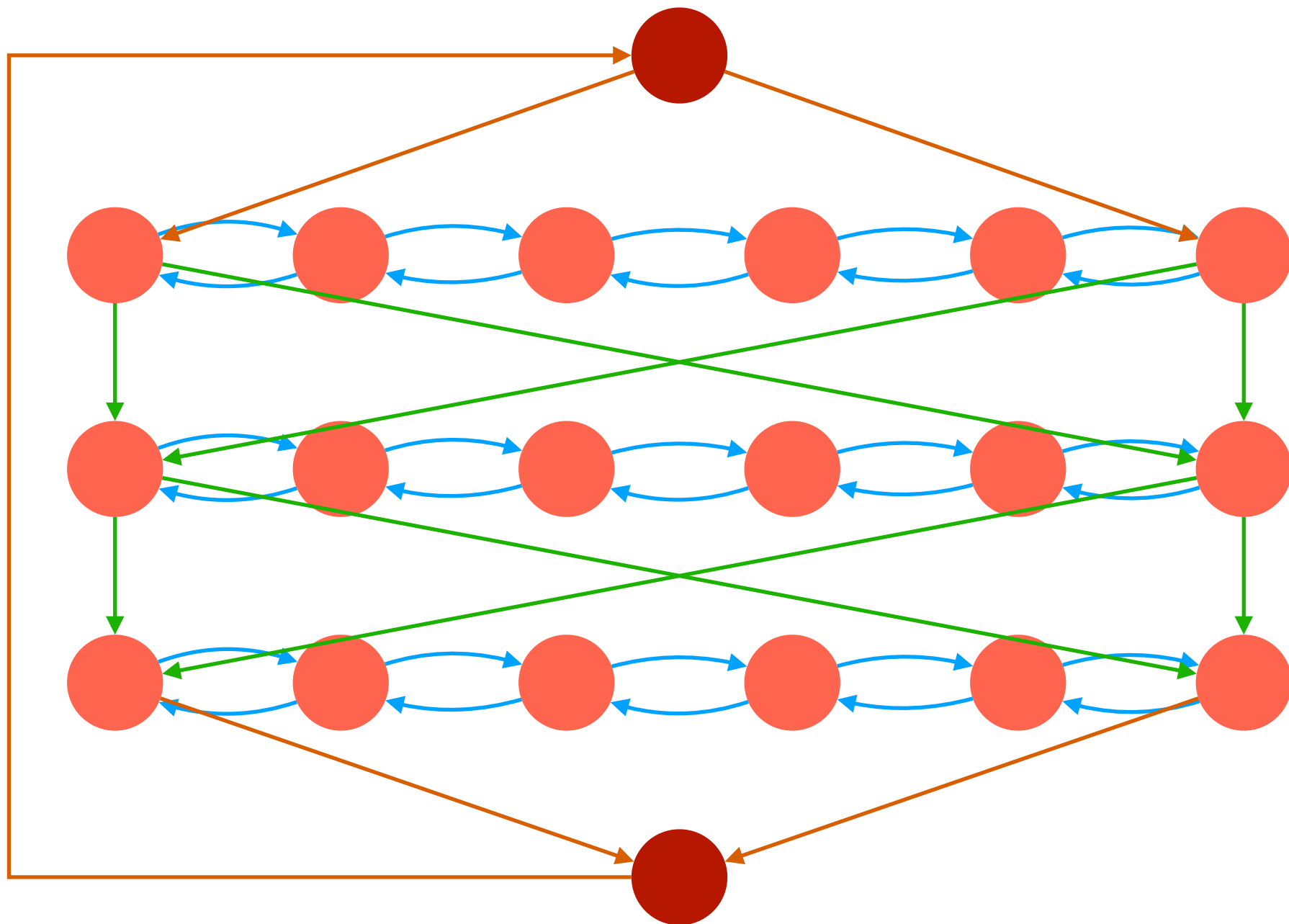


Questa direzione di visita indica che
l'assegnamento di questa variabile è a **false**

Notate che per ottenere un ciclo hamiltoniano non è possibile “cambiare direzione” durante una visita dei nodi (si passerebbe due volte per lo stesso nodo)

3-SAT \leq_P Ciclo hamiltoniano

Collegiamo i grafi delle diverse variabili in modo che arrivati "alla fine" sia possibile scegliere se visitare la variabile successiva da sinistra a destra o da destra a sinistra. Questo ci indica che valore assumerà la variabile



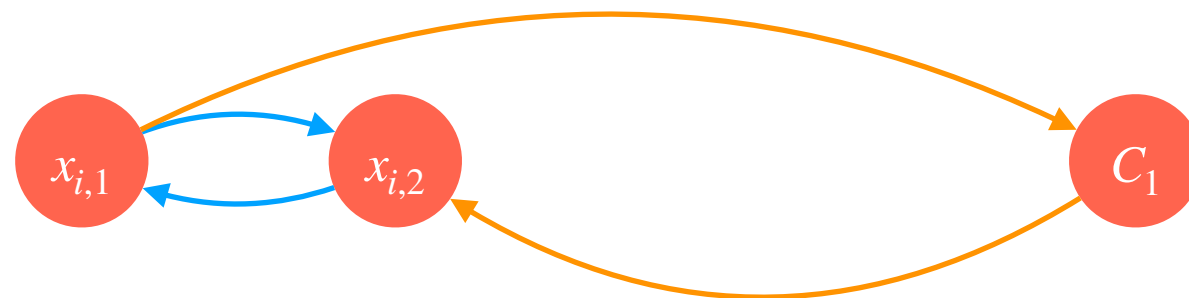
3-SAT \leq_P Ciclo hamiltoniano

Data la seguente formula in 3-CNF:

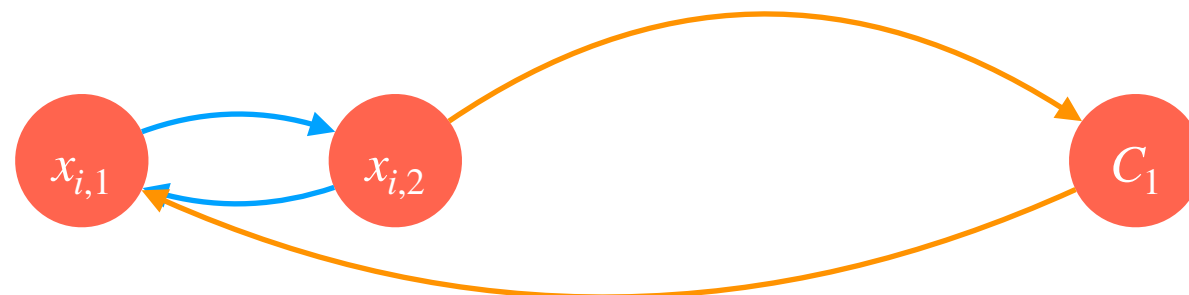
$$(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$$

Aggiungiamo un nodo per ogni clausola. Nell'esempio C_1, C_2, C_3

Se l' i -esima variabile appare non negata nella clausola colleghiamo in questo modo:

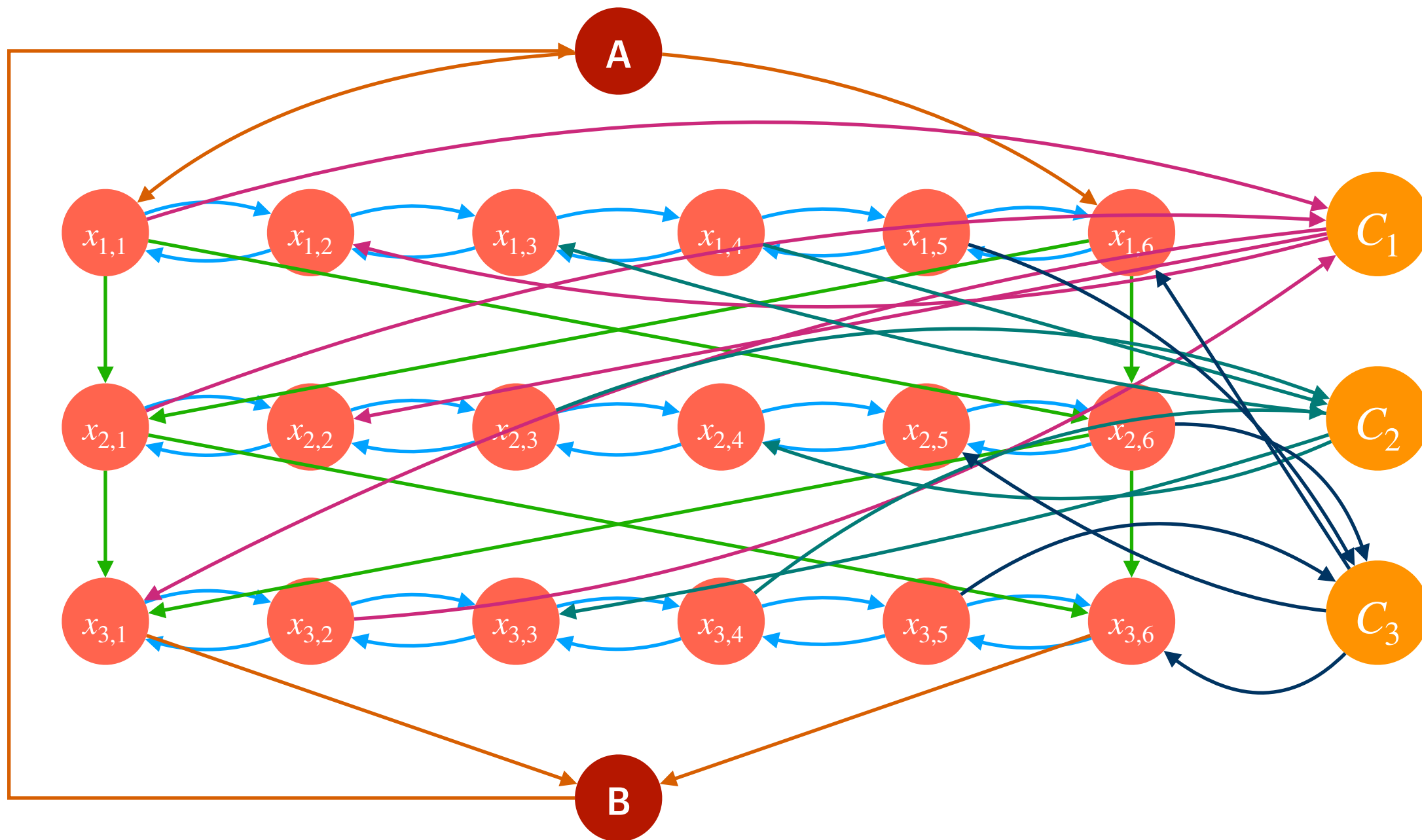


Altrimenti se l' i -esima variabile appare negata colleghiamo in direzione opposta:



3-SAT \leq_P Ciclo hamiltoniano

Ne segue che la formula $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee x_3)$ è codificata dal seguente grafo:

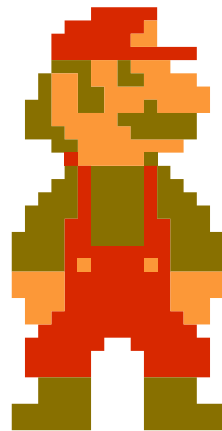


Super Mario e riduzioni

- Mostriamo che un problema legato a “Super Mario” è NP-hard
- Per questo mostriamo come codificare una formula in 3-CNF all’interno di un livello del gioco
- Le slide seguenti sono adattate da quelle preparate da Antonio E. Porreca, McF all’Università Aix-Marseille (Francia)
- <https://aeporreca.org>



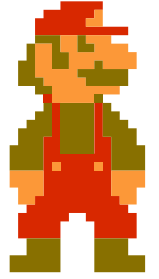
**Il gioco di
Super Mario
è NP-hard!**



Riferimenti

- Greg Aloupisa, Erik D. Demaine, Alan Guob, Giovanni Viglietta
Classic Nintendo games are (computationally) hard,
Theoretical Computer Science
doi.org/10.1016/j.tcs.2015.02.037
(scaricabile da arxiv.org/abs/1203.1895)
- Cory Chang (aka Undefined Behavior)
What Makes Mario NP-Hard? (Polynomial Reductions)
youtu.be/oS8m9fSk-Wk

Definizione del problema



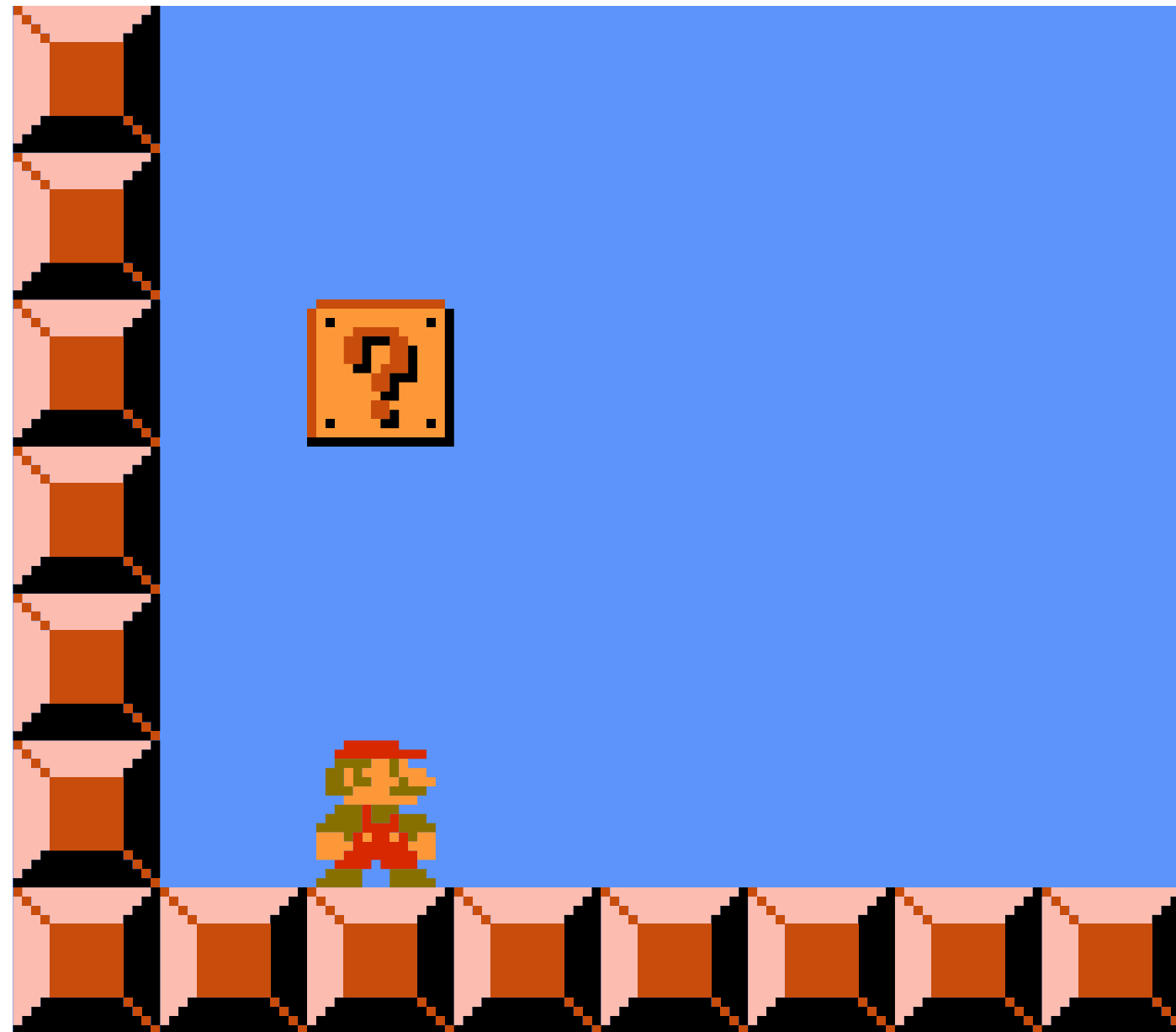
- **Input:** un livello di Super Mario
- **Problema di decisione:**
è possibile arrivare alla fine del livello?

3-SAT

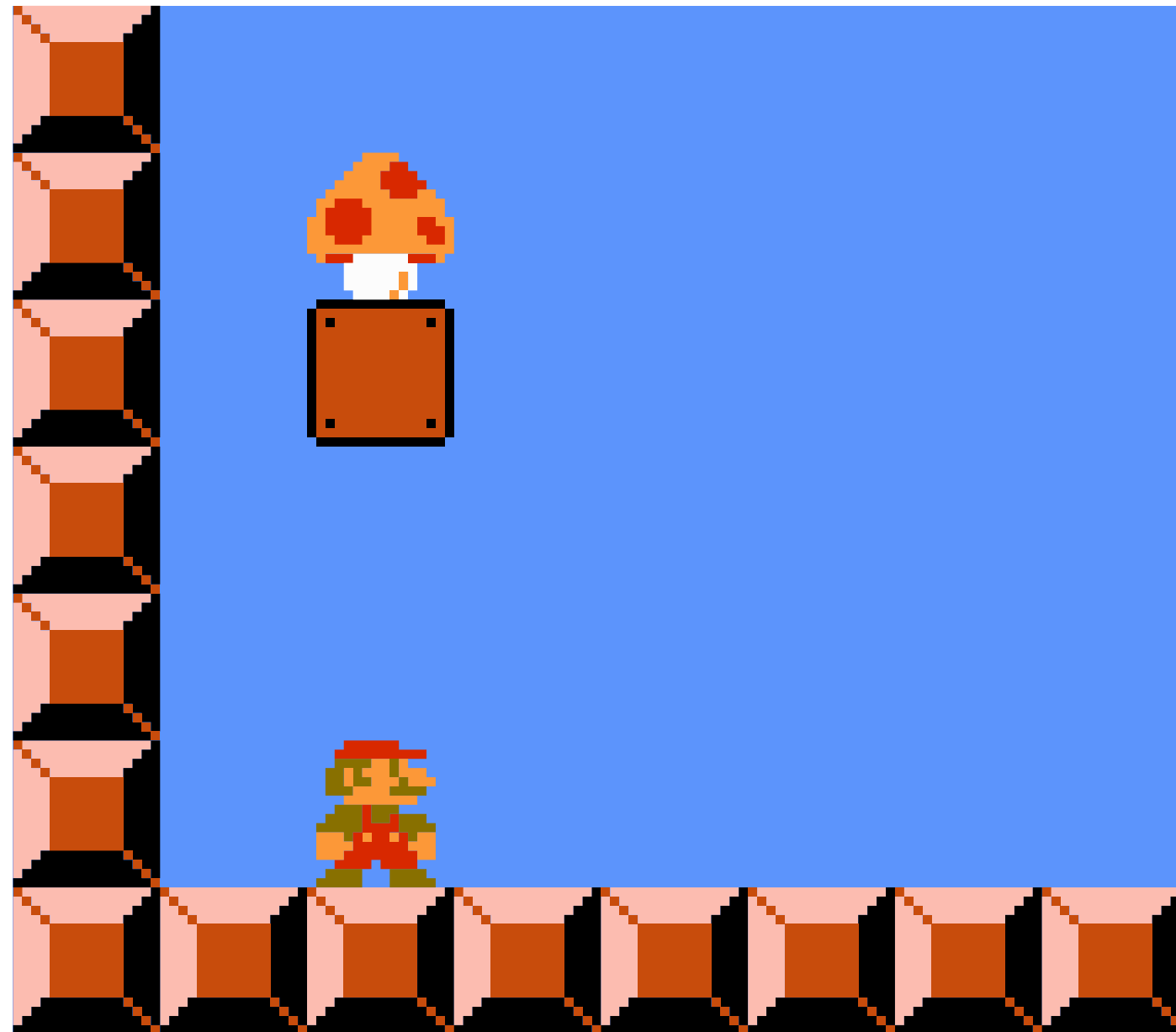
\leq_P



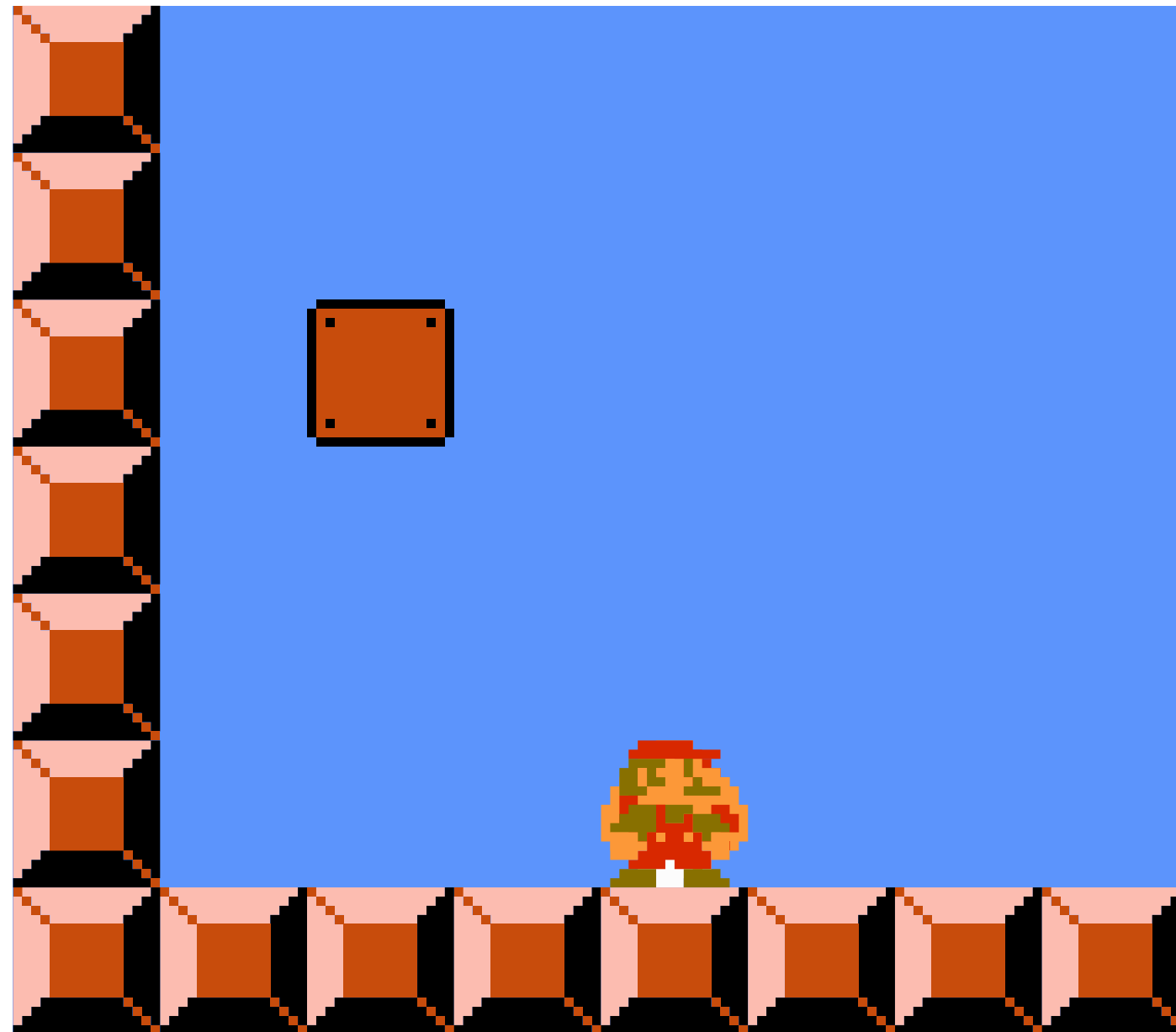
Gadget per la partenza



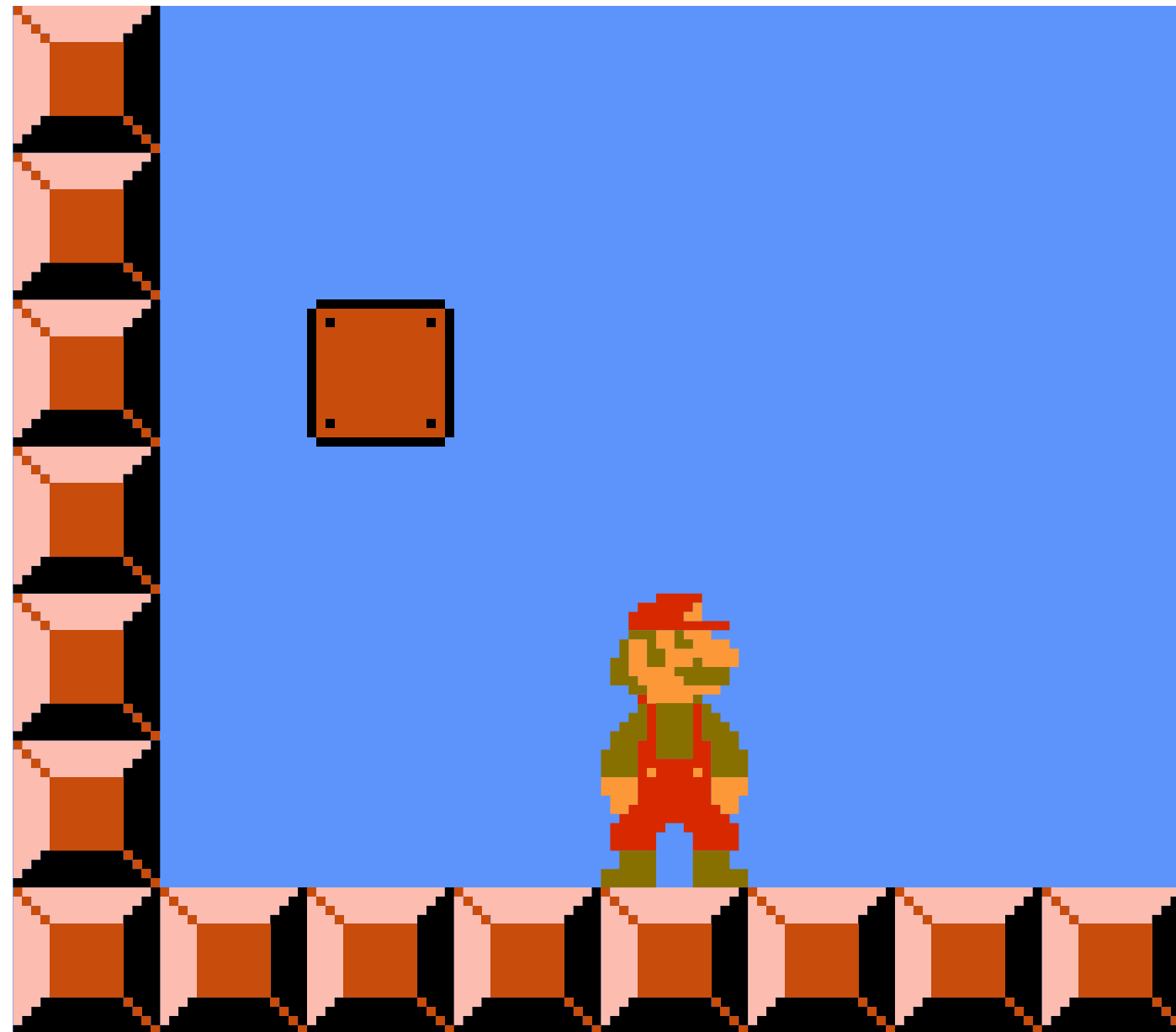
Gadget per la partenza



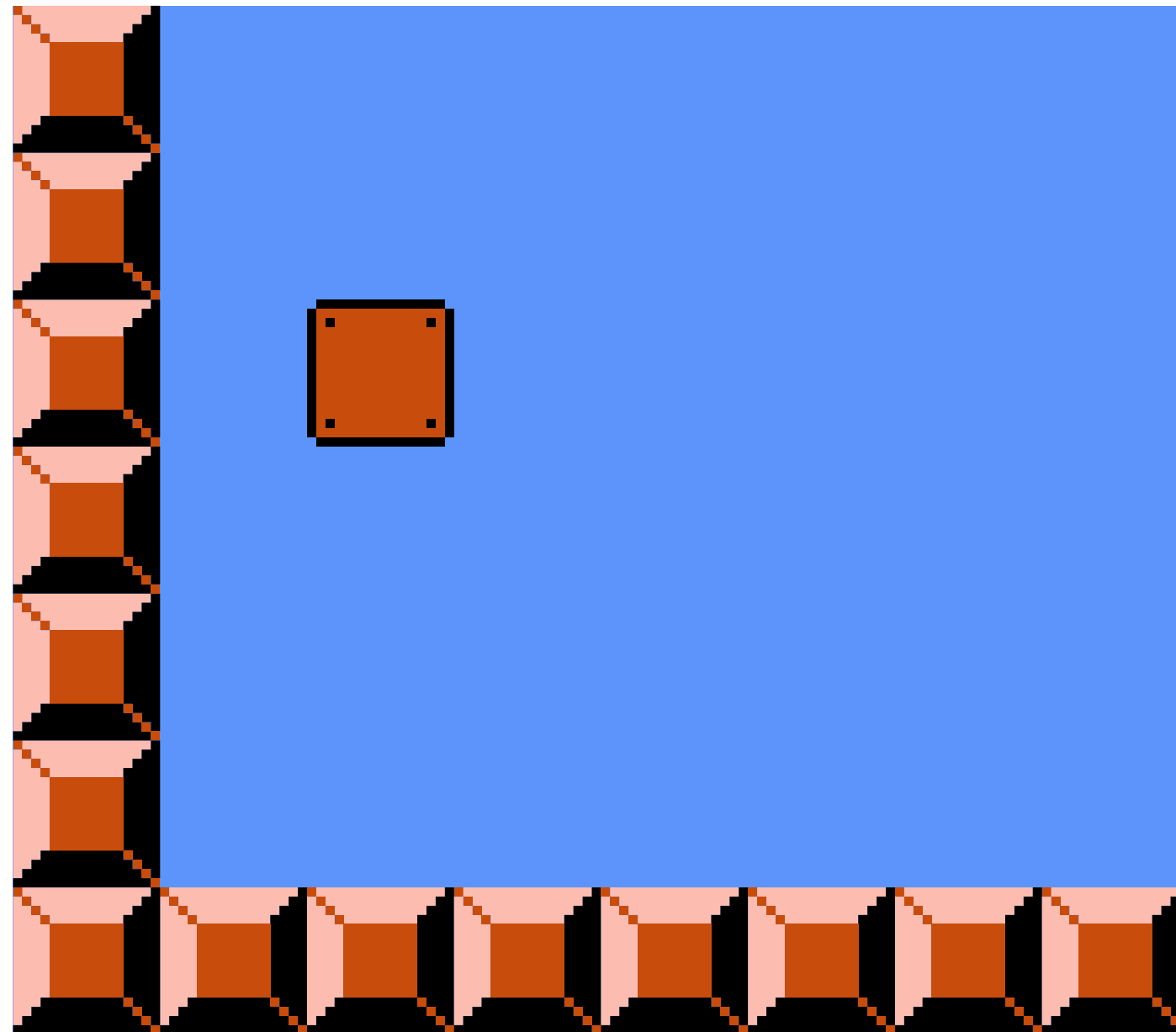
Gadget per la partenza



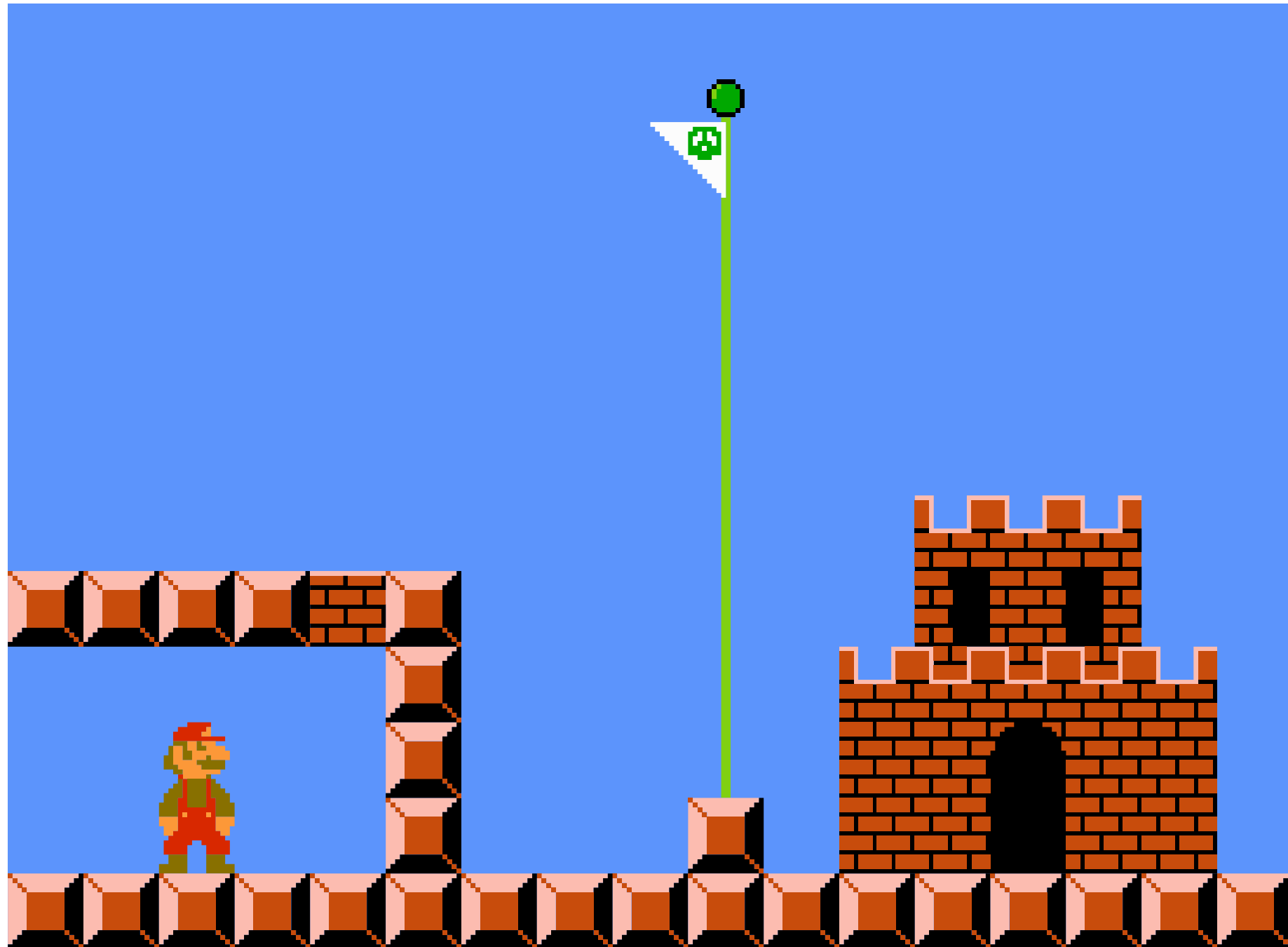
Gadget per la partenza



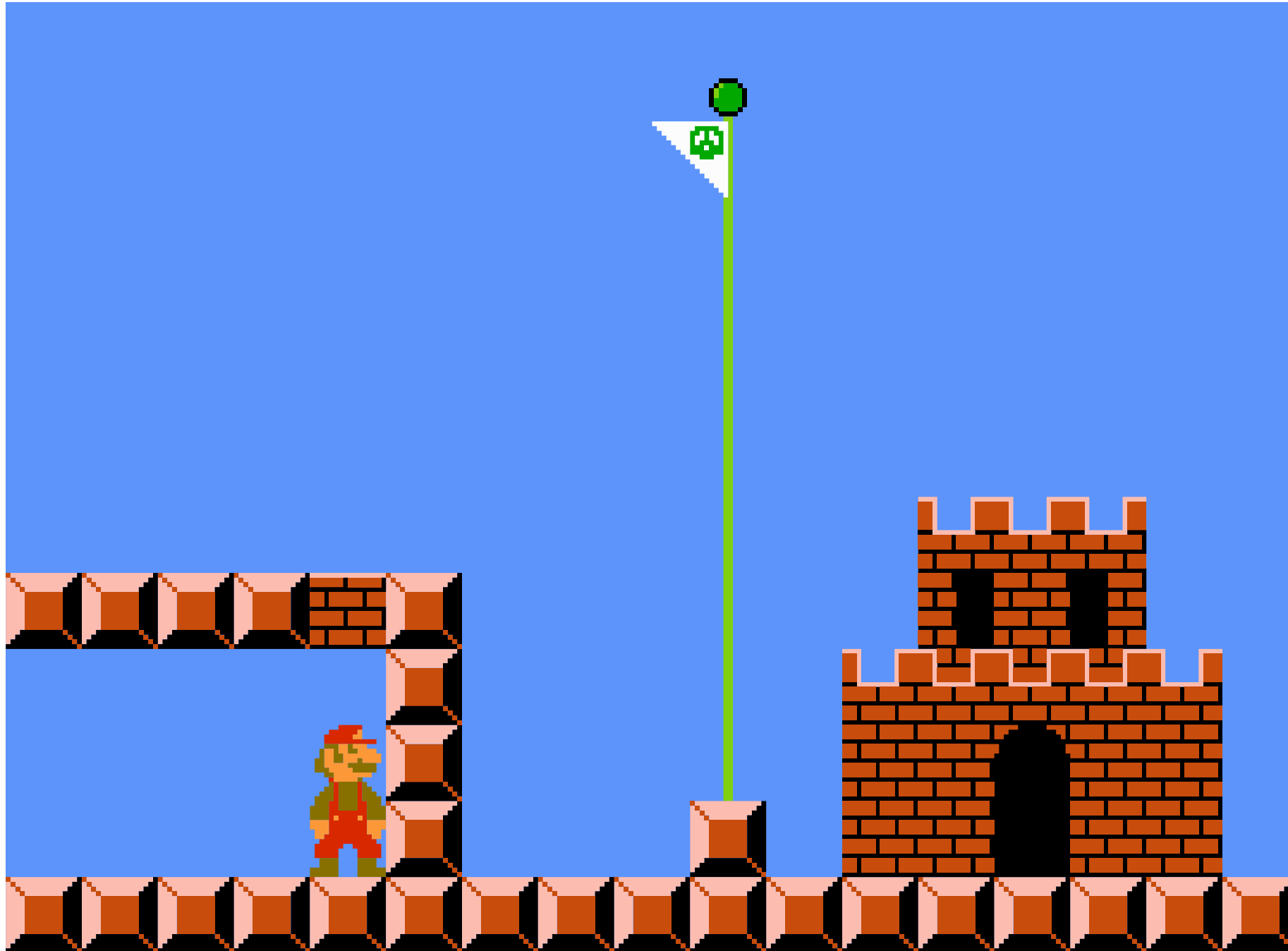
Gadget per la partenza



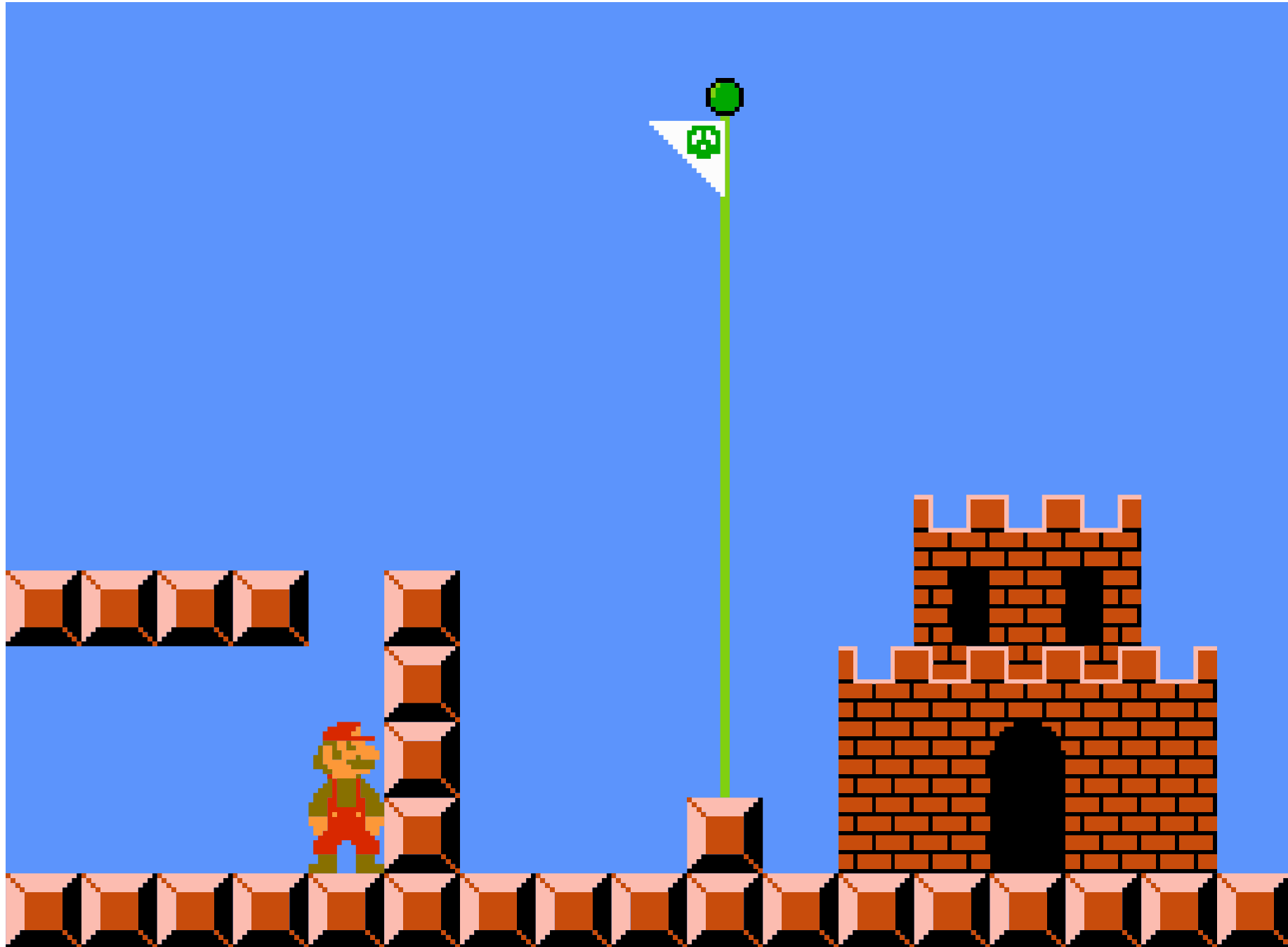
Gadget per l'arrivo



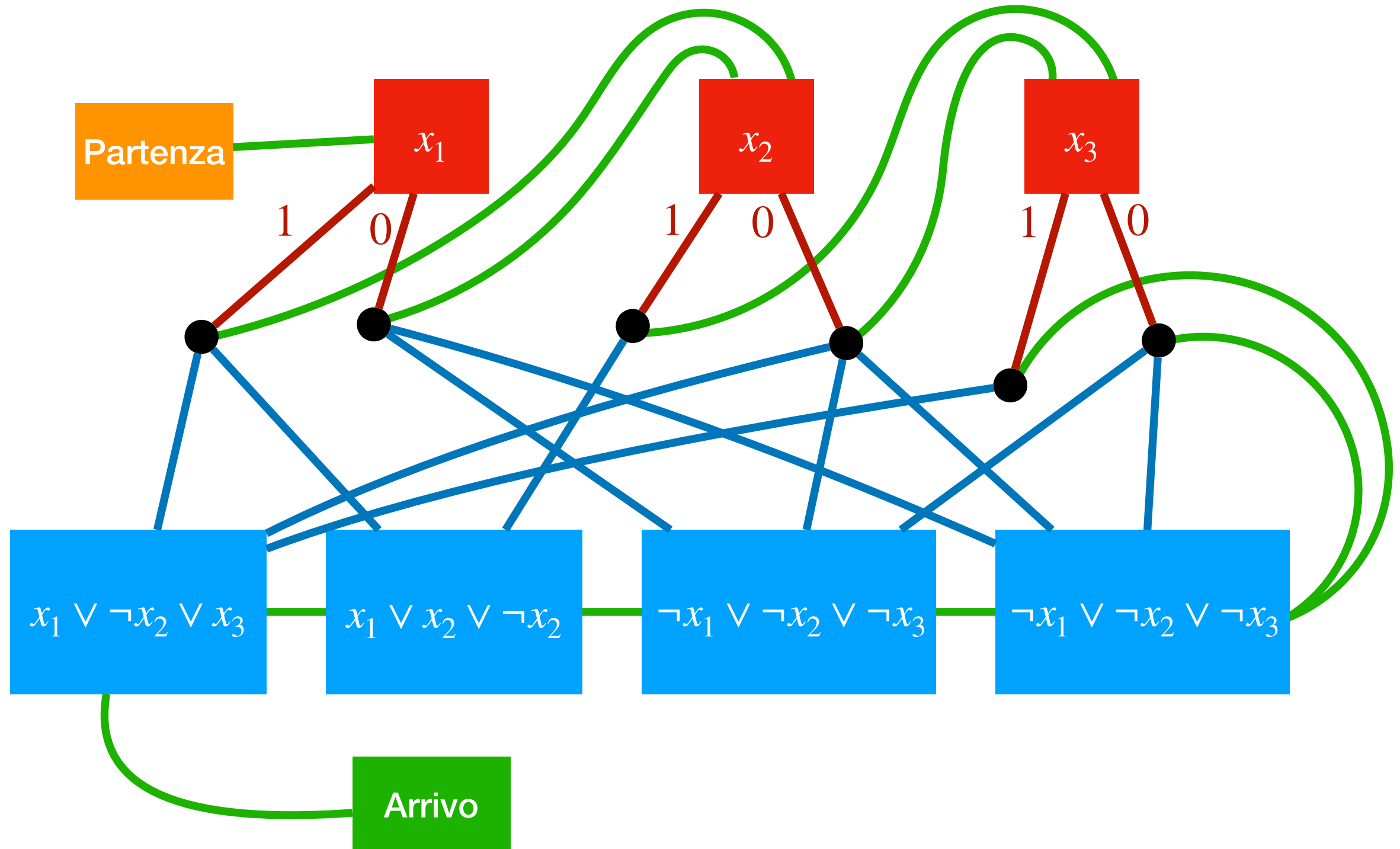
Gadget d'arrivée



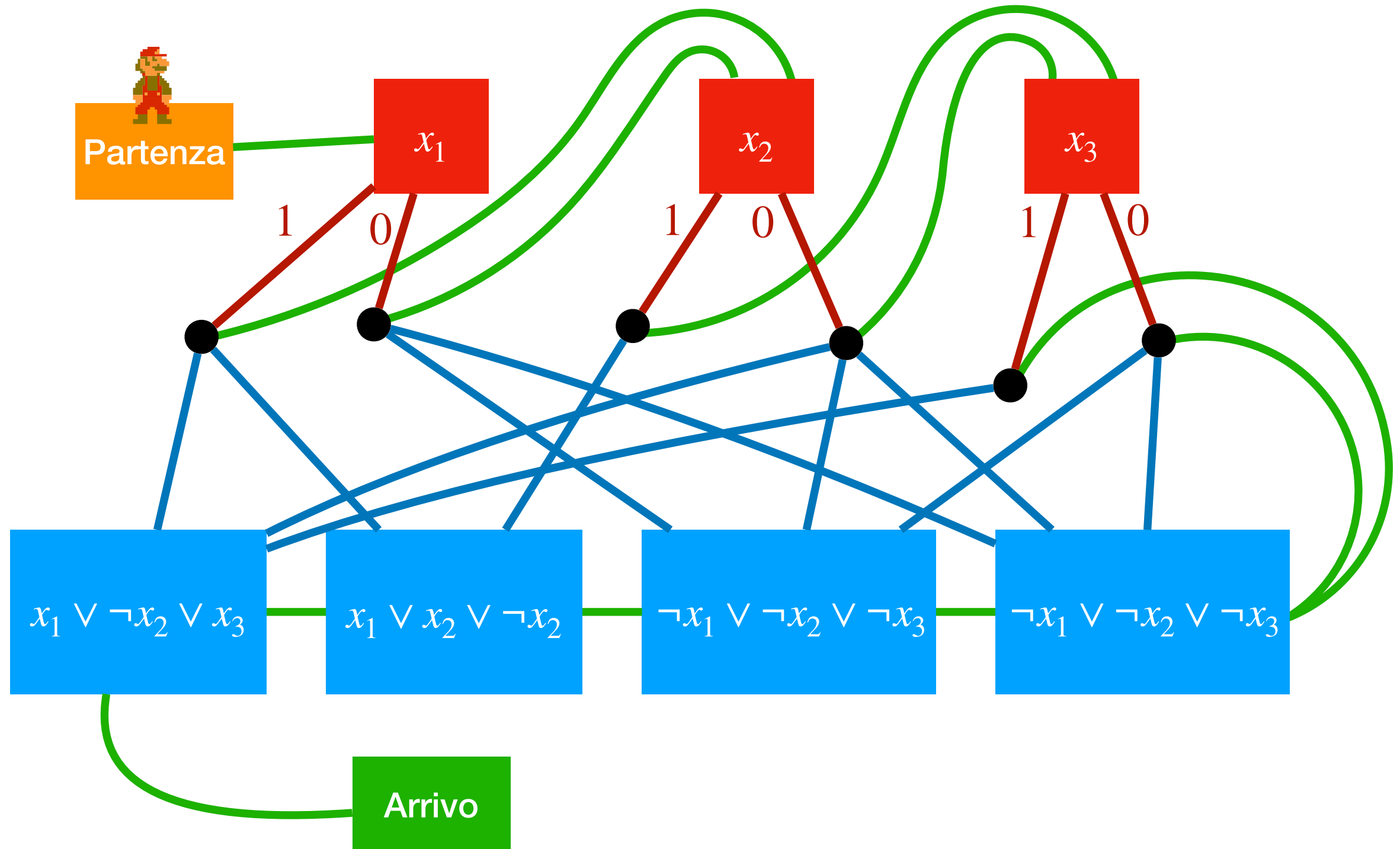
Gadget d'arrivée



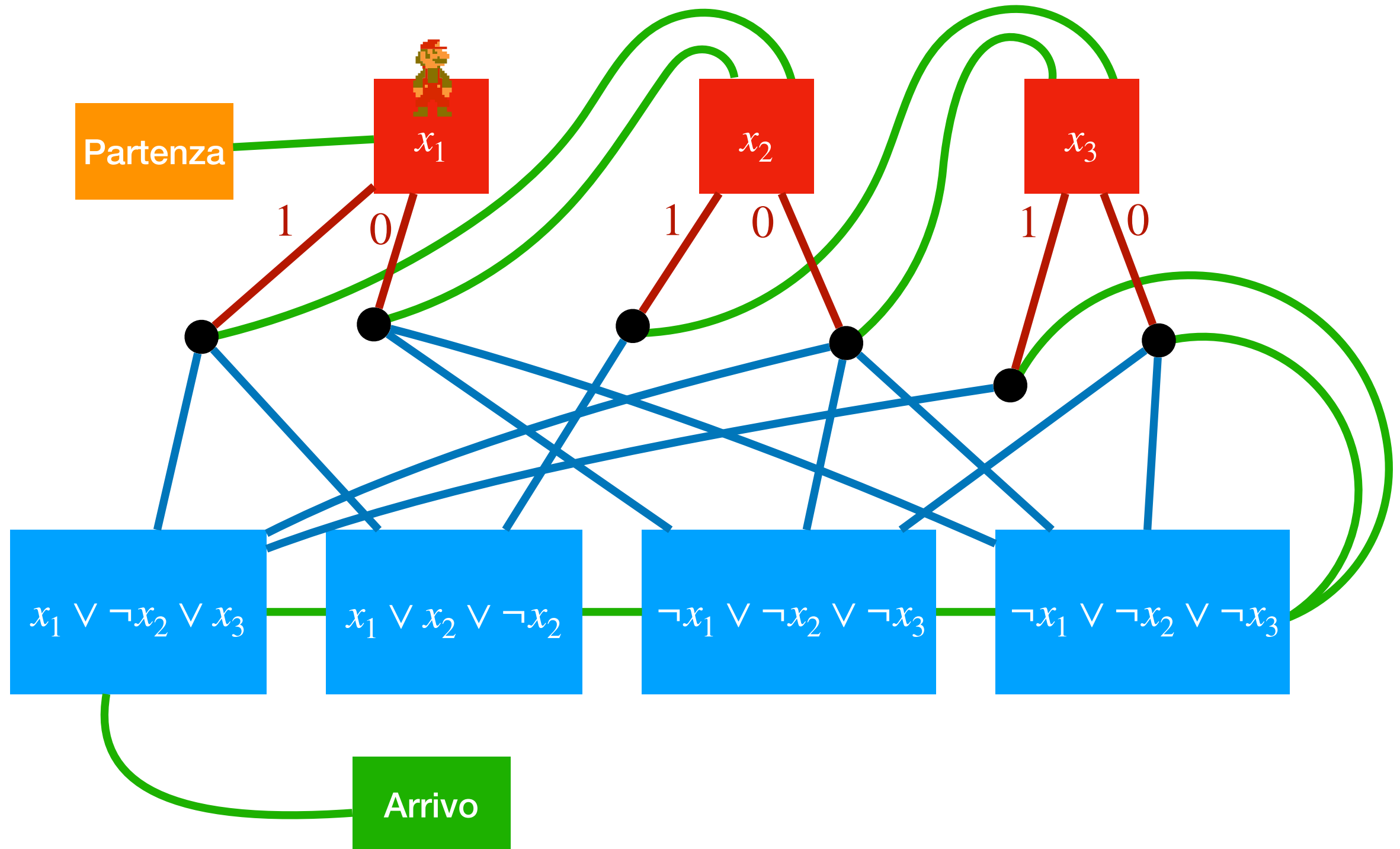
Percorso di Mario



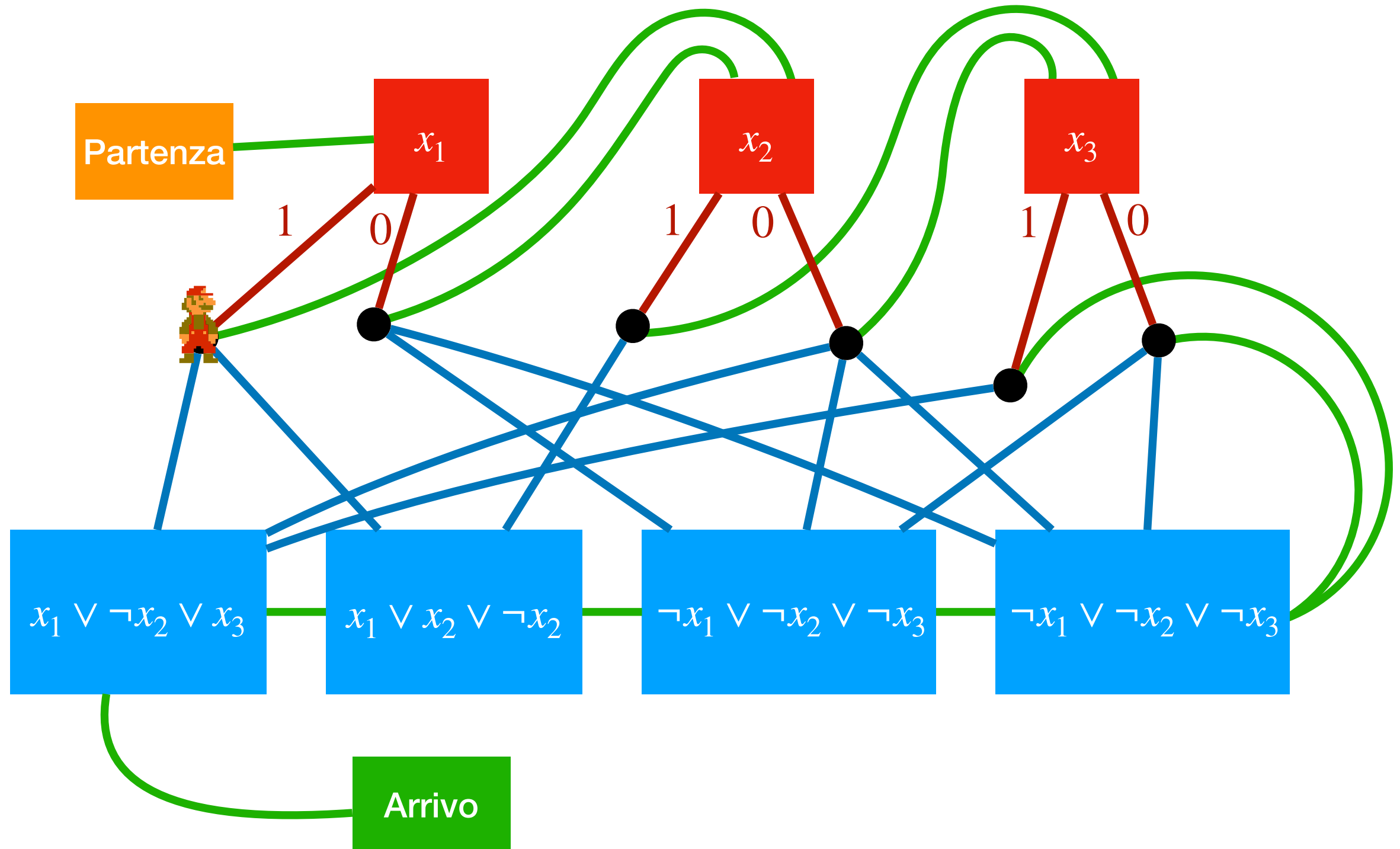
Percorso di Mario



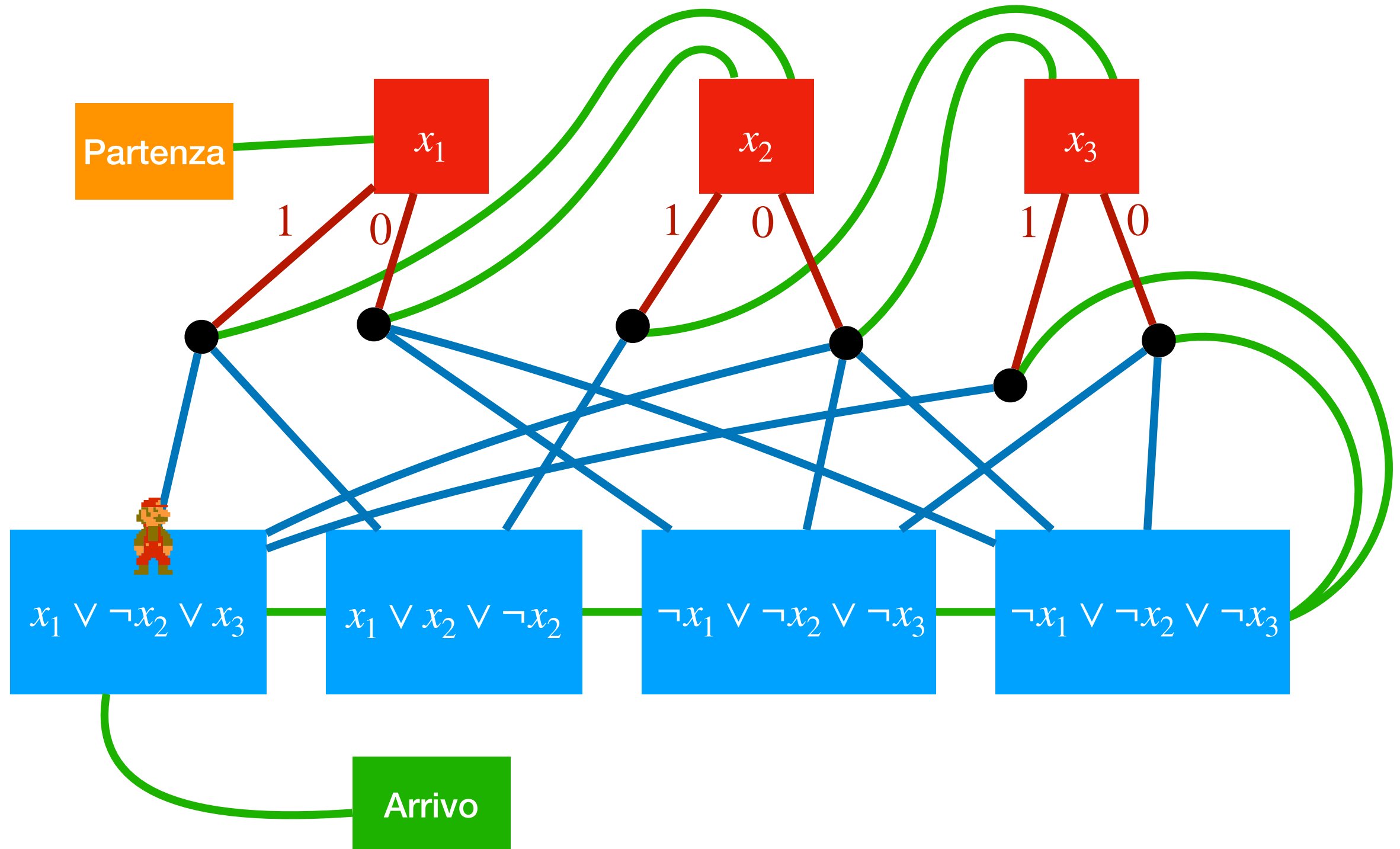
Percorso di Mario



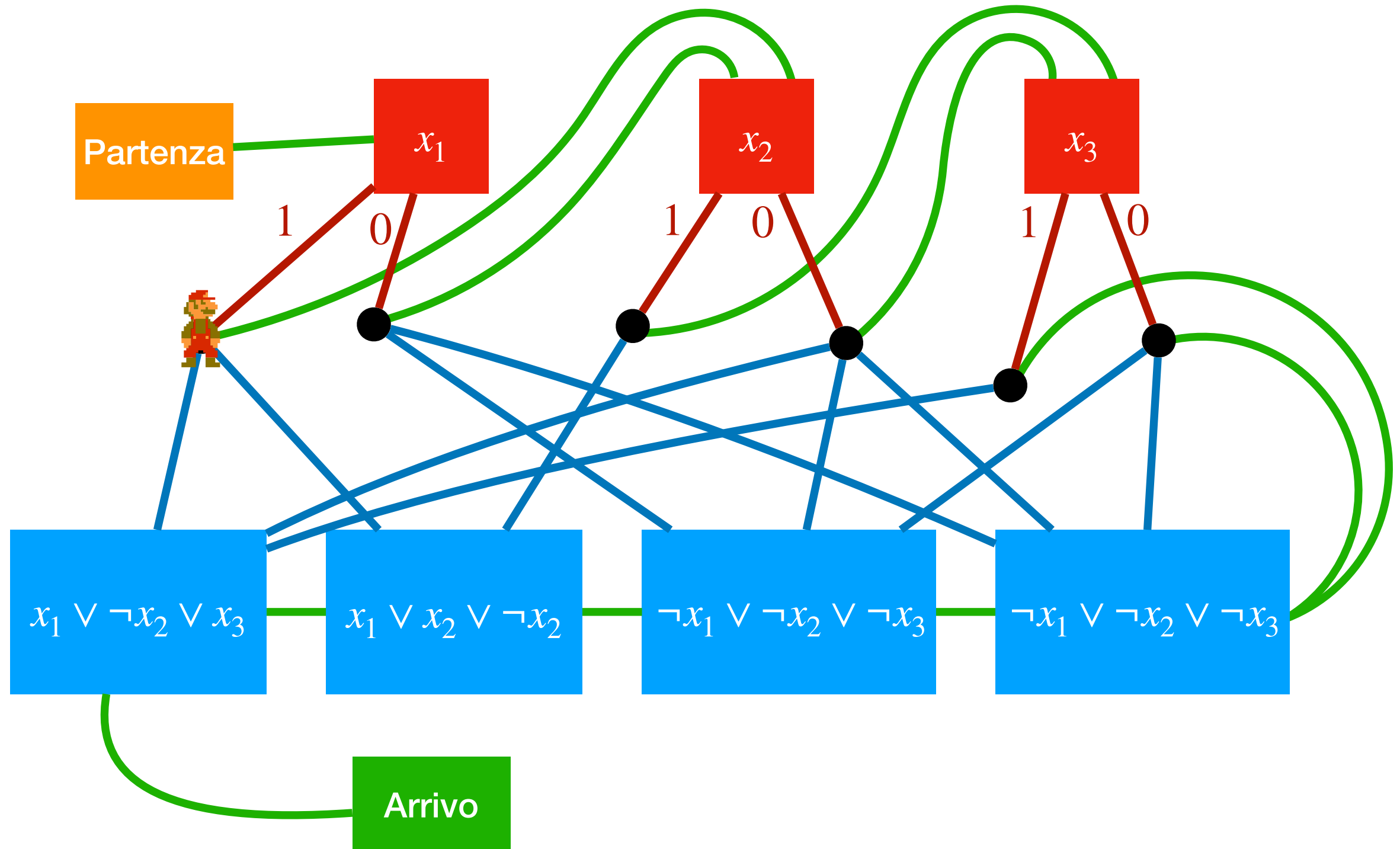
Percorso di Mario



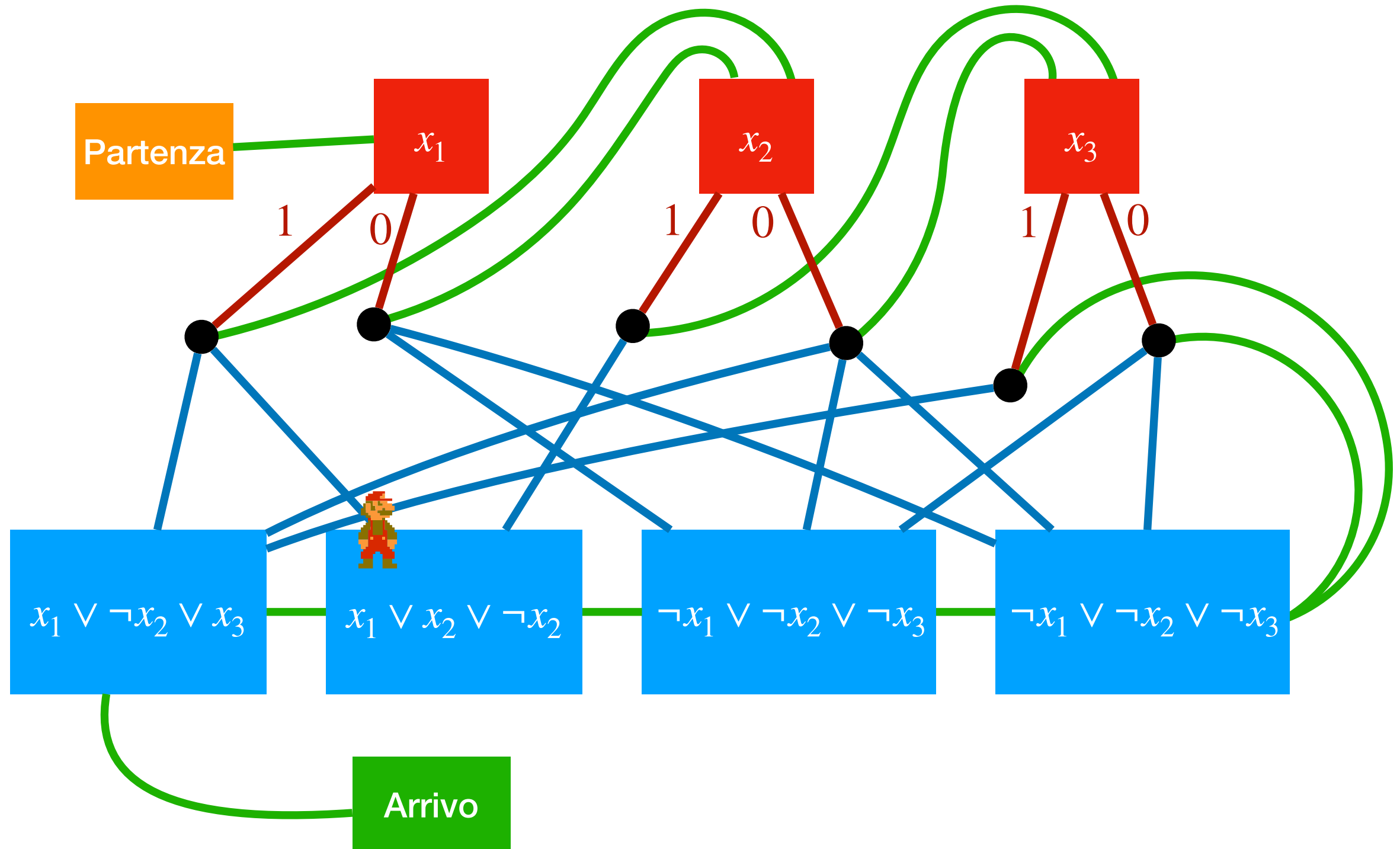
Percorso di Mario



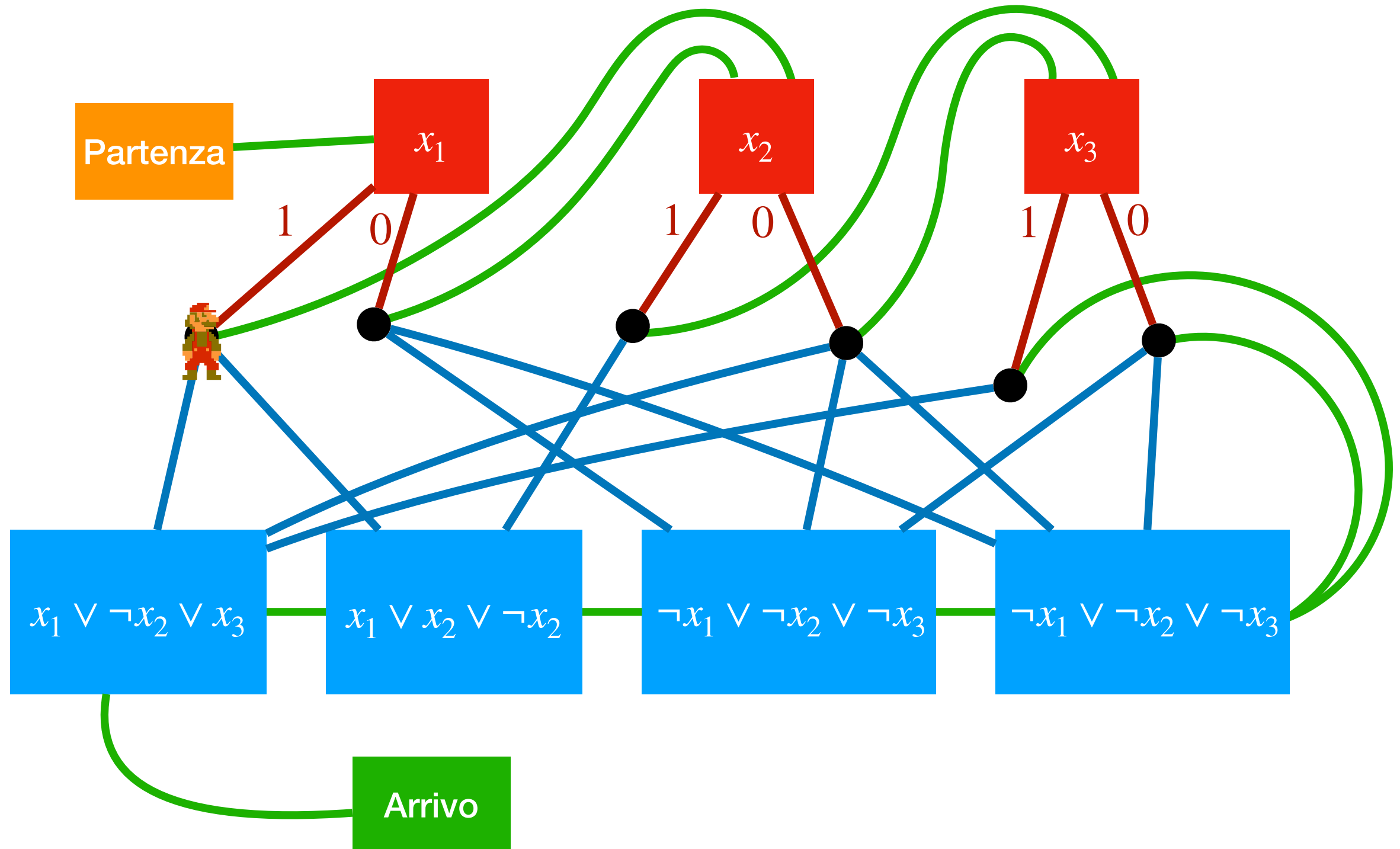
Percorso di Mario



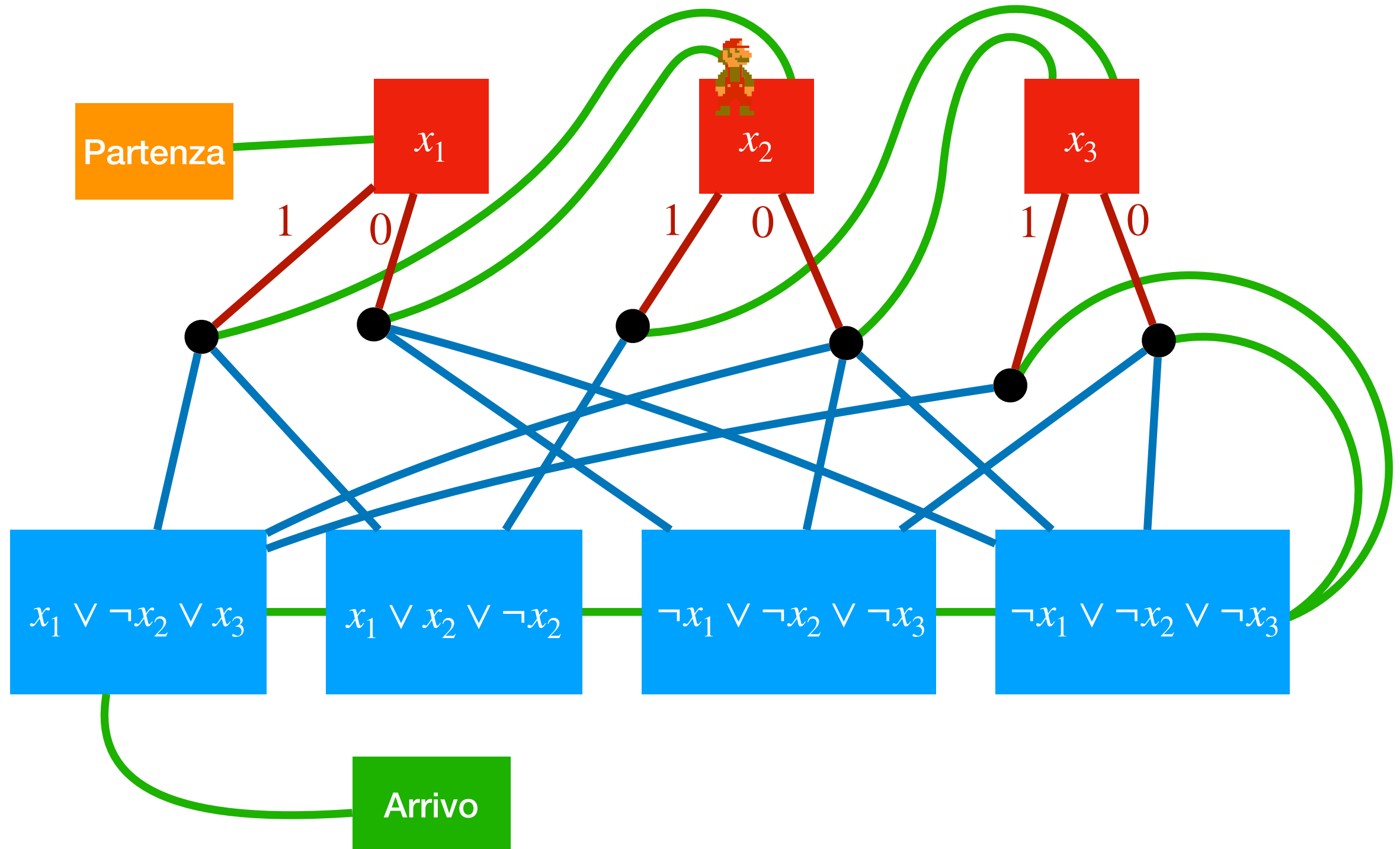
Percorso di Mario



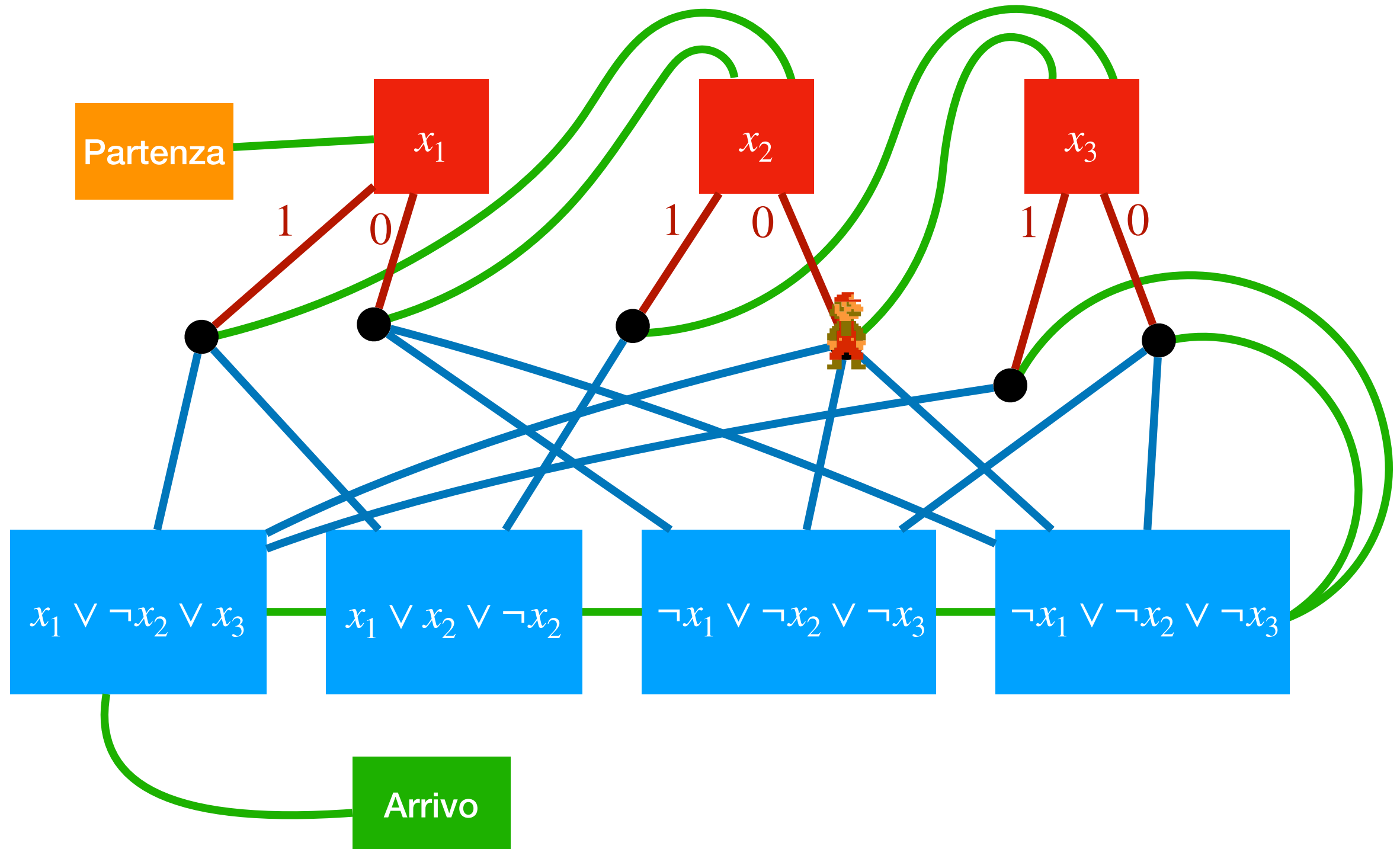
Percorso di Mario



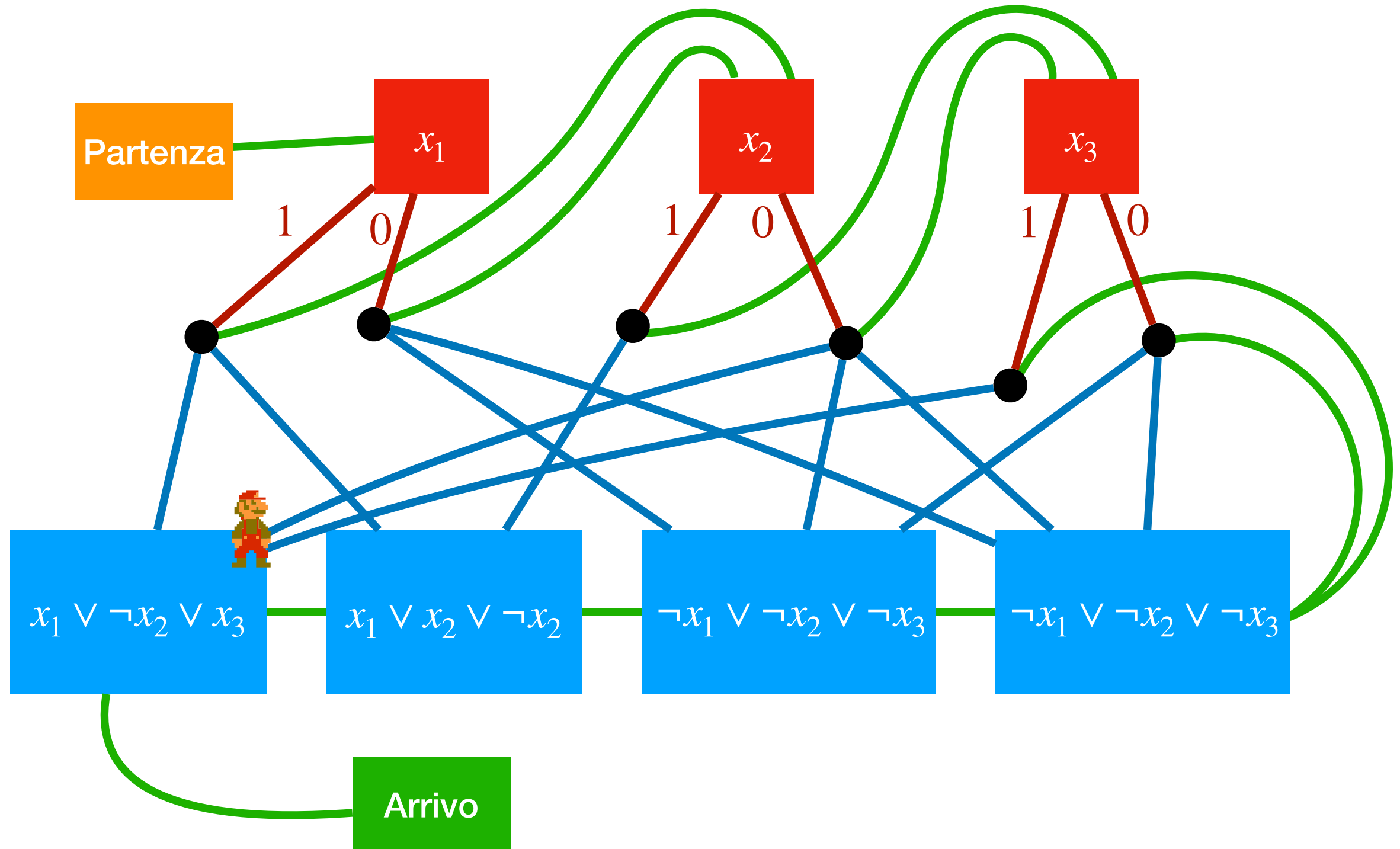
Percorso di Mario



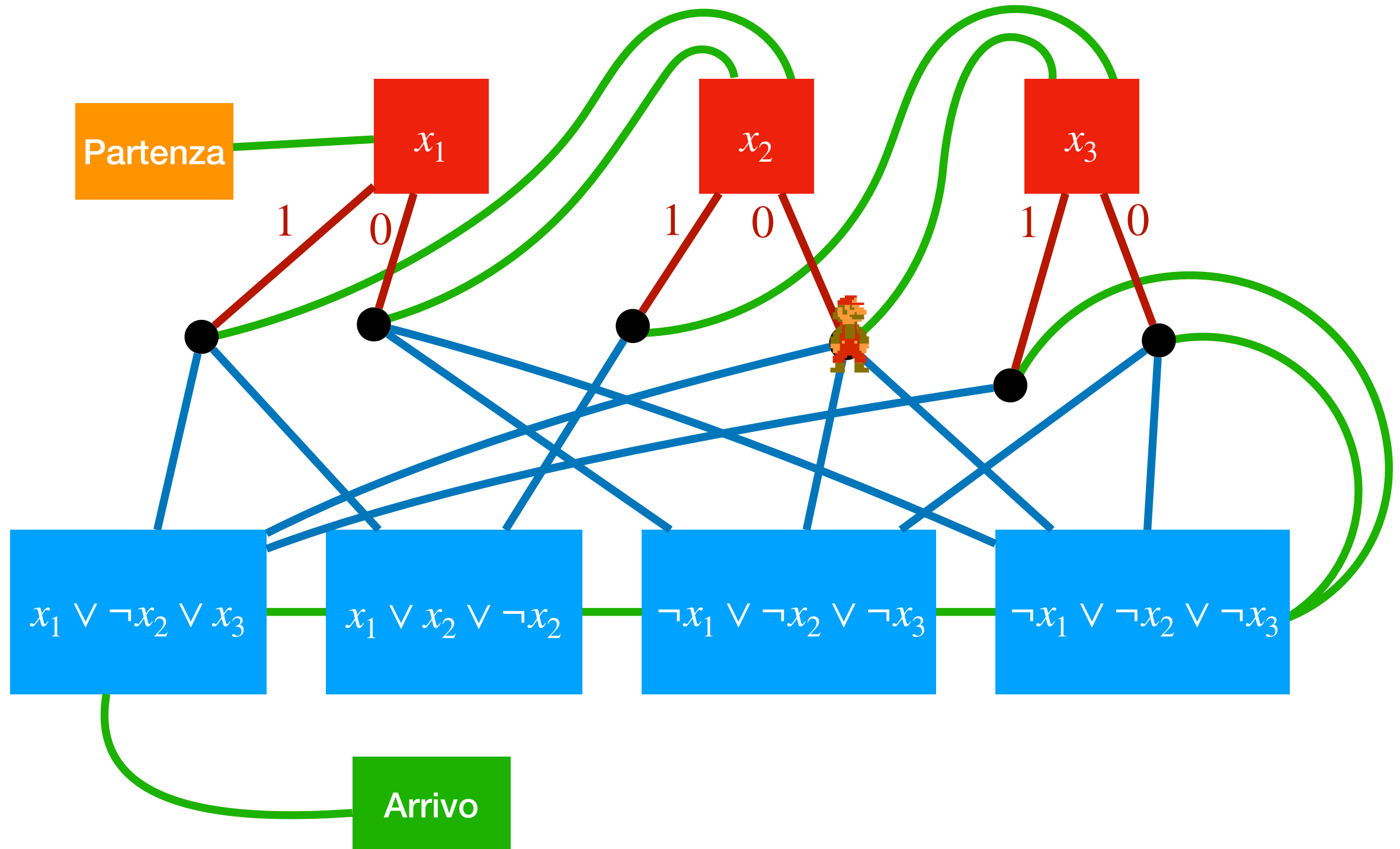
Percorso di Mario



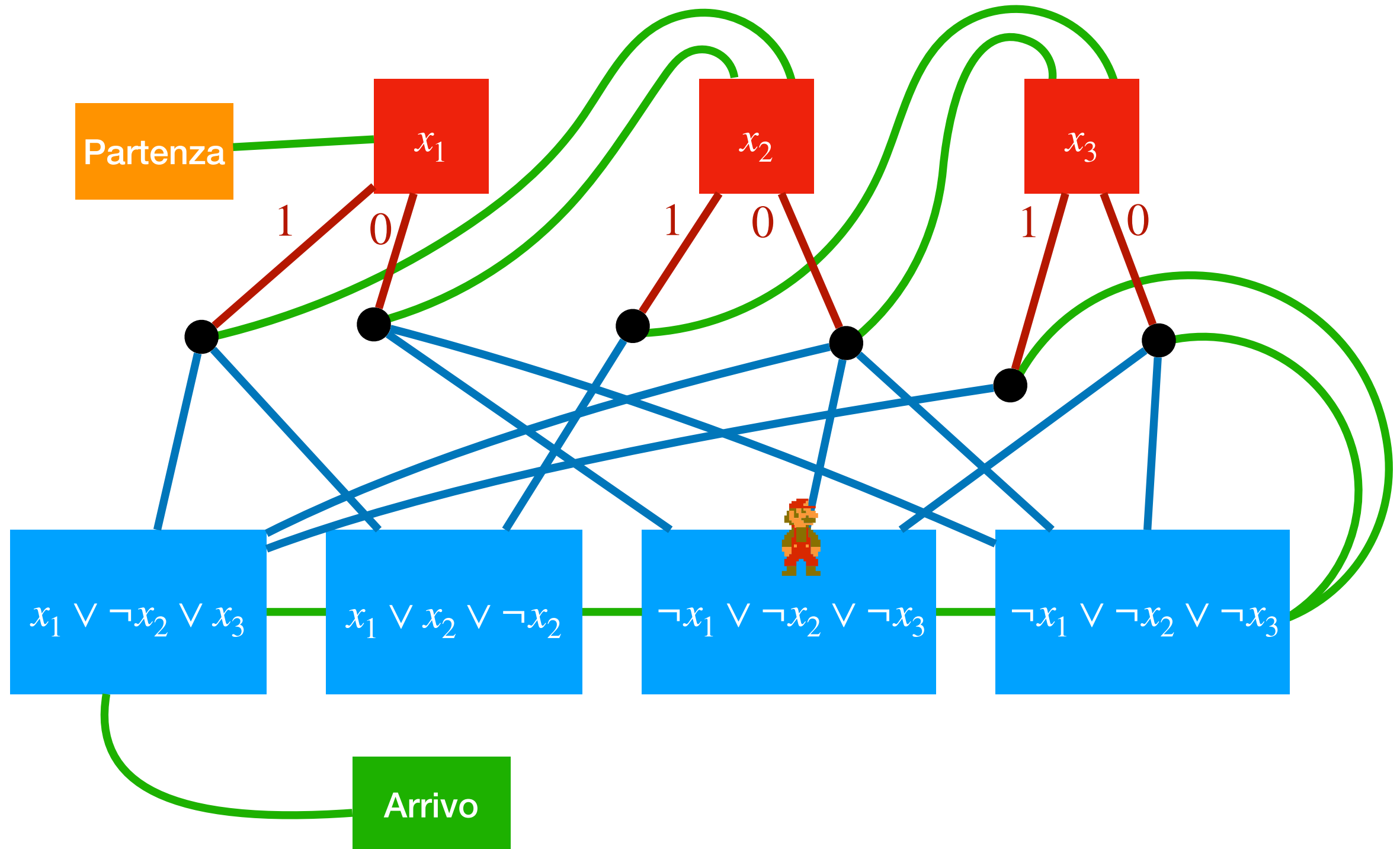
Percorso di Mario



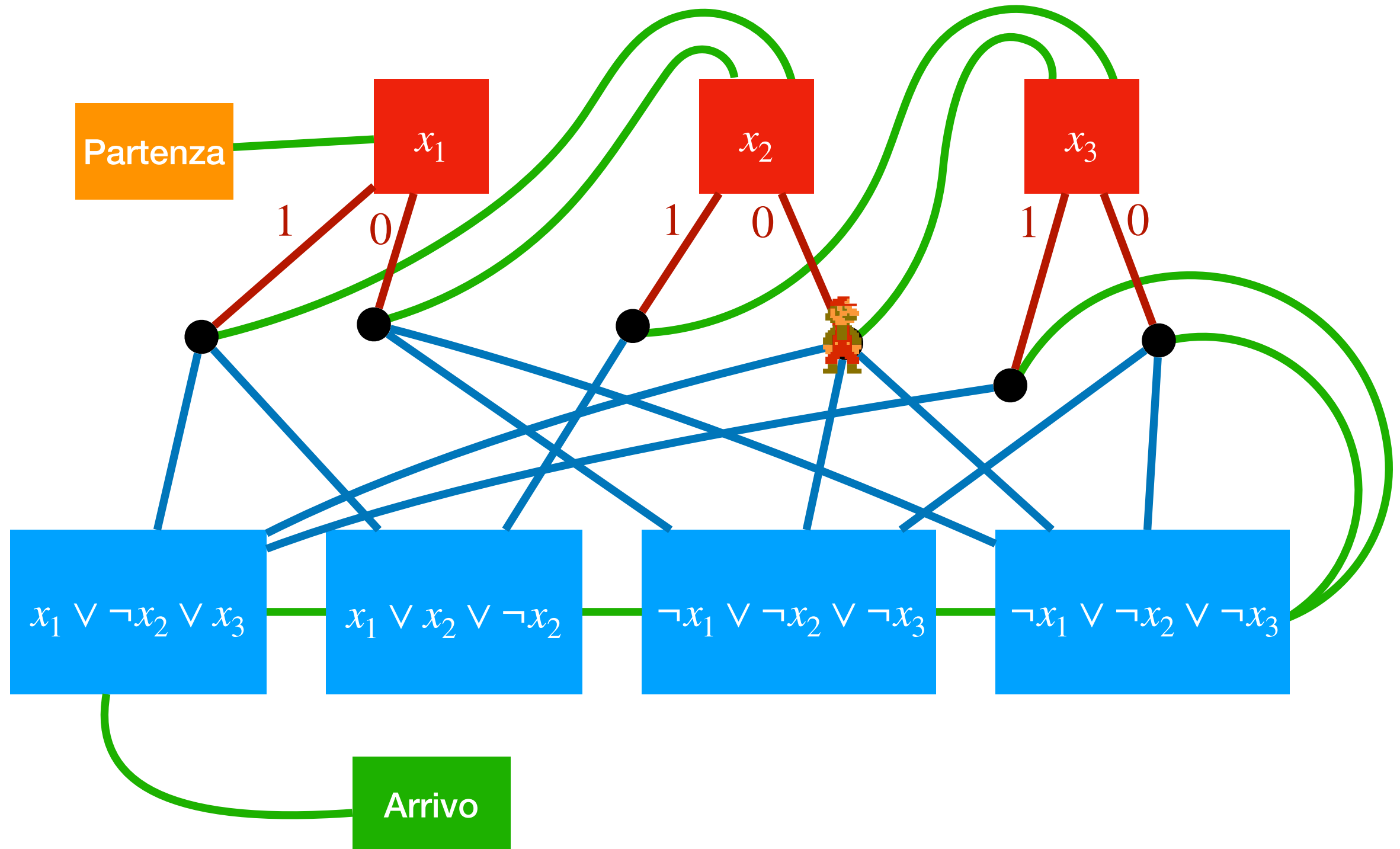
Percorso di Mario



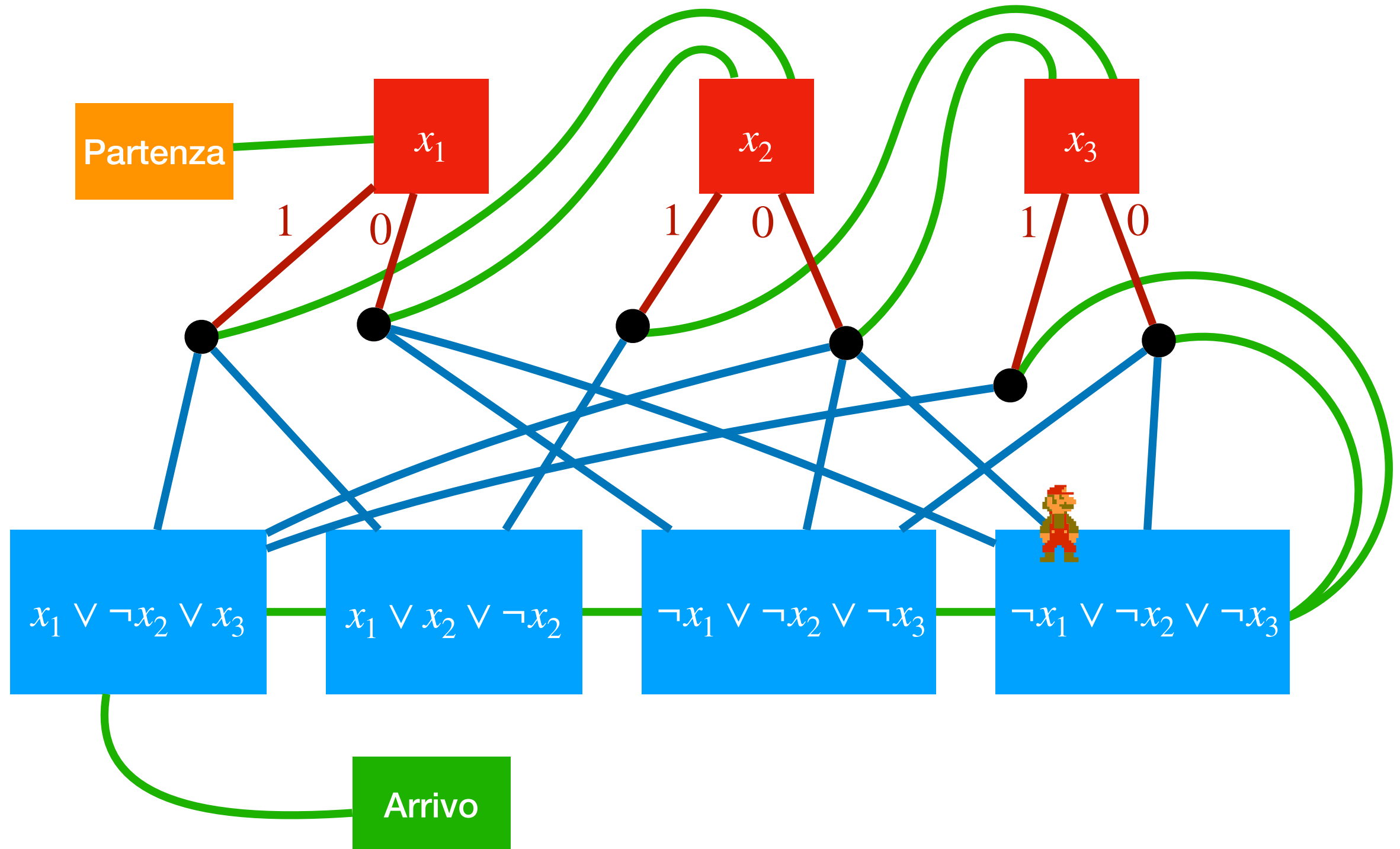
Percorso di Mario



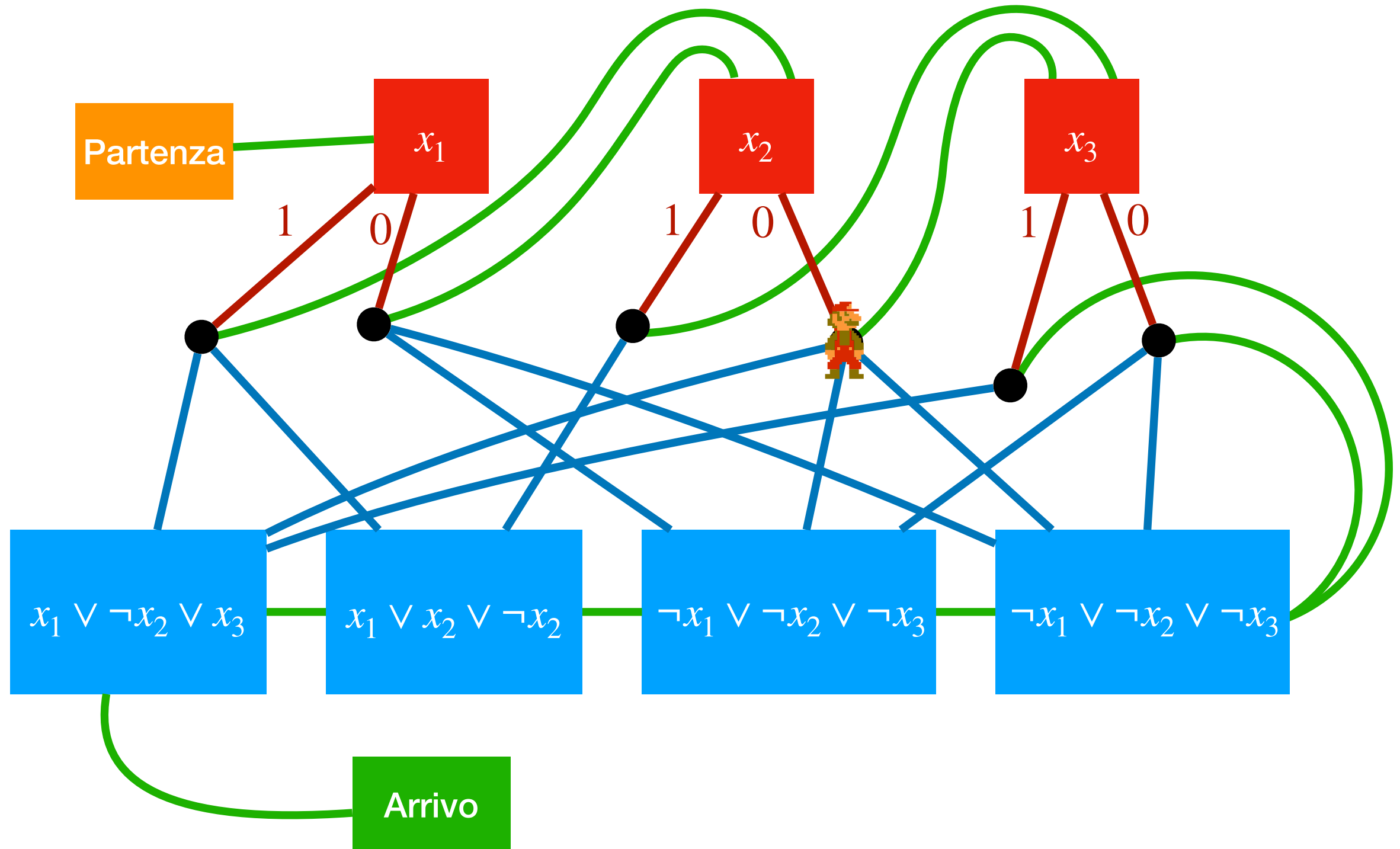
Percorso di Mario



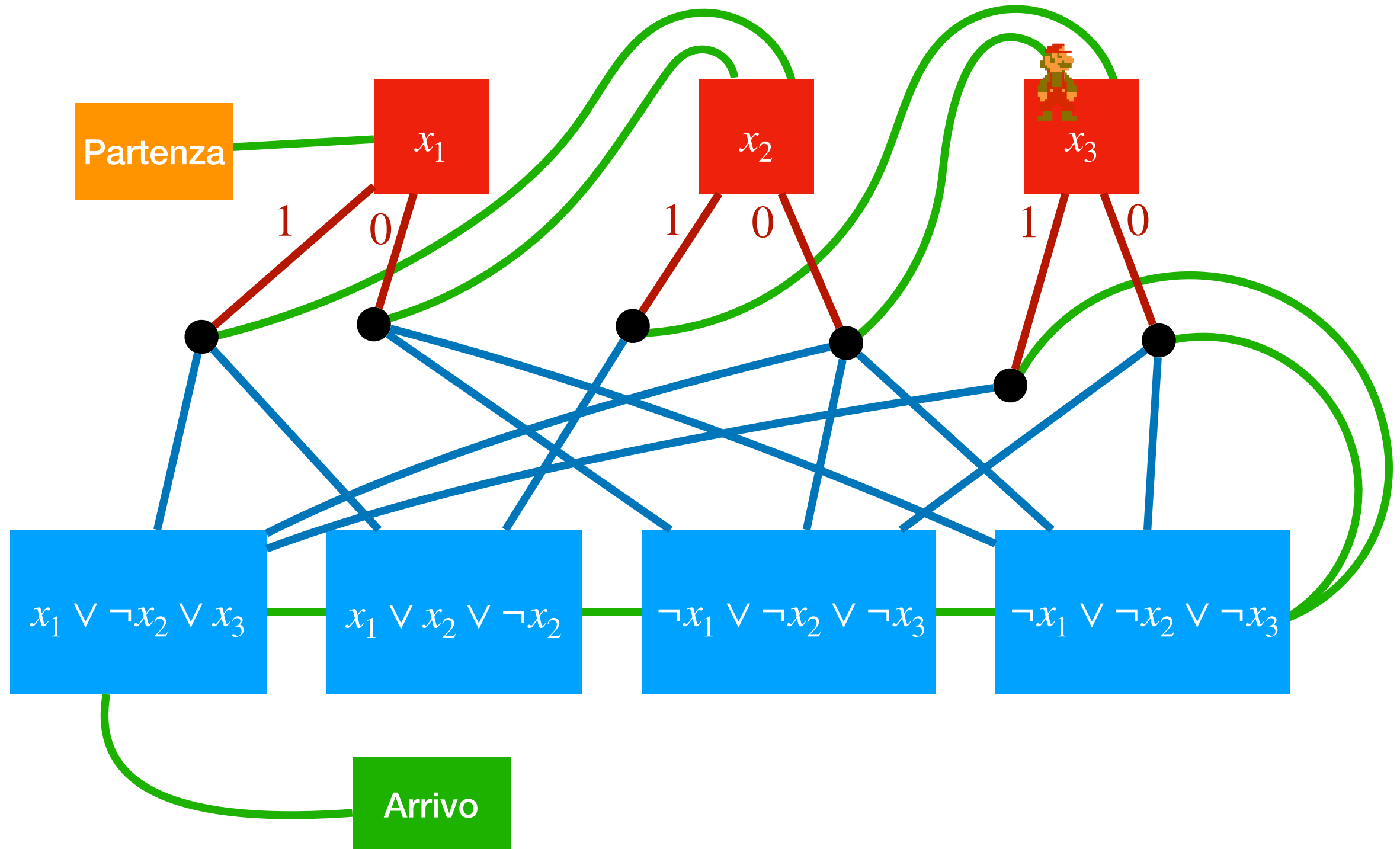
Percorso di Mario



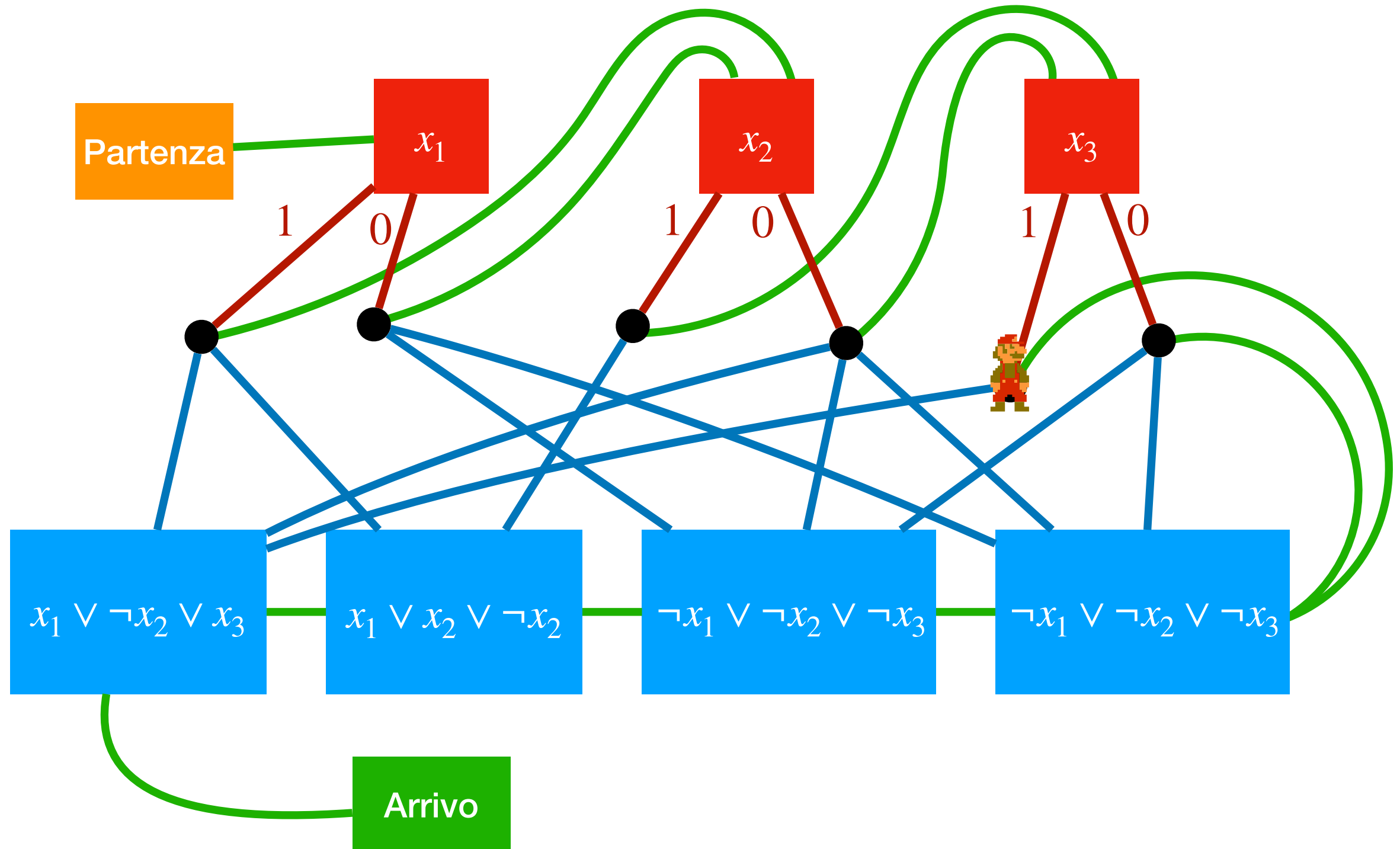
Percorso di Mario



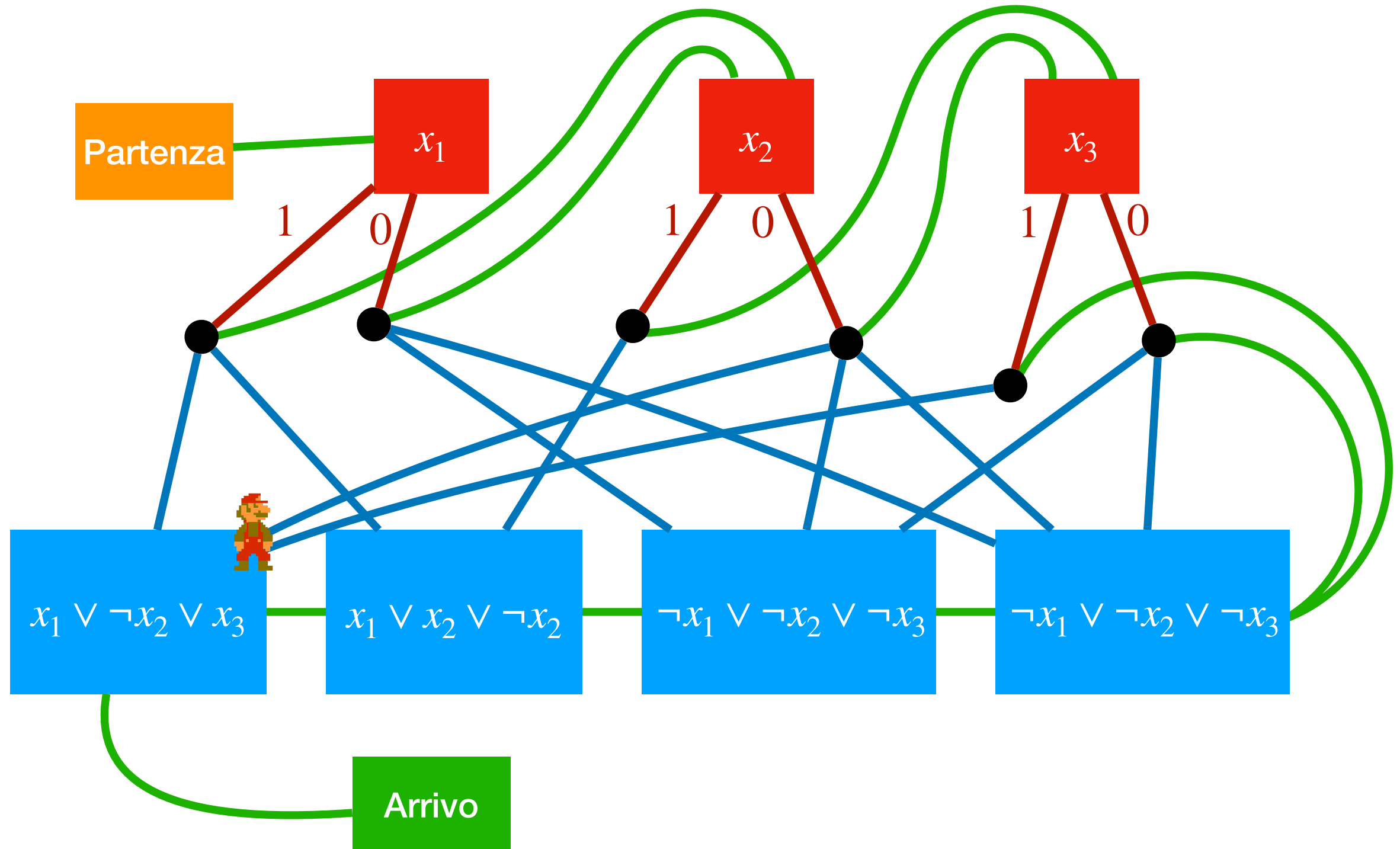
Percorso di Mario



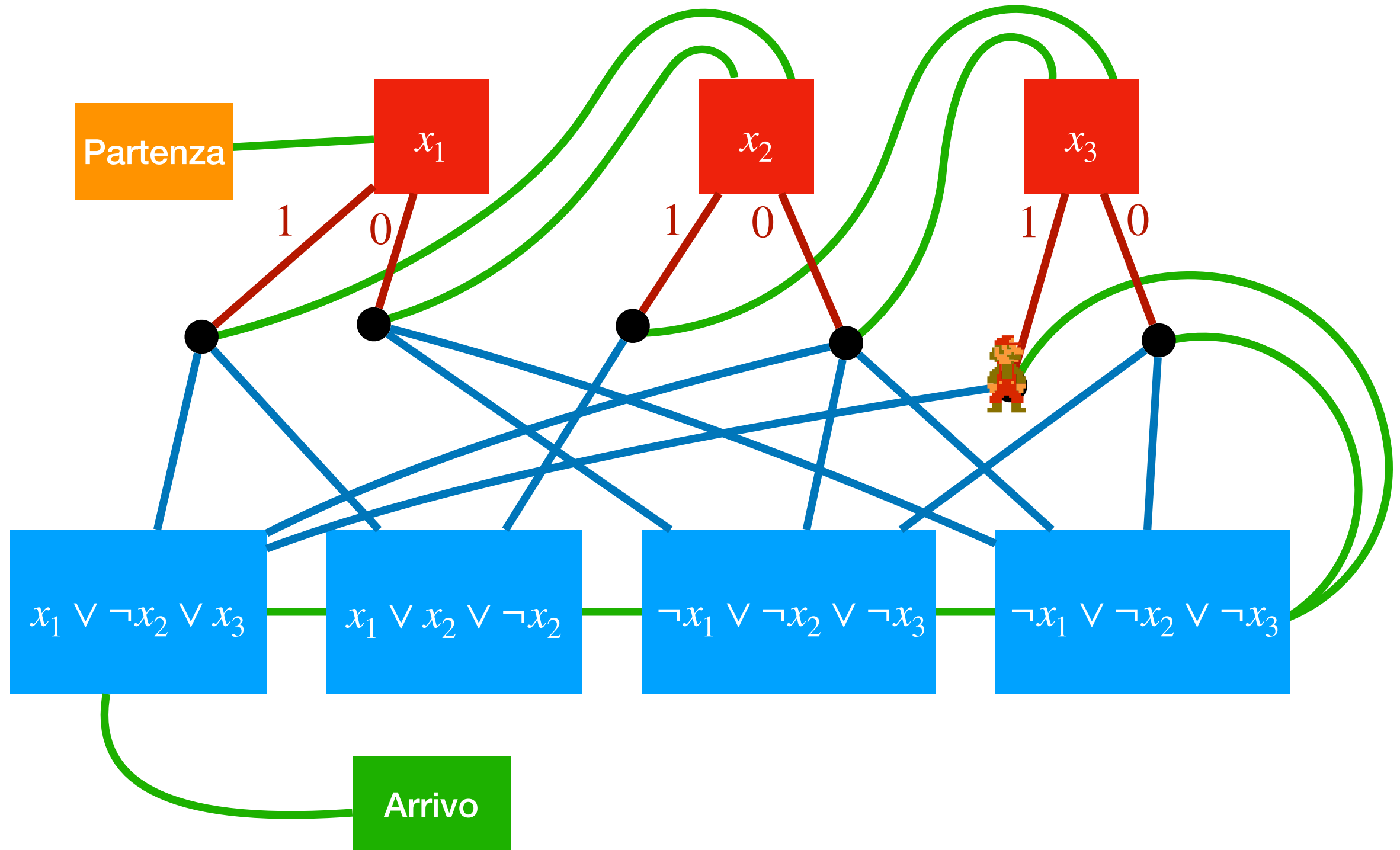
Percorso di Mario



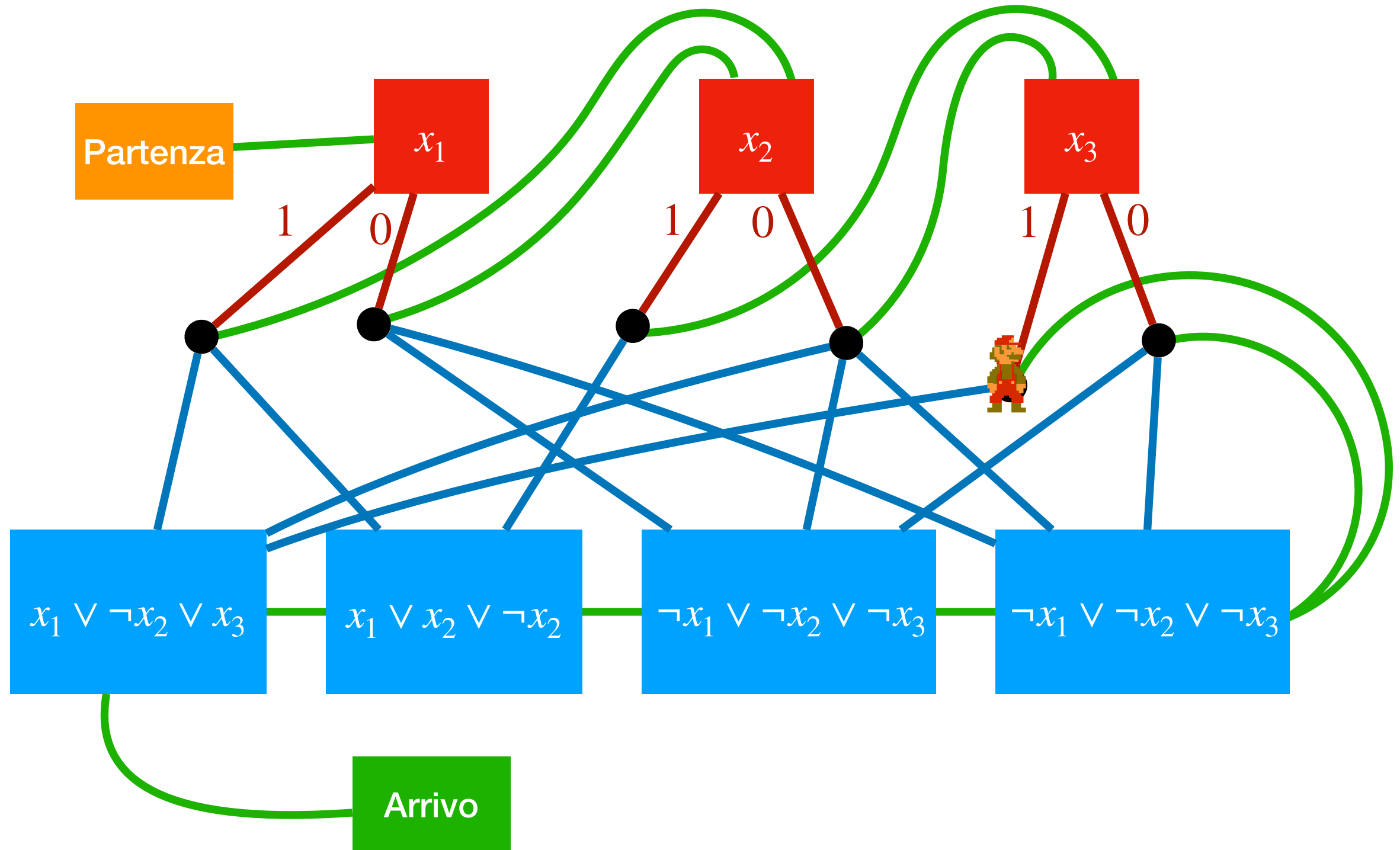
Percorso di Mario



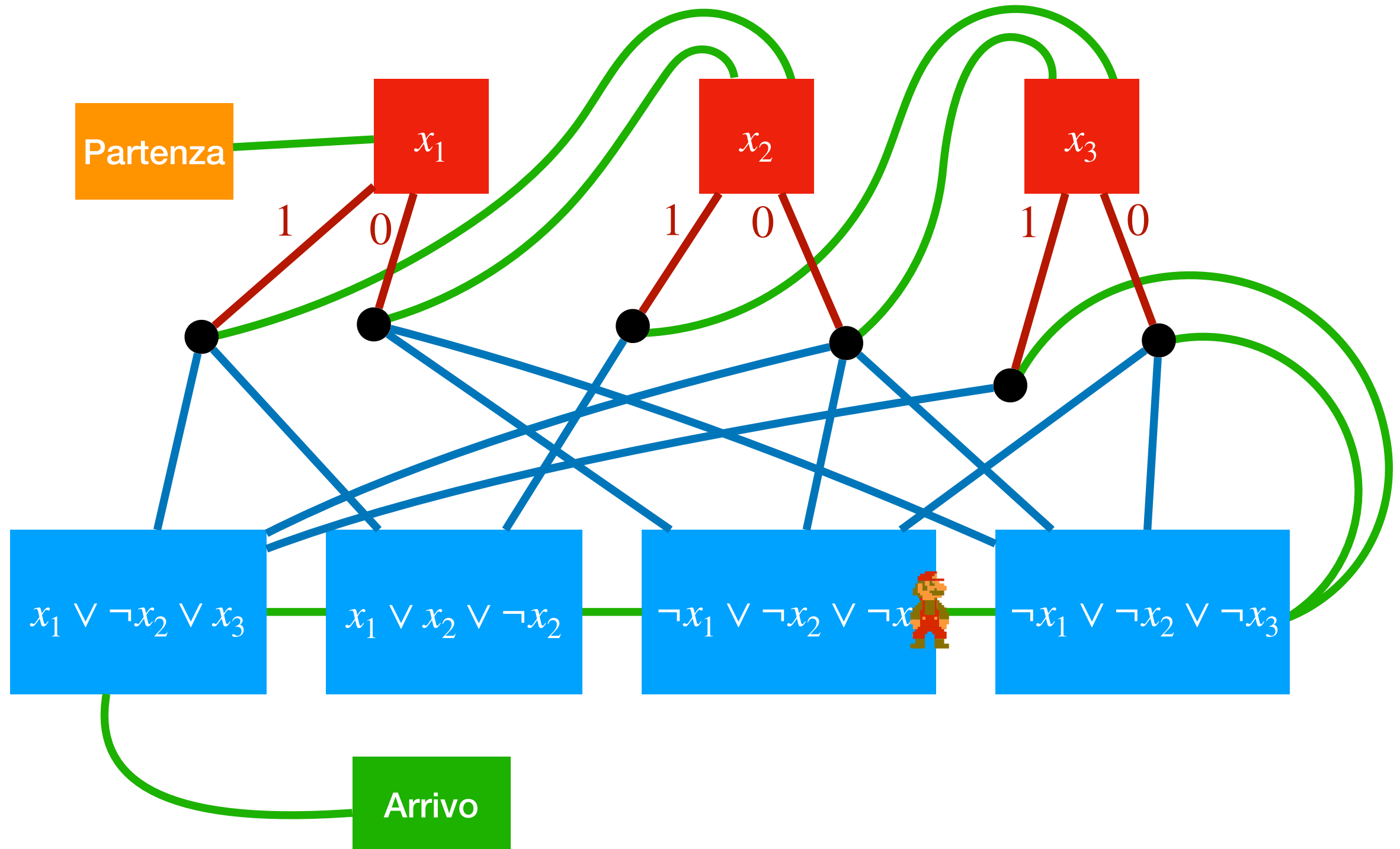
Percorso di Mario



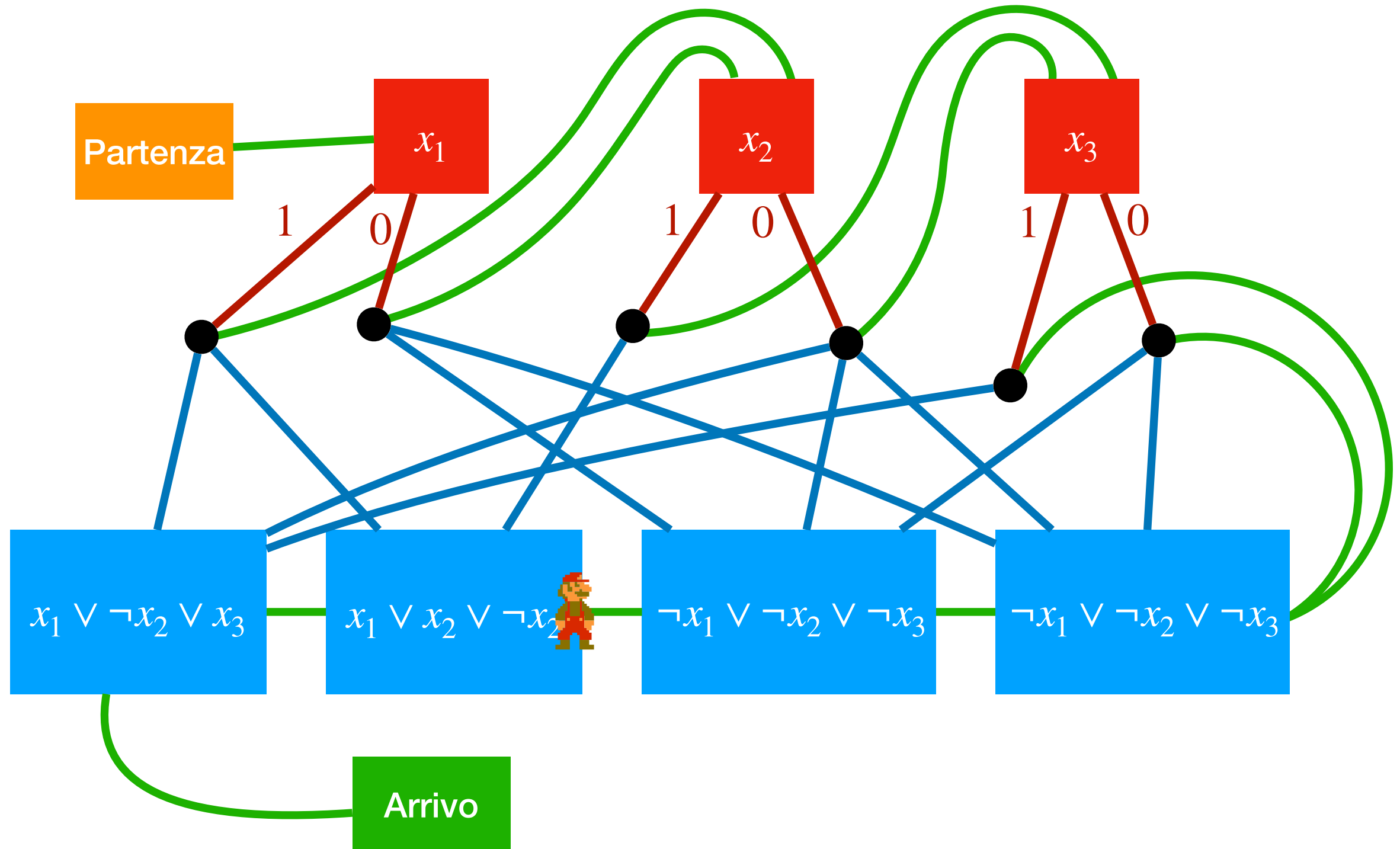
Percorso di Mario



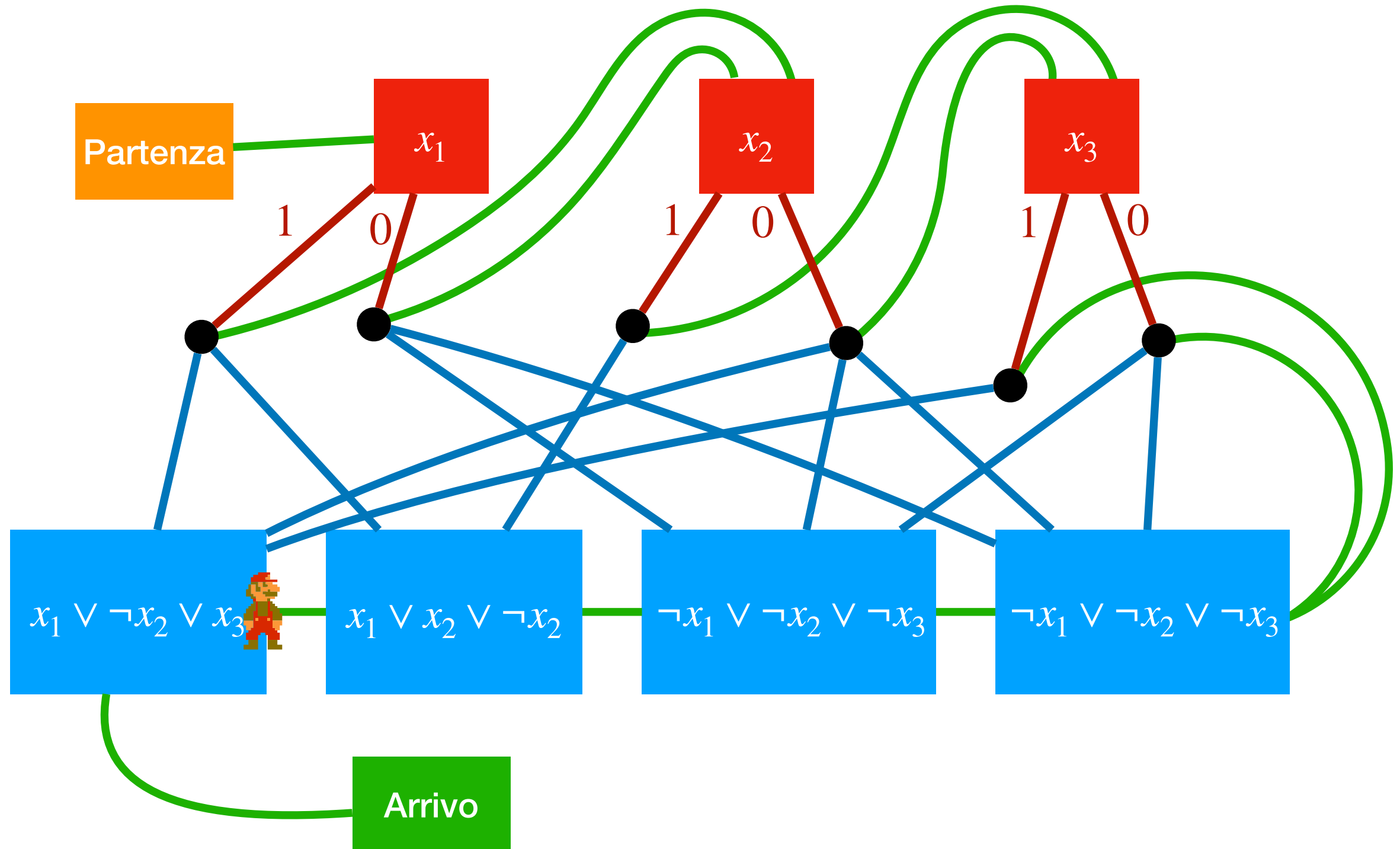
Percorso di Mario



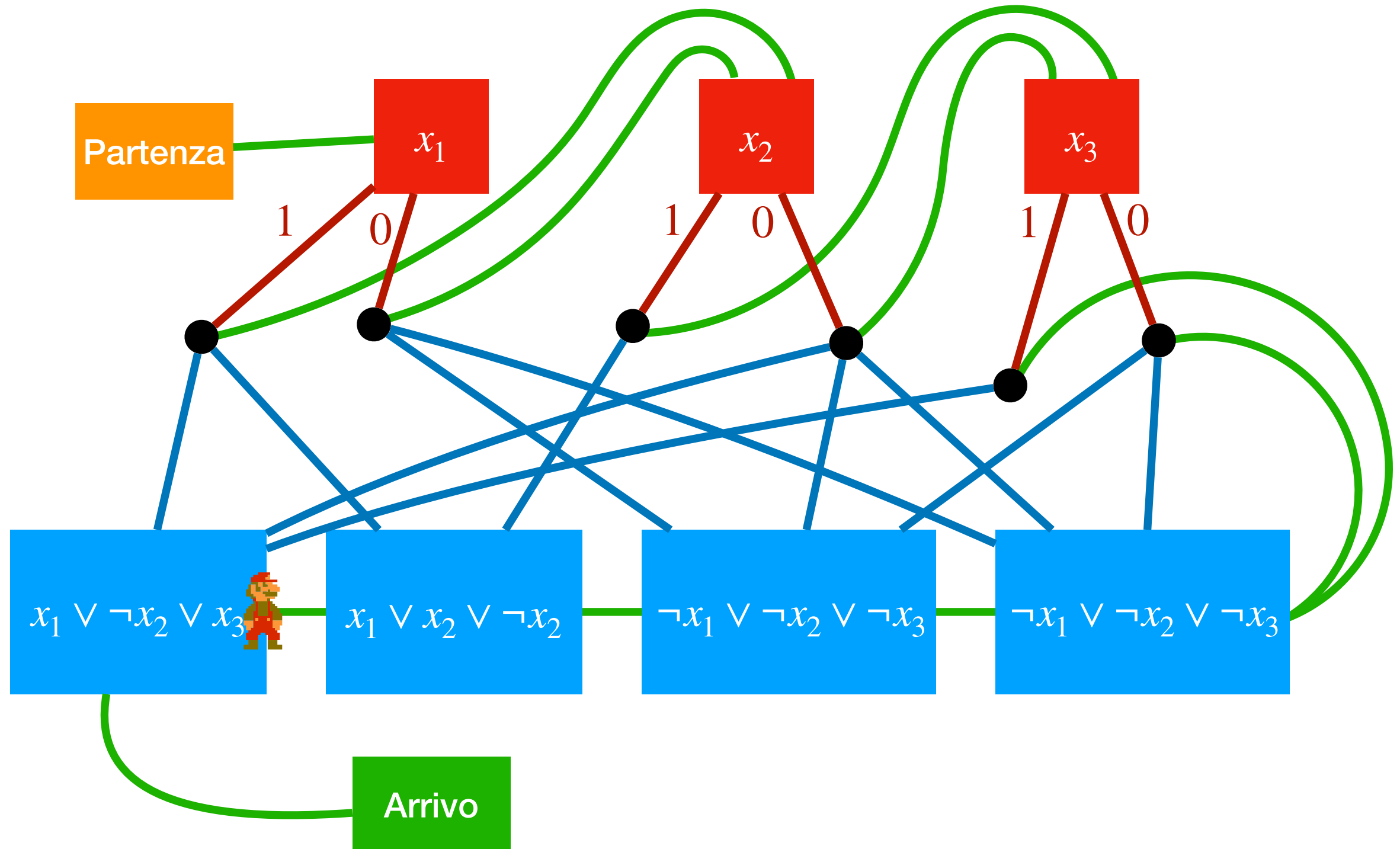
Percorso di Mario



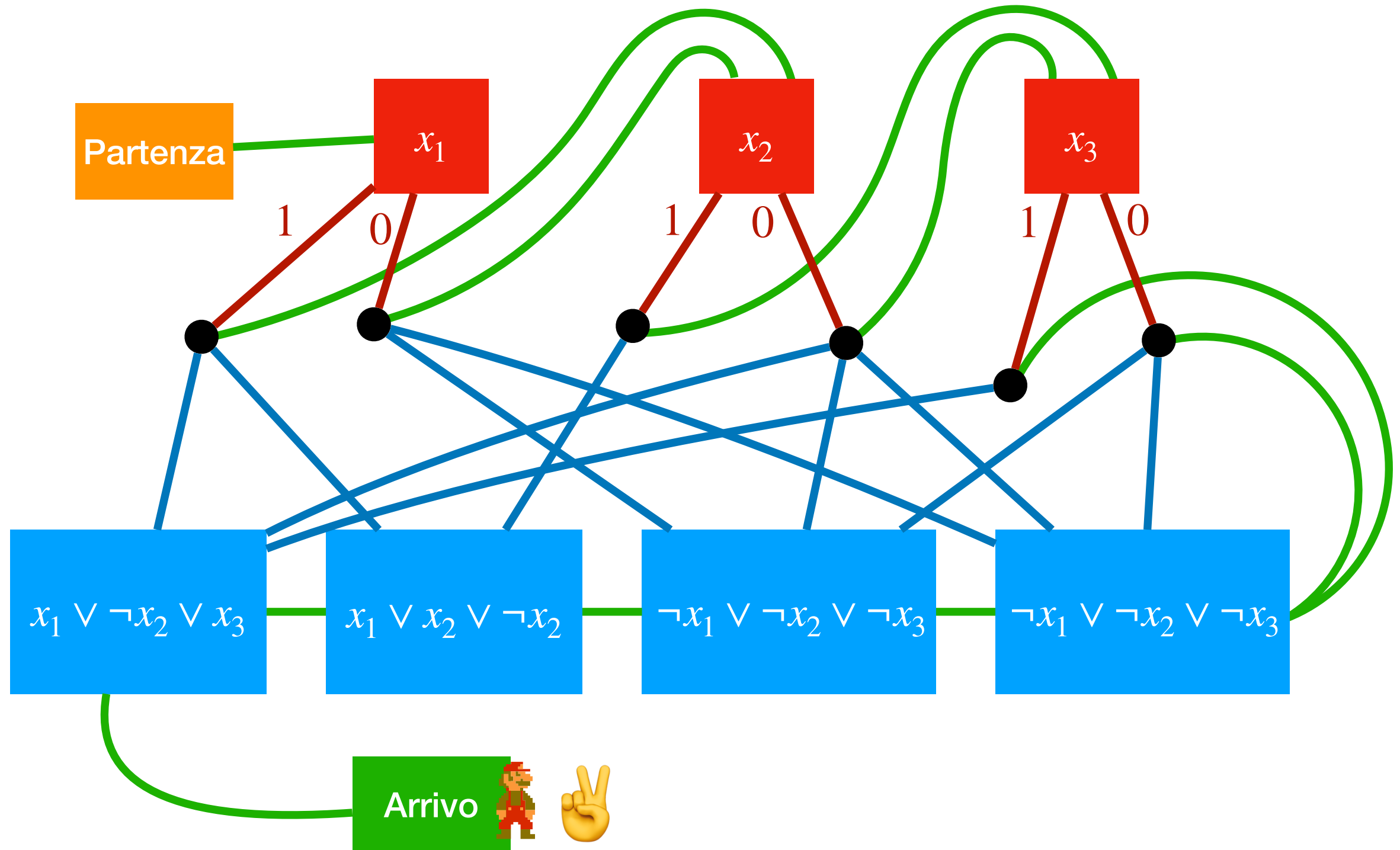
Percorso di Mario



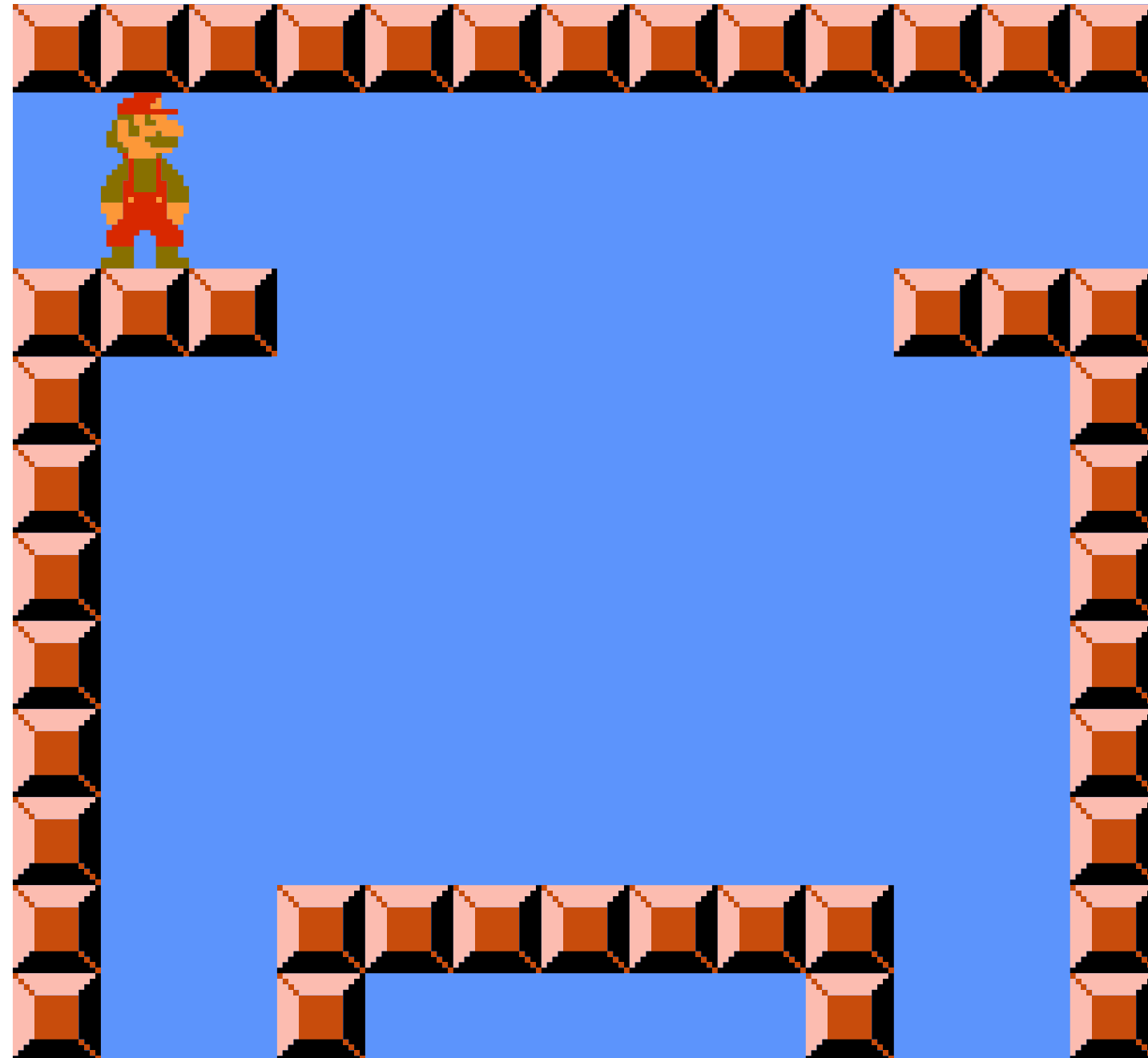
Percorso di Mario



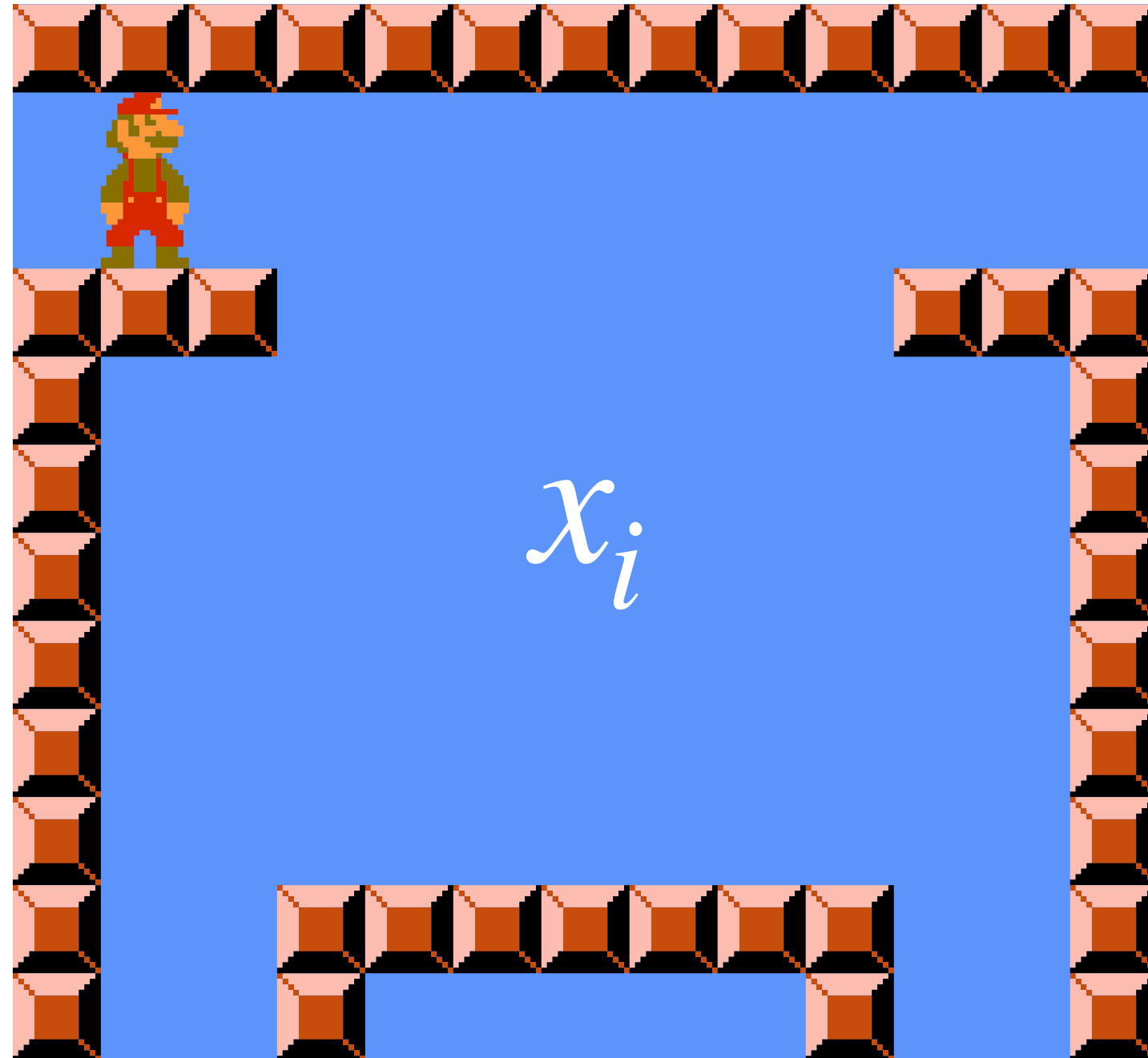
Percorso di Mario



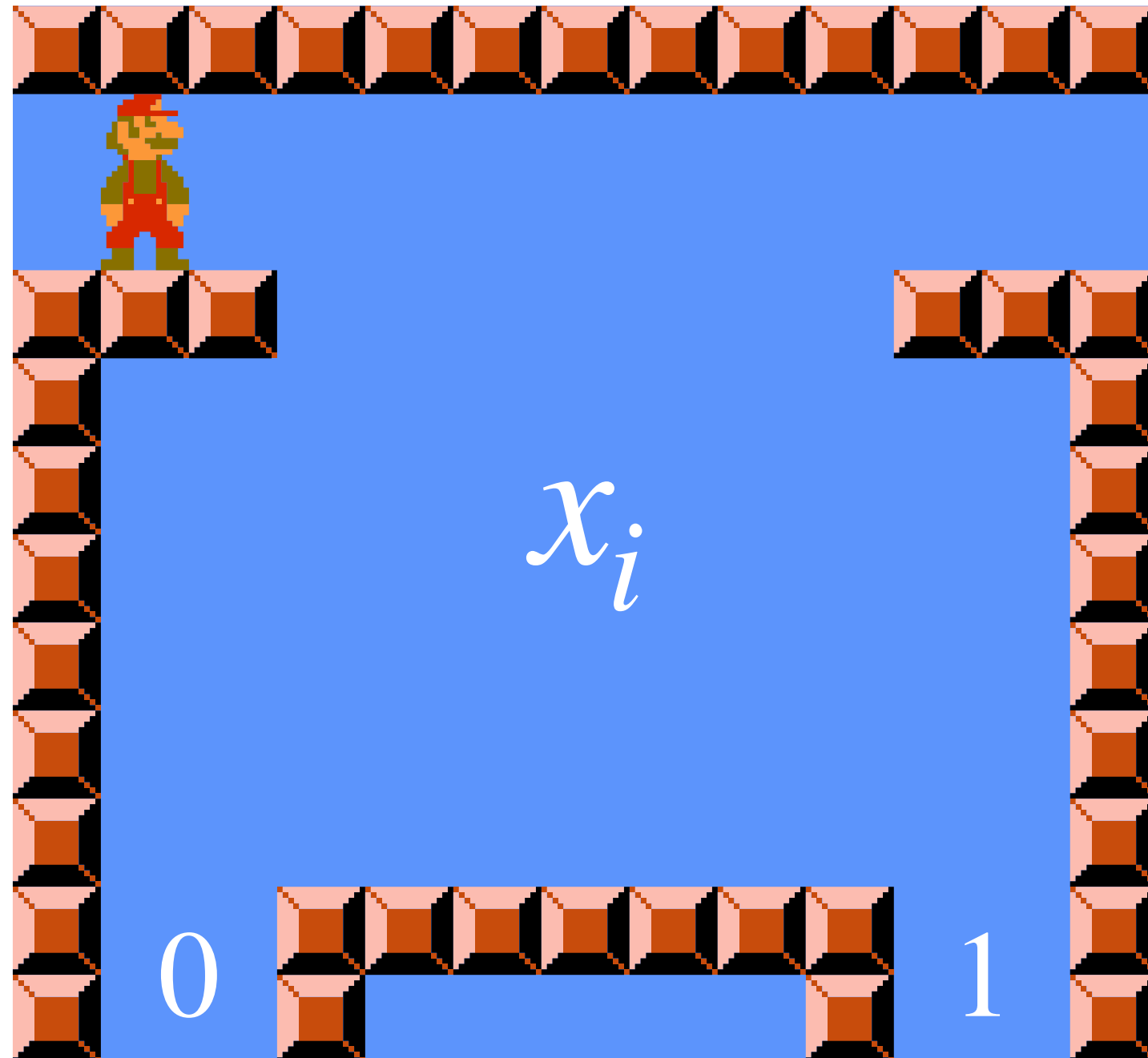
Gadget per le variabili



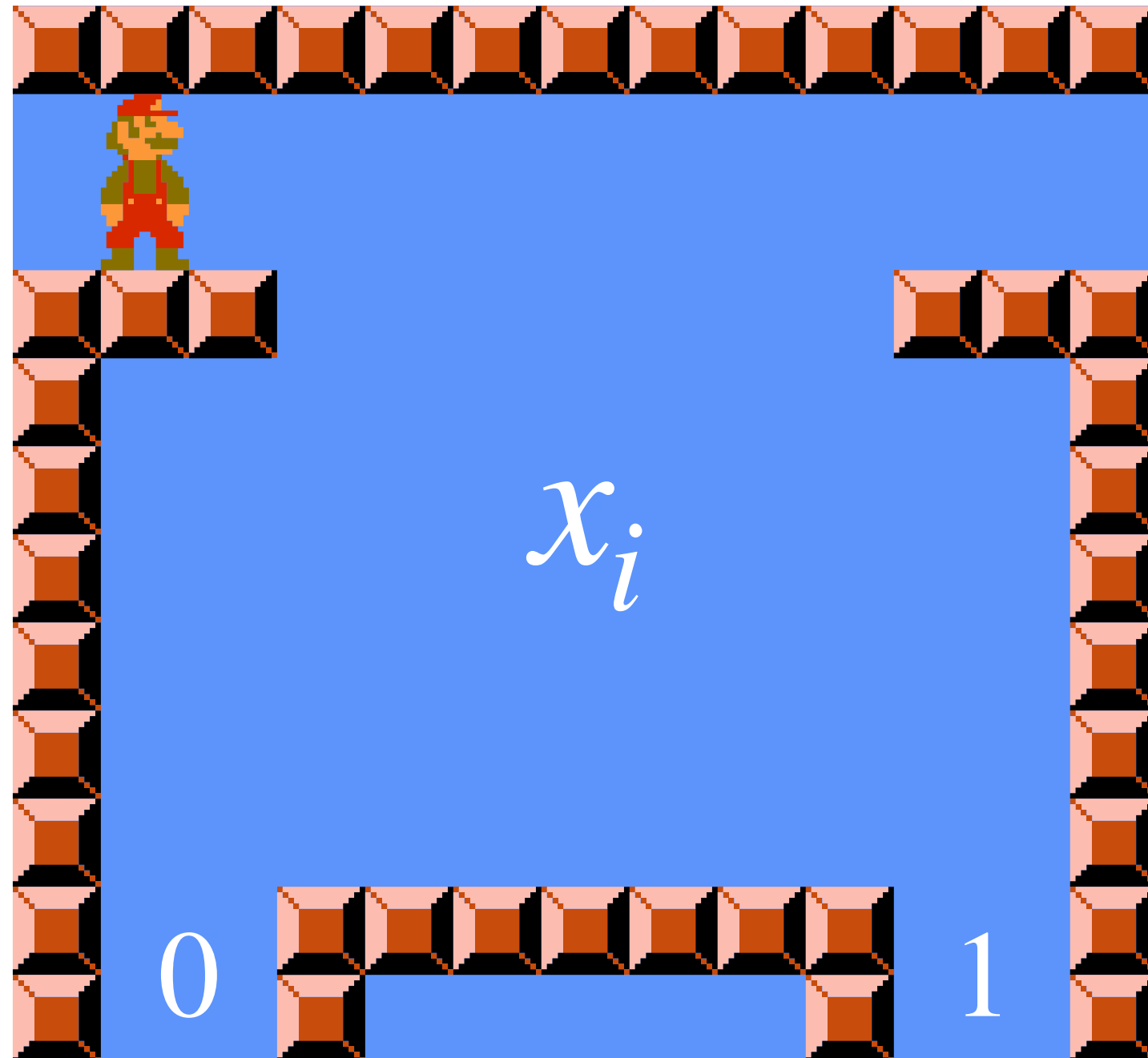
Gadget per le variabili



Gadget per le variabili

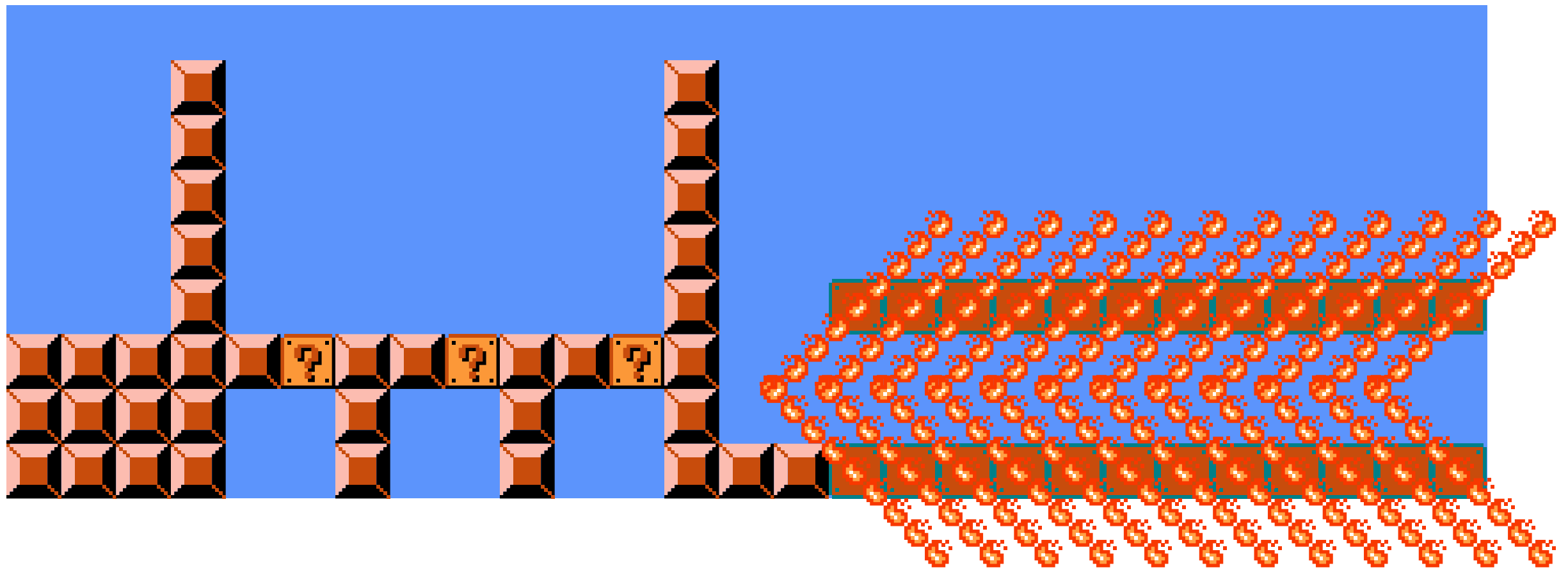


Gadget per le variabili



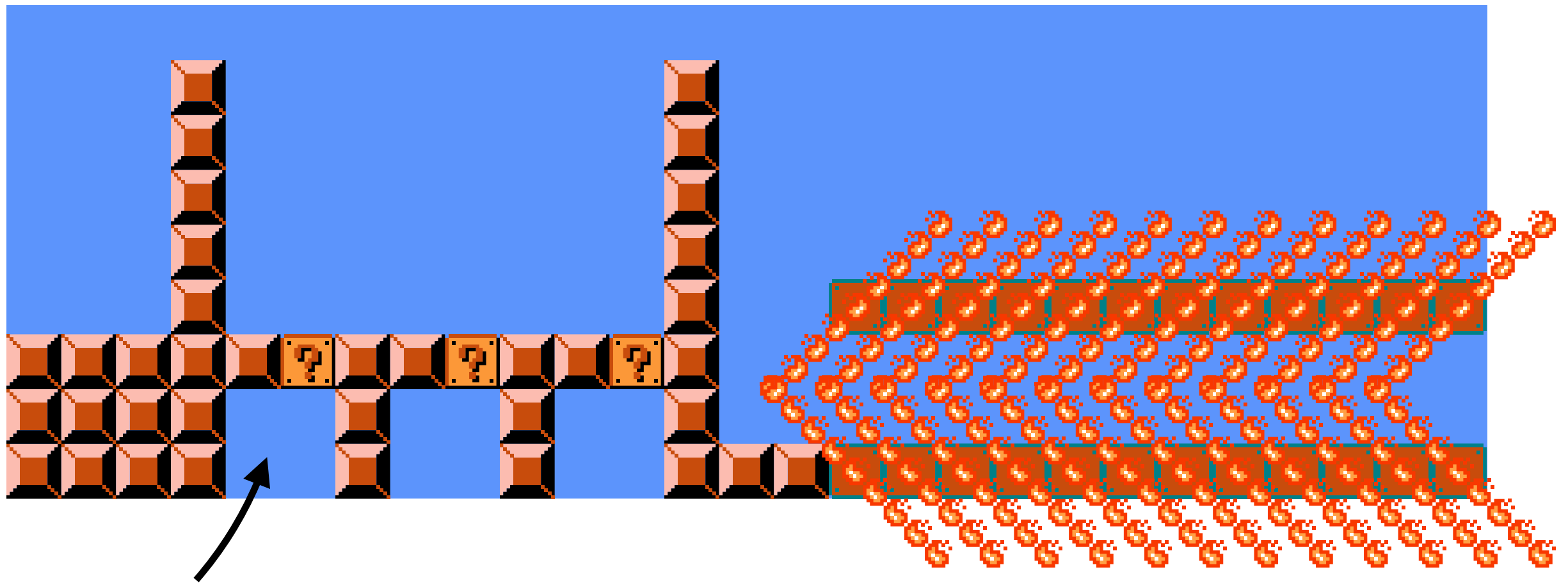
Gadget per le clausole

$$x_1 \vee \neg x_3 \vee x_6$$



Gadget per le clausole

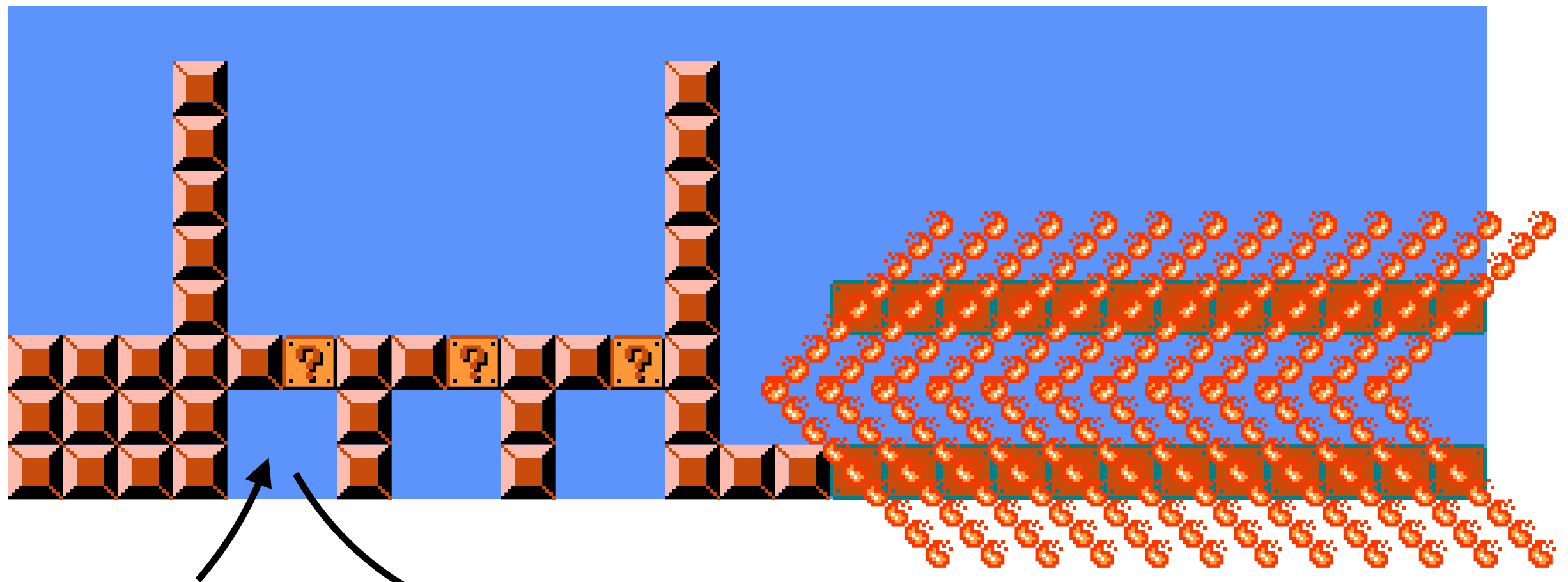
$$x_1 \vee \neg x_3 \vee x_6$$



Arrivo dall'uscita $x_1 = 1$
del gadget della variabile x_1

Gadget per le clausole

$$x_1 \vee \neg x_3 \vee x_6$$

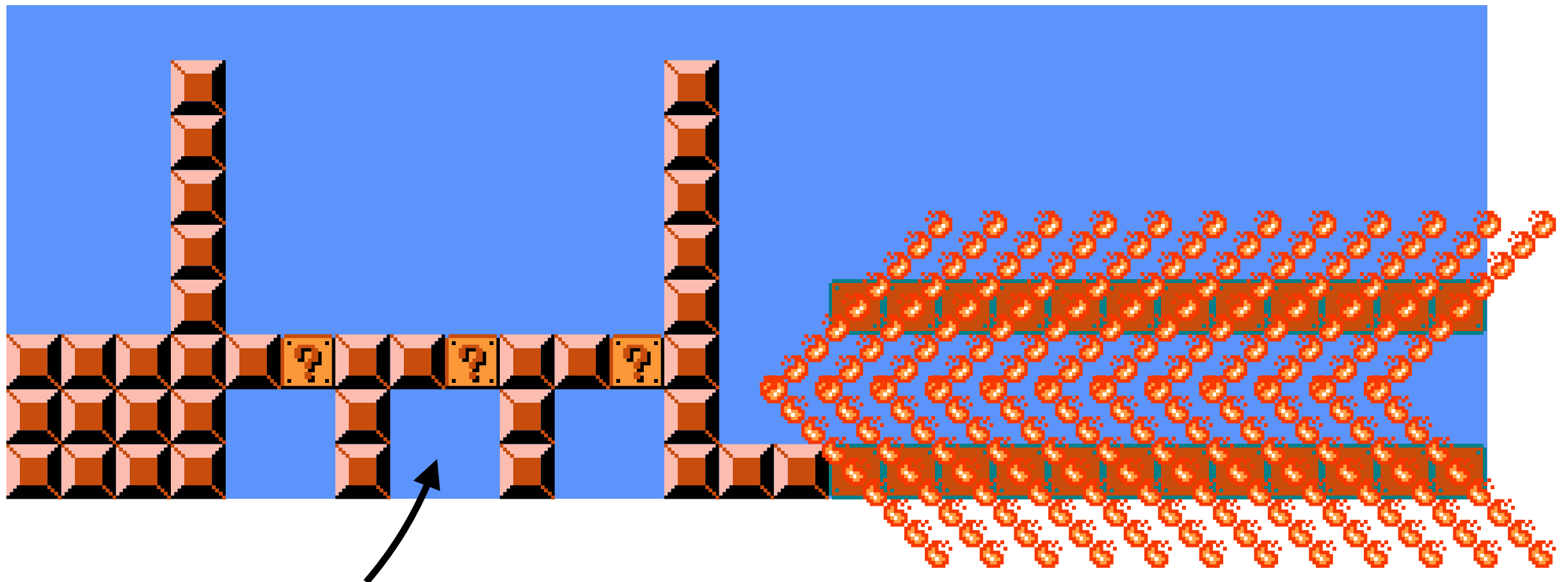


Arrivo dall'uscita $x_1 = 1$
del gadget della variabile x_1

Verso l'entrata del gadget per
la successiva variabile (ovvero x_2)

Gadget per le clausole

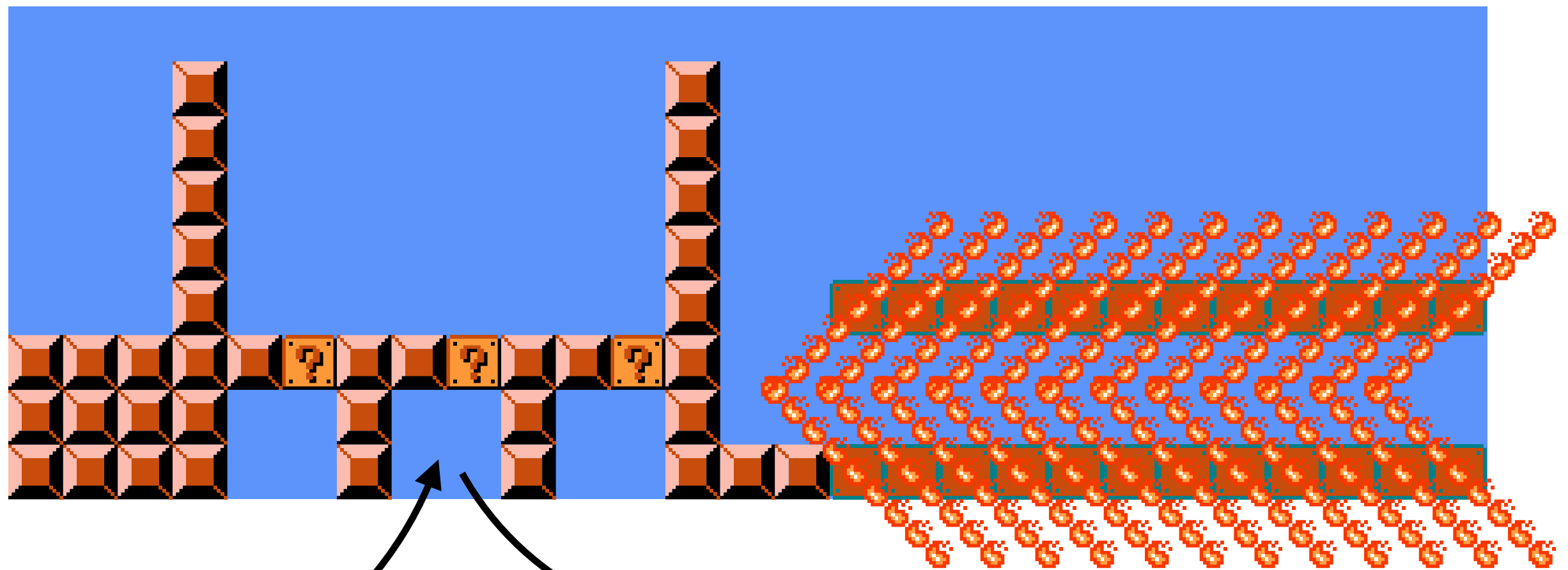
$$x_1 \vee \neg x_3 \vee x_6$$



Arrivo dall'uscita $x_3 = 0$
del gadget della variabile x_3

Gadget per le clausole

$$x_1 \vee \neg x_3 \vee x_6$$

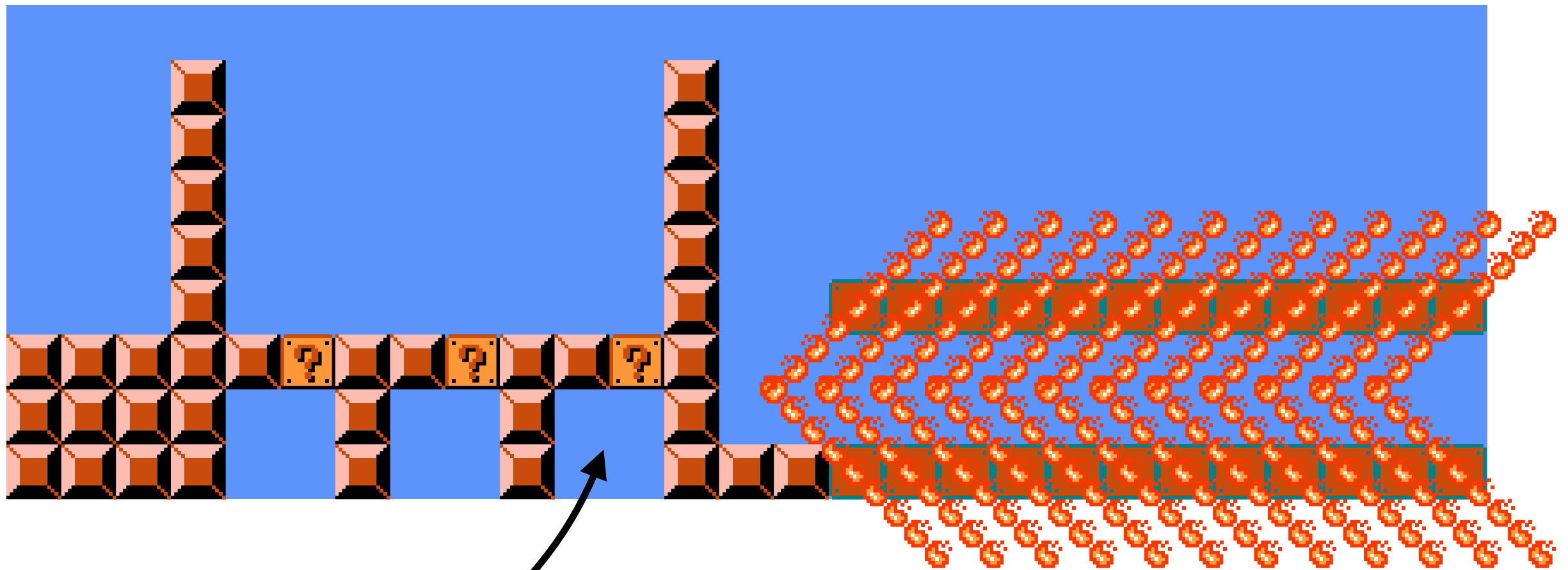


Arrivo dall'uscita $x_3 = 0$
del gadget della variabile x_3

Verso l'entrata del gadget per
la successiva variabile (ovvero x_4)

Gadget per le clausole

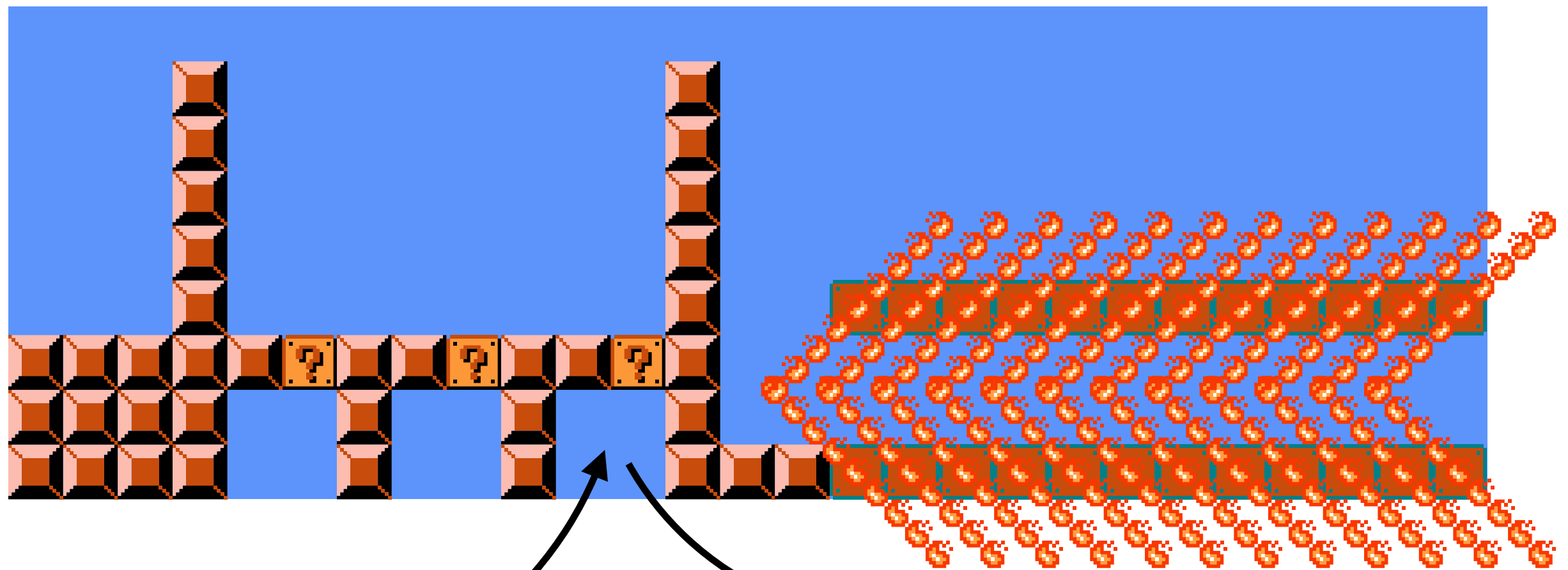
$$x_1 \vee \neg x_3 \vee x_6$$



Arrivo dall'uscita $x_6 = 1$
del gadget della variabile x_6

Gadget per le clausole

$$x_1 \vee \neg x_3 \vee x_6$$

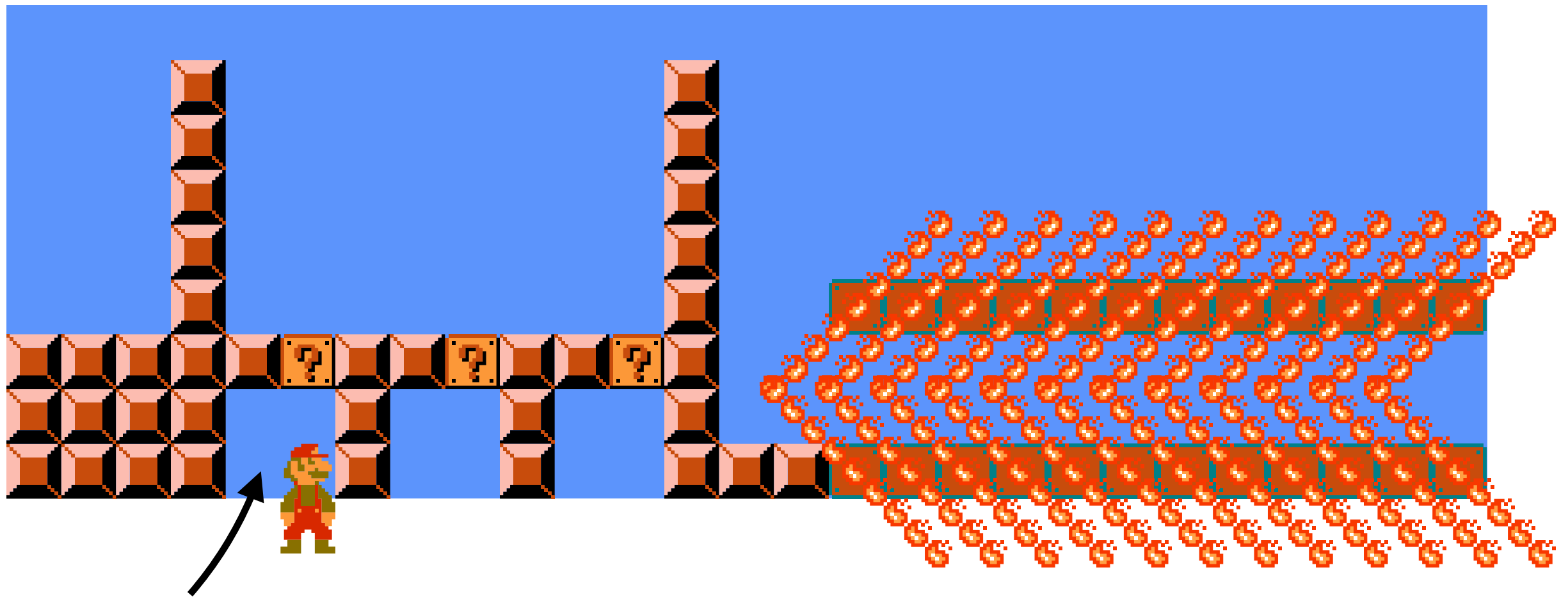


Arrivo dall'uscita $x_6 = 1$
del gadget della variabile x_6

Verso l'entrata del gadget per
la successiva variabile (ovvero x_7)

Gadget per le clausole

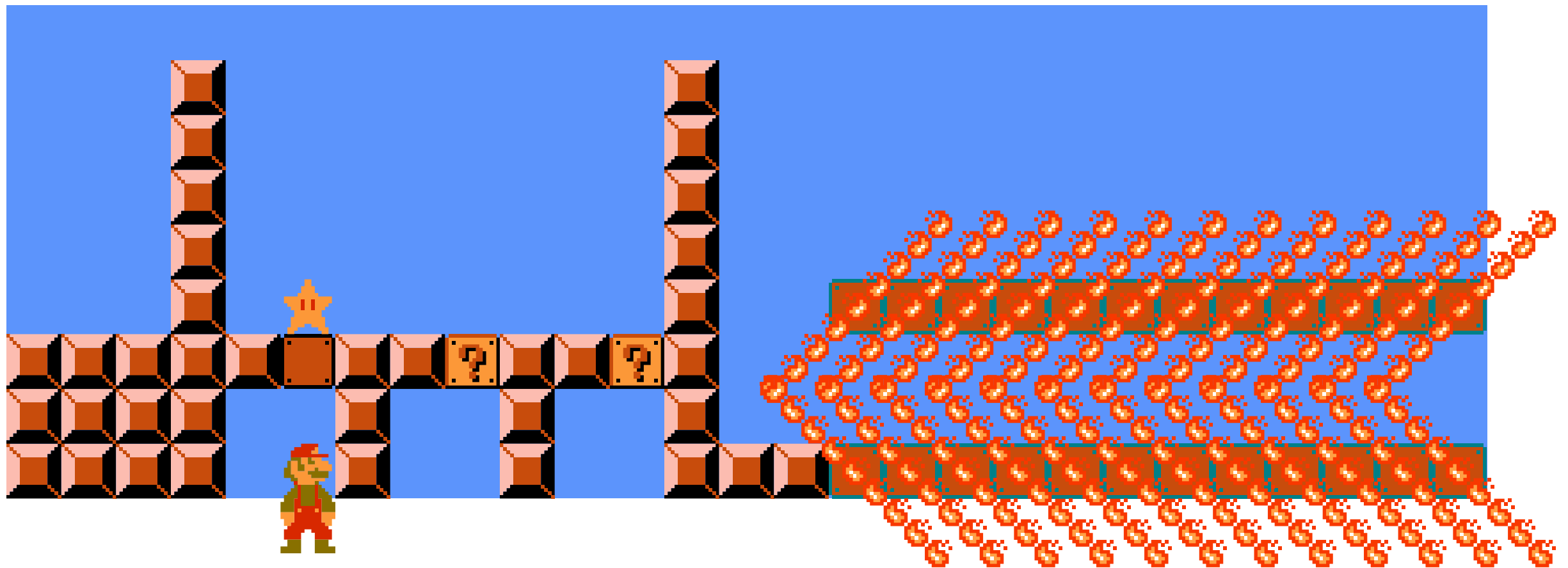
$$x_1 \vee \neg x_3 \vee x_6$$



Arrivo dall'uscita $x_1 = 1$
del gadget della variabile x_1

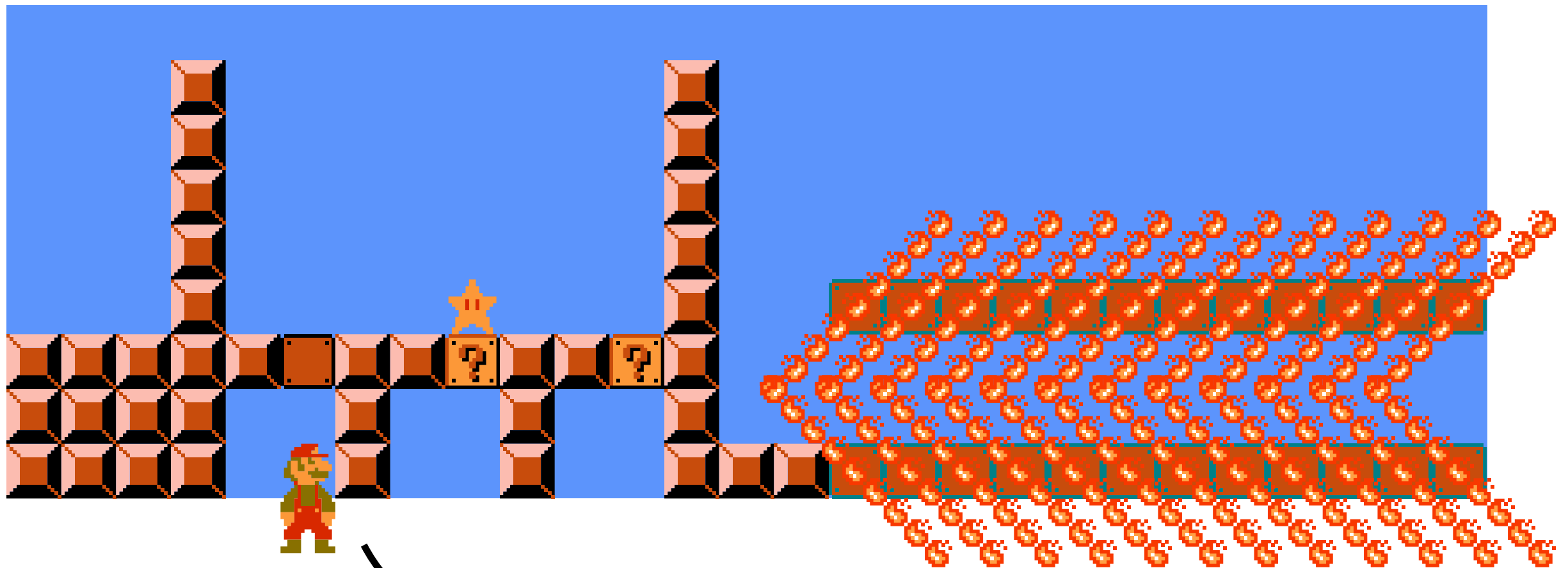
Gadget per le clausole

$$x_1 \vee \neg x_3 \vee x_6$$



Gadget per le clausole

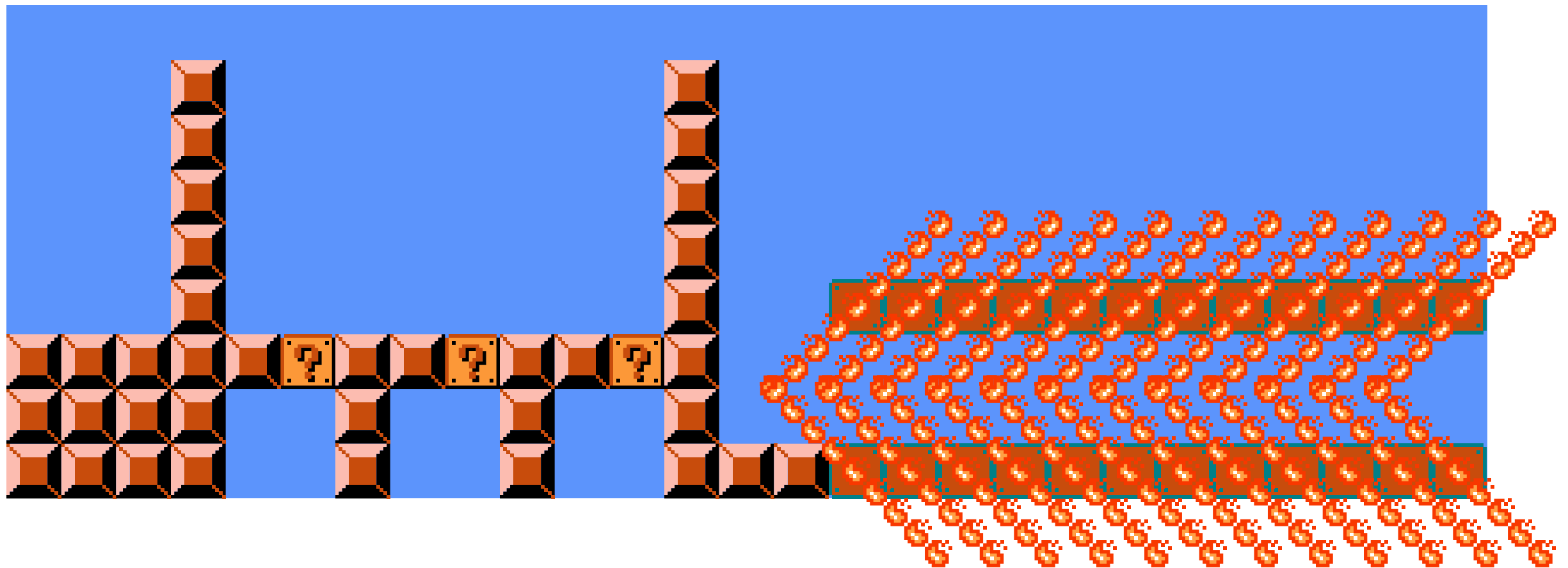
$$x_1 \vee \neg x_3 \vee x_6$$



Verso l'entrata del gadget
per la successiva variabile (ovvero x_2)

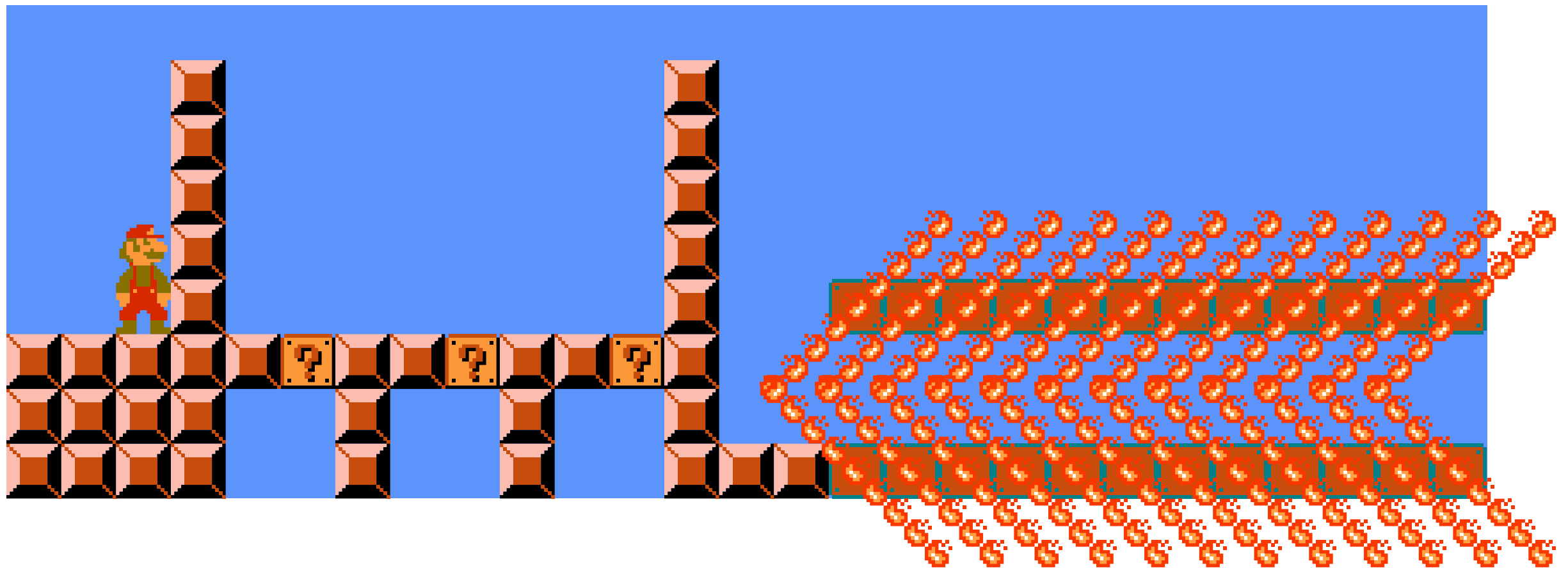
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 0$$



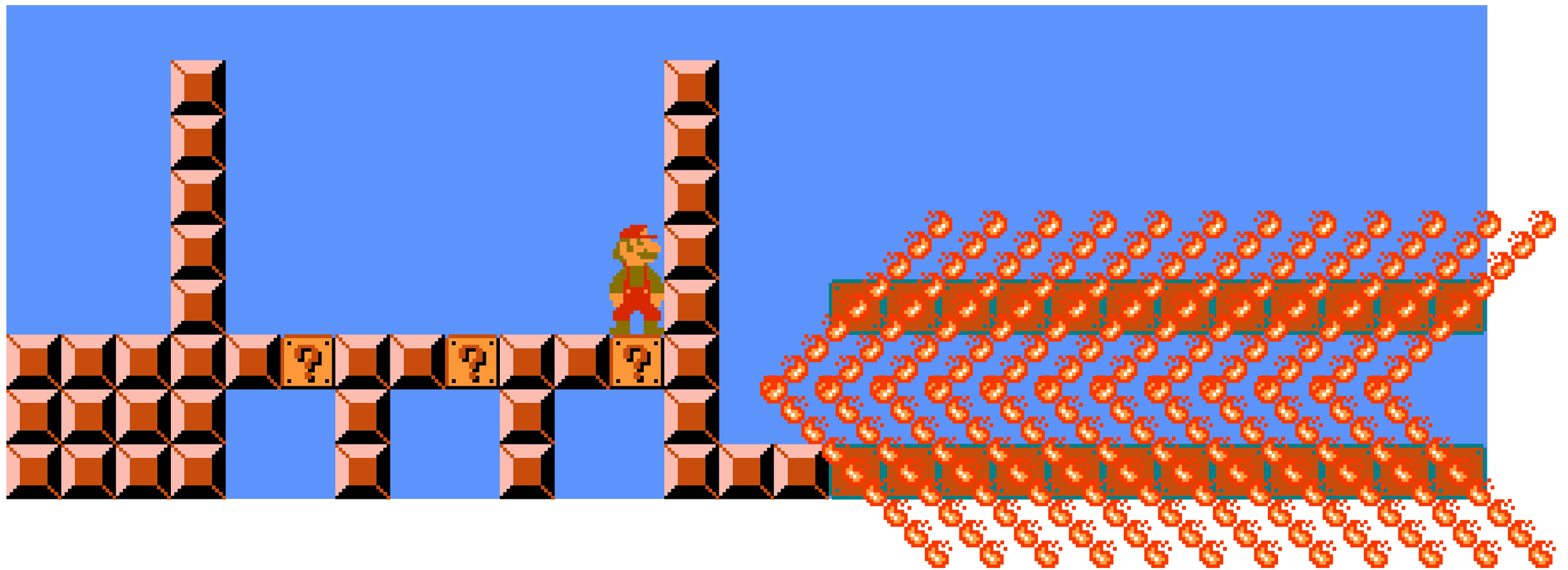
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 0$$



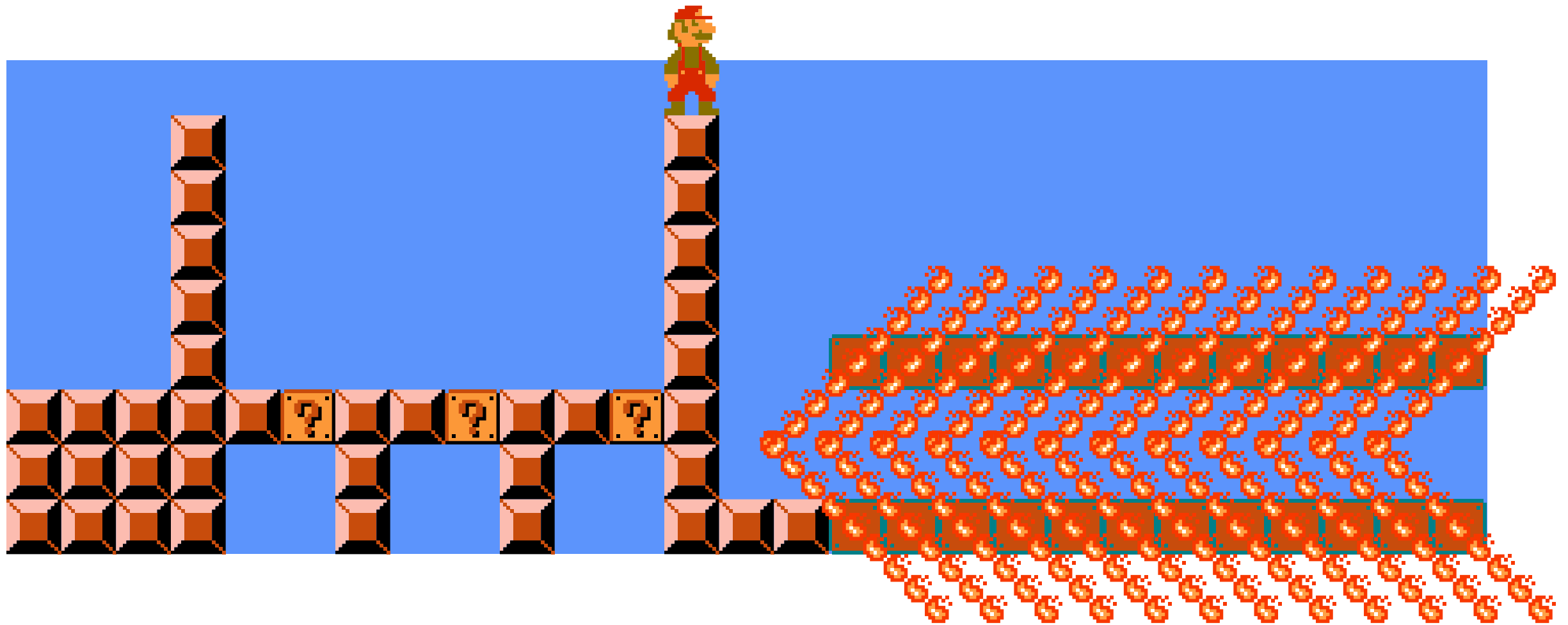
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 0$$



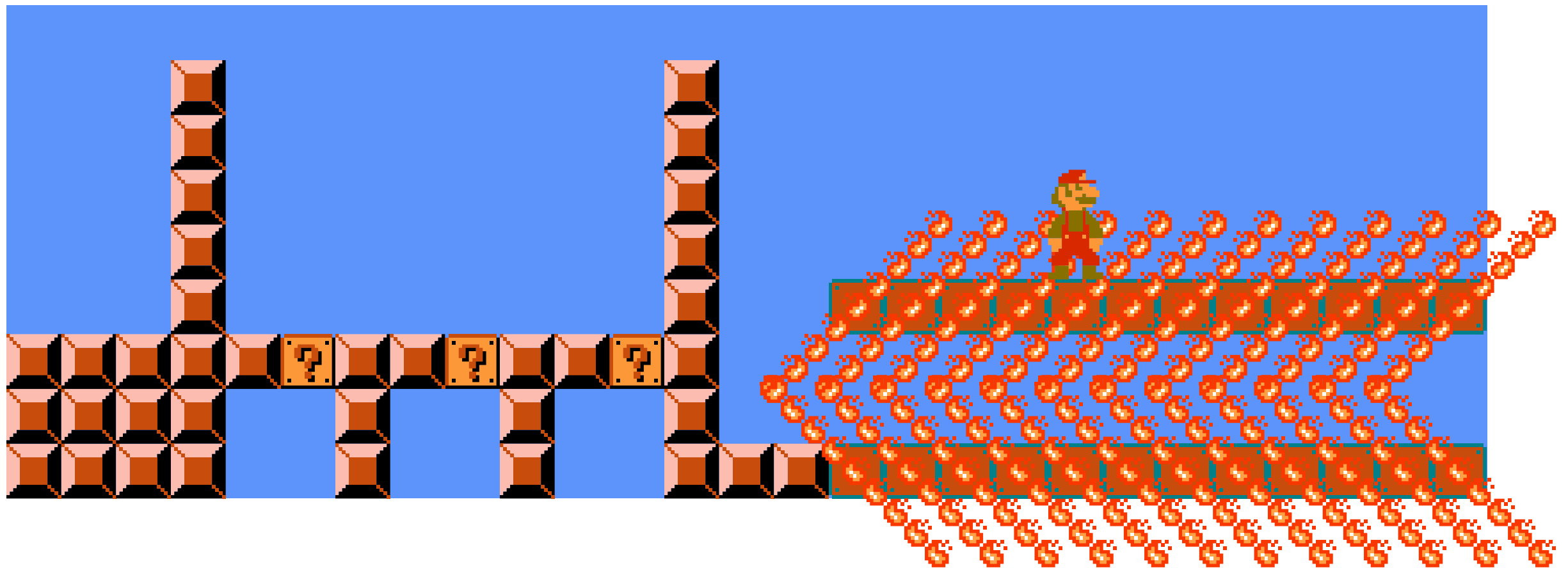
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 0$$



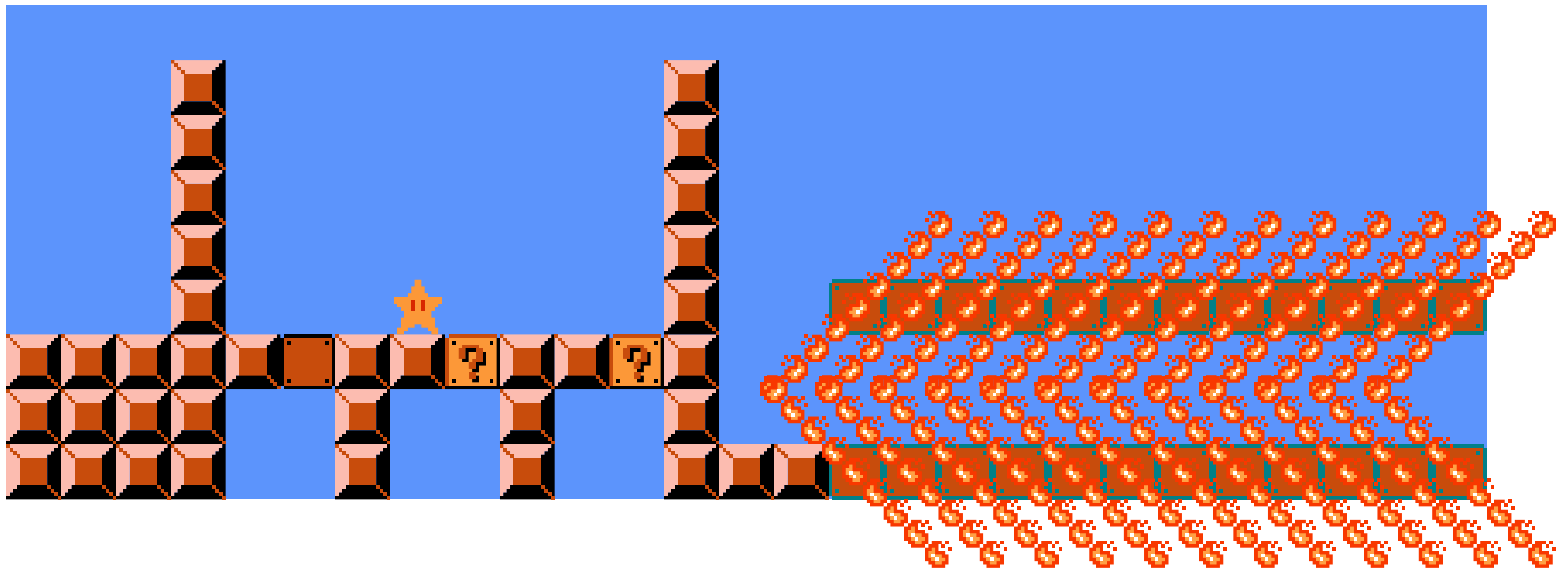
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 0$$



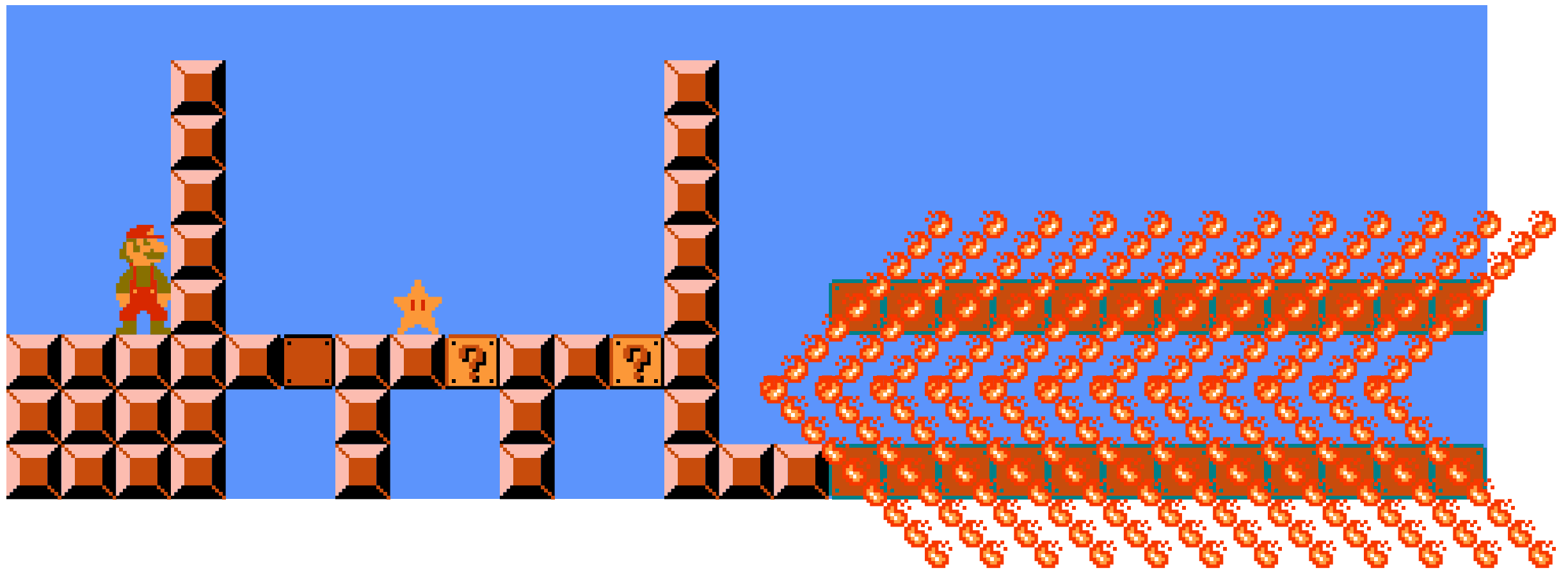
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$



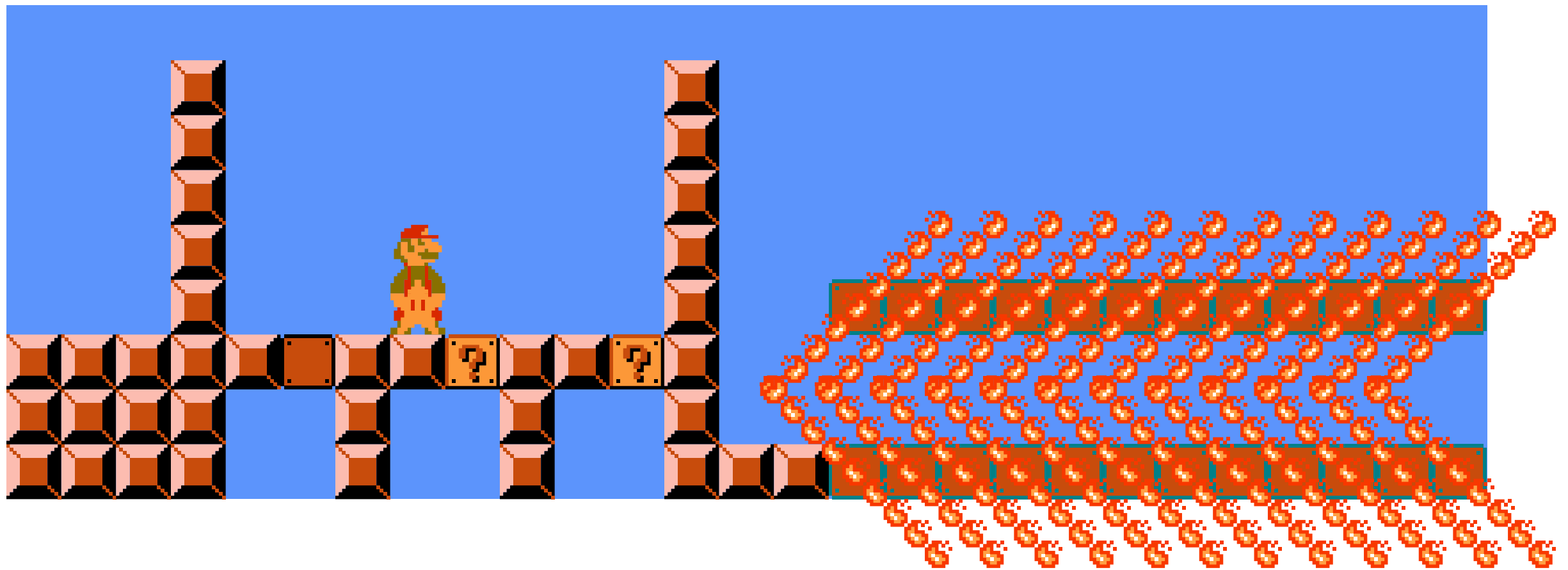
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$



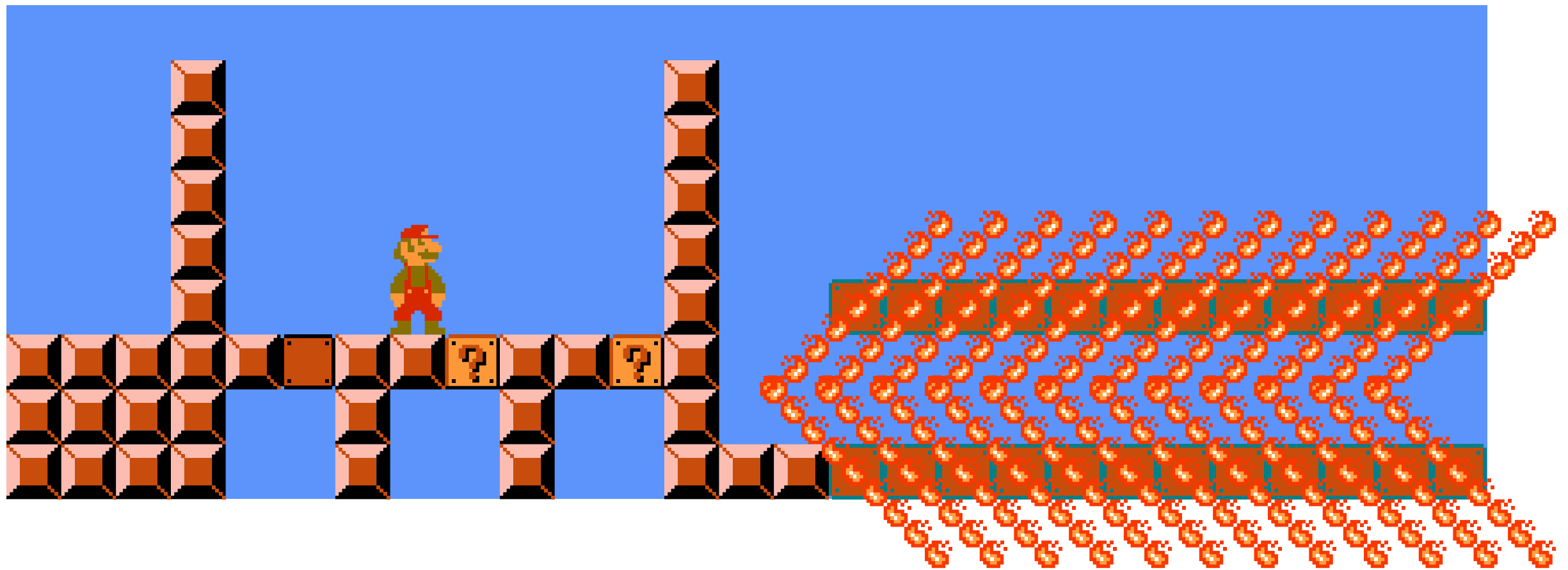
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$



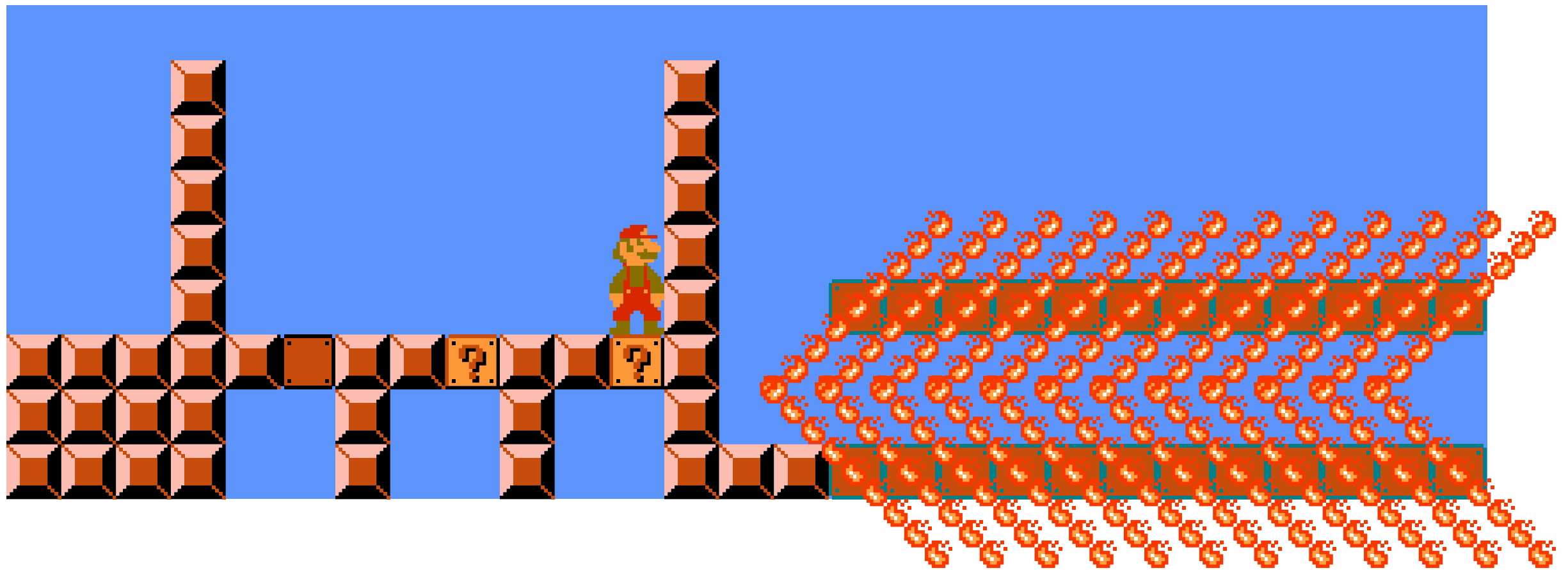
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$



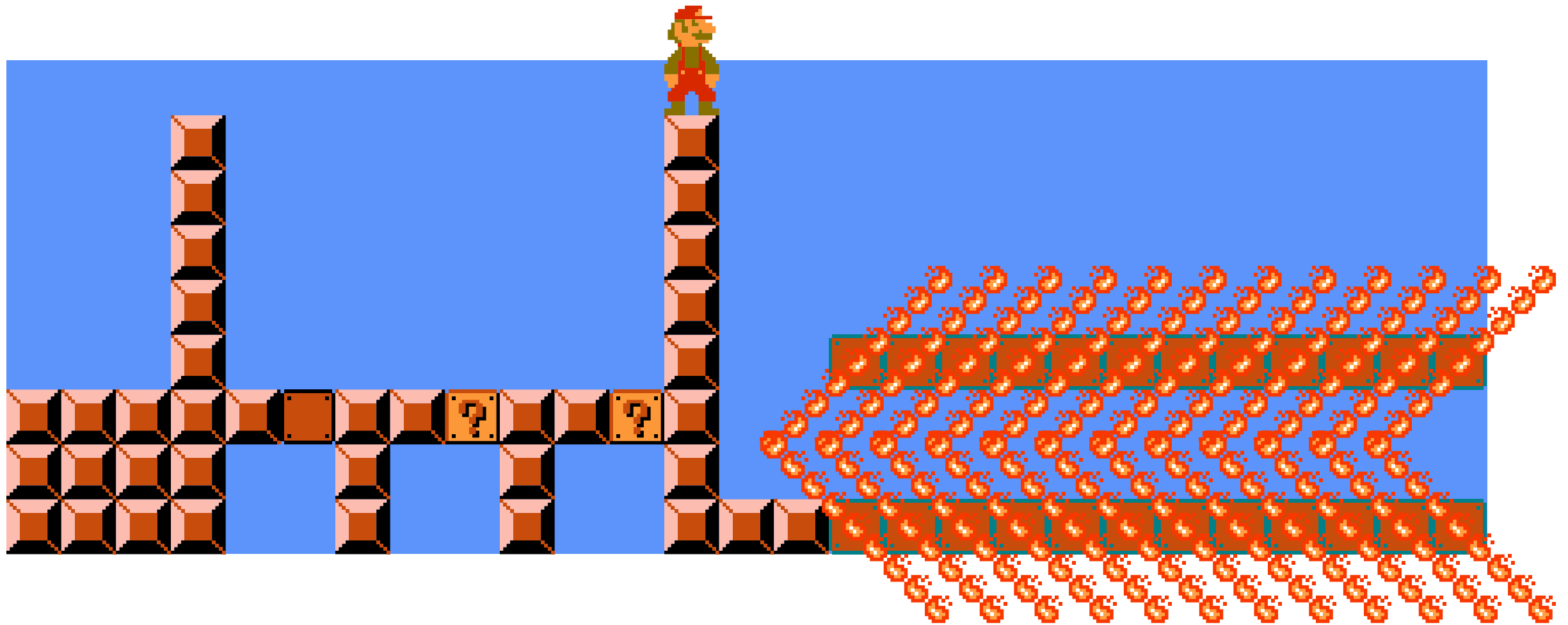
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$



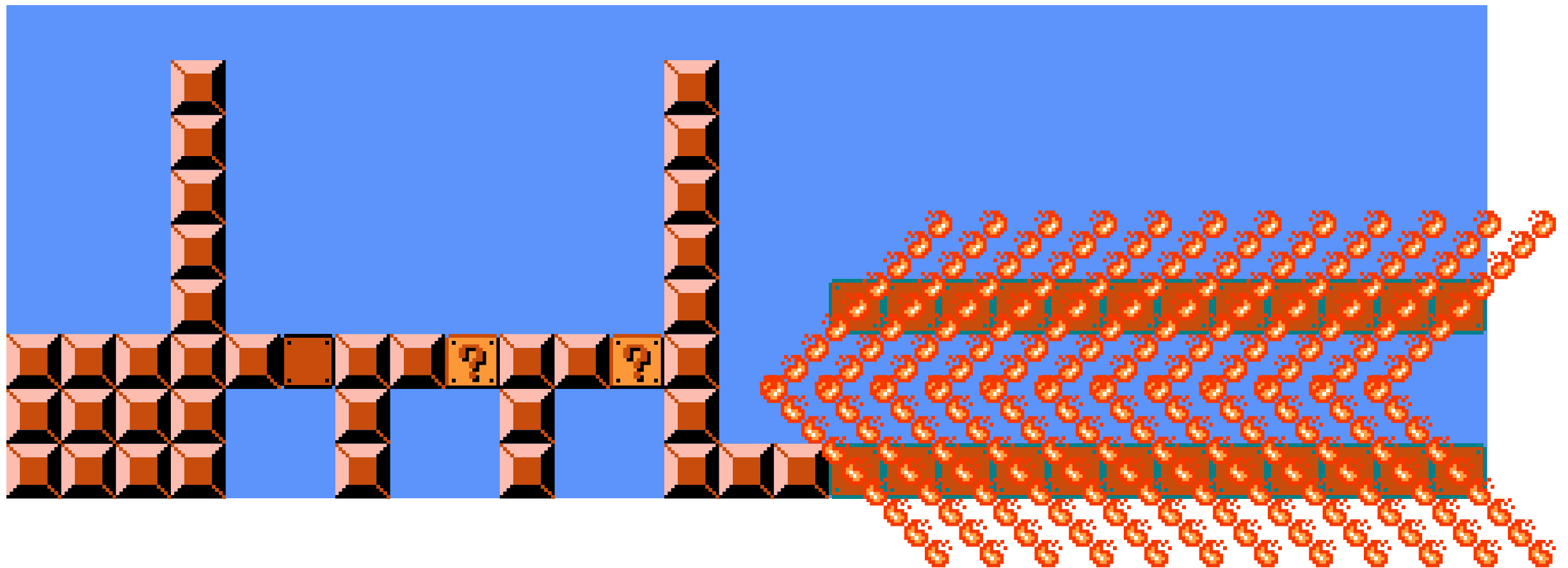
Valutazione delle clausole

$$x_1 \vee \neg x_3 \vee x_6 = 1$$

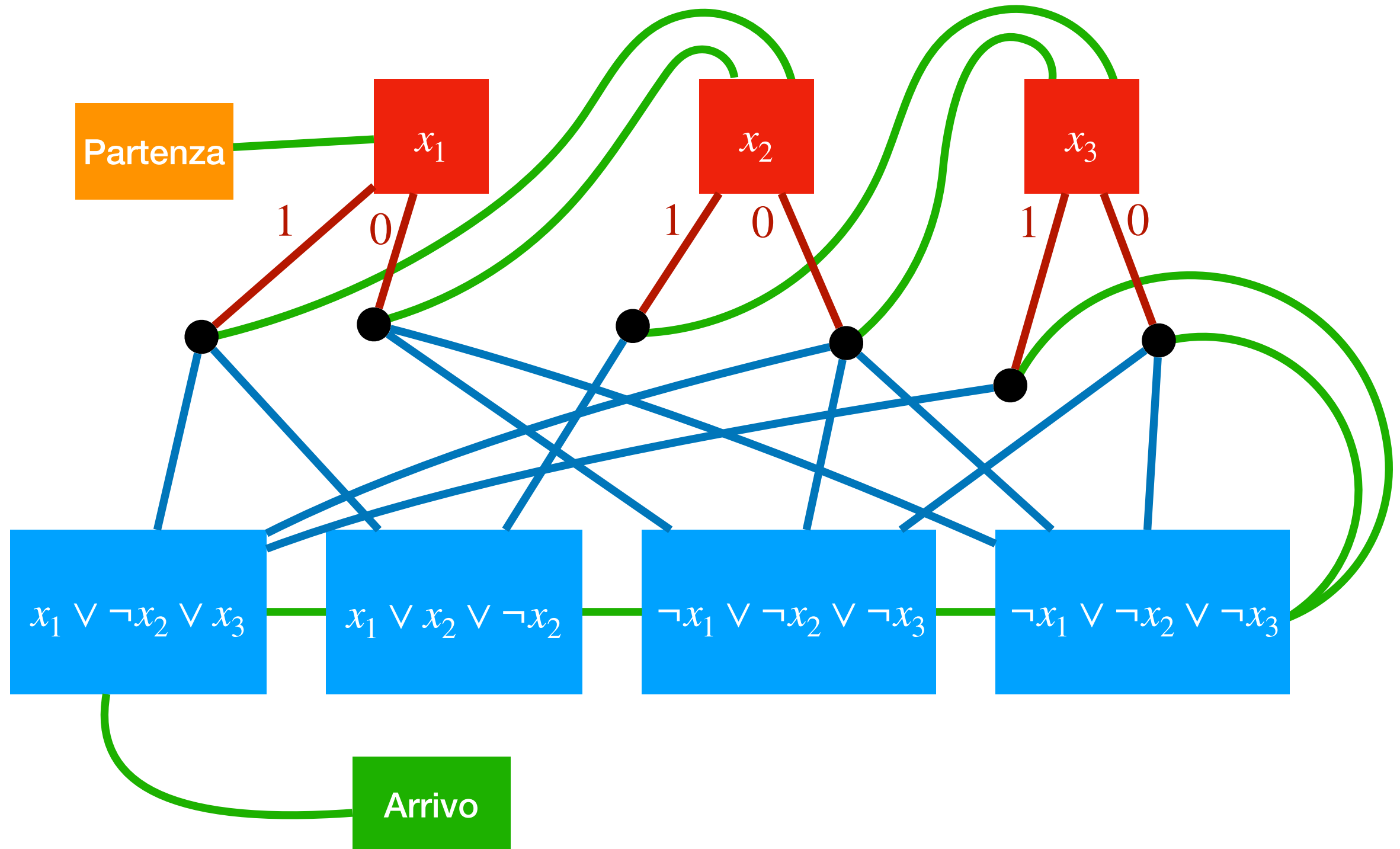


Valutazione delle clausole

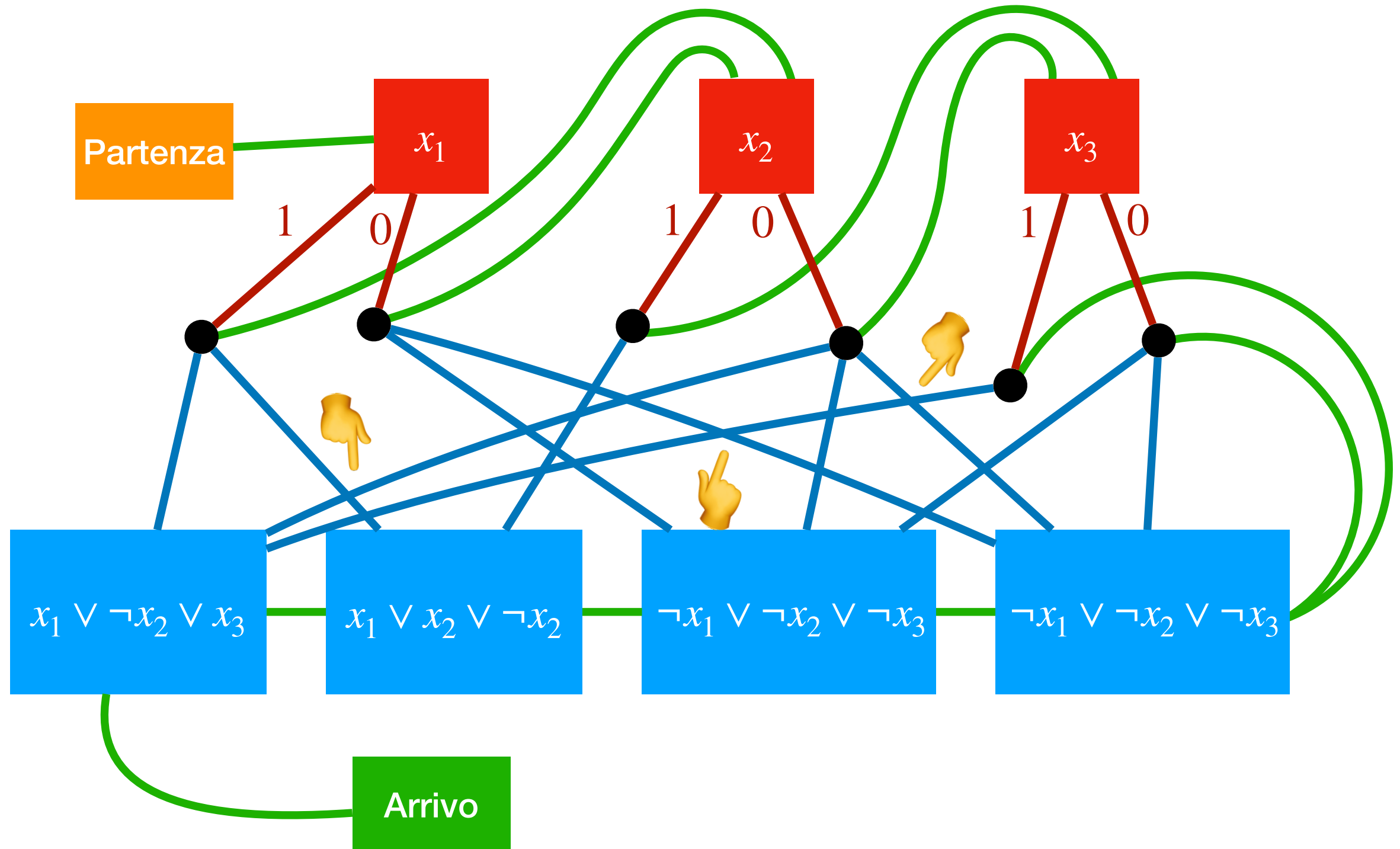
$$x_1 \vee \neg x_3 \vee x_6 = 1$$



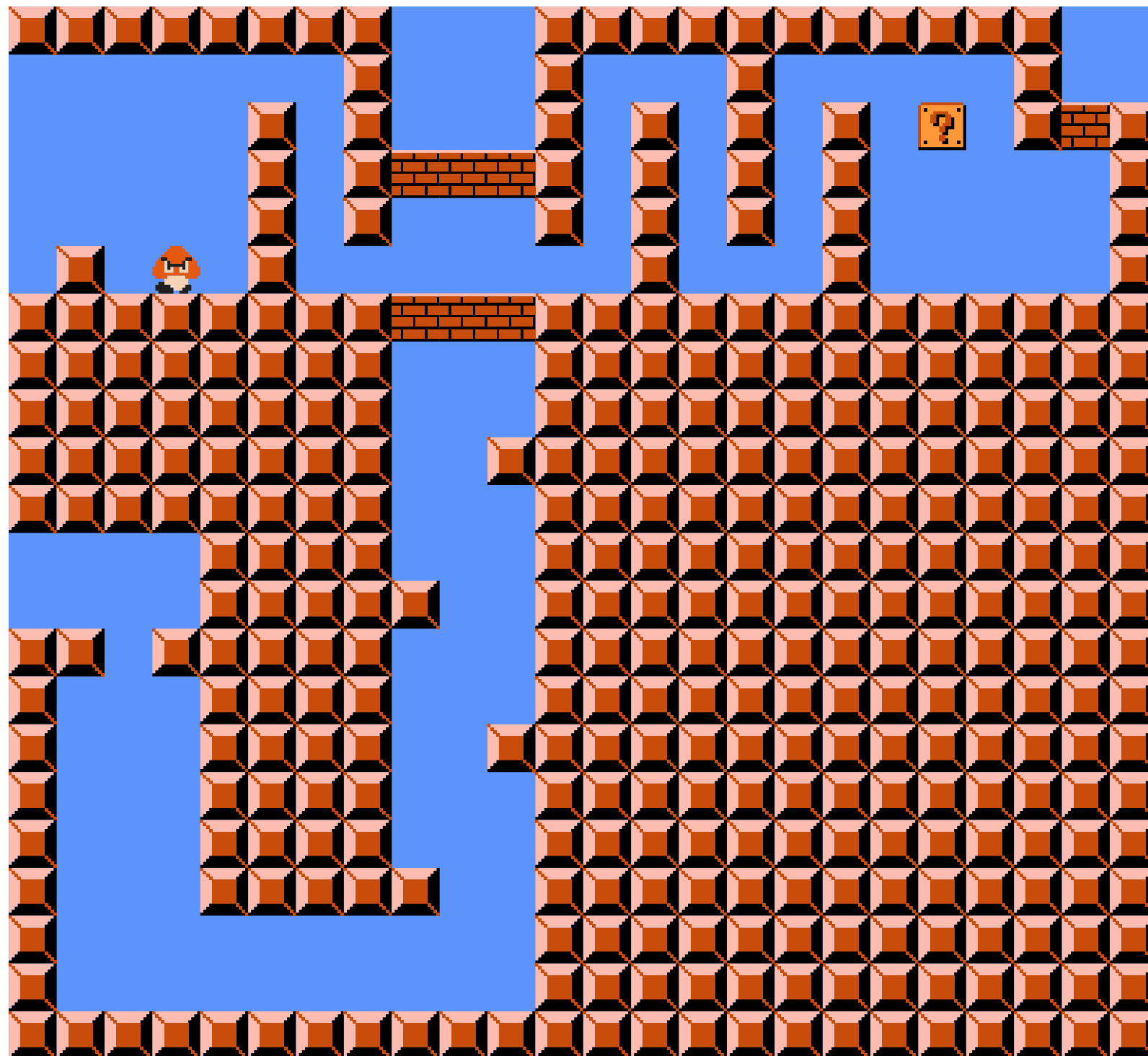
E gli incroci?



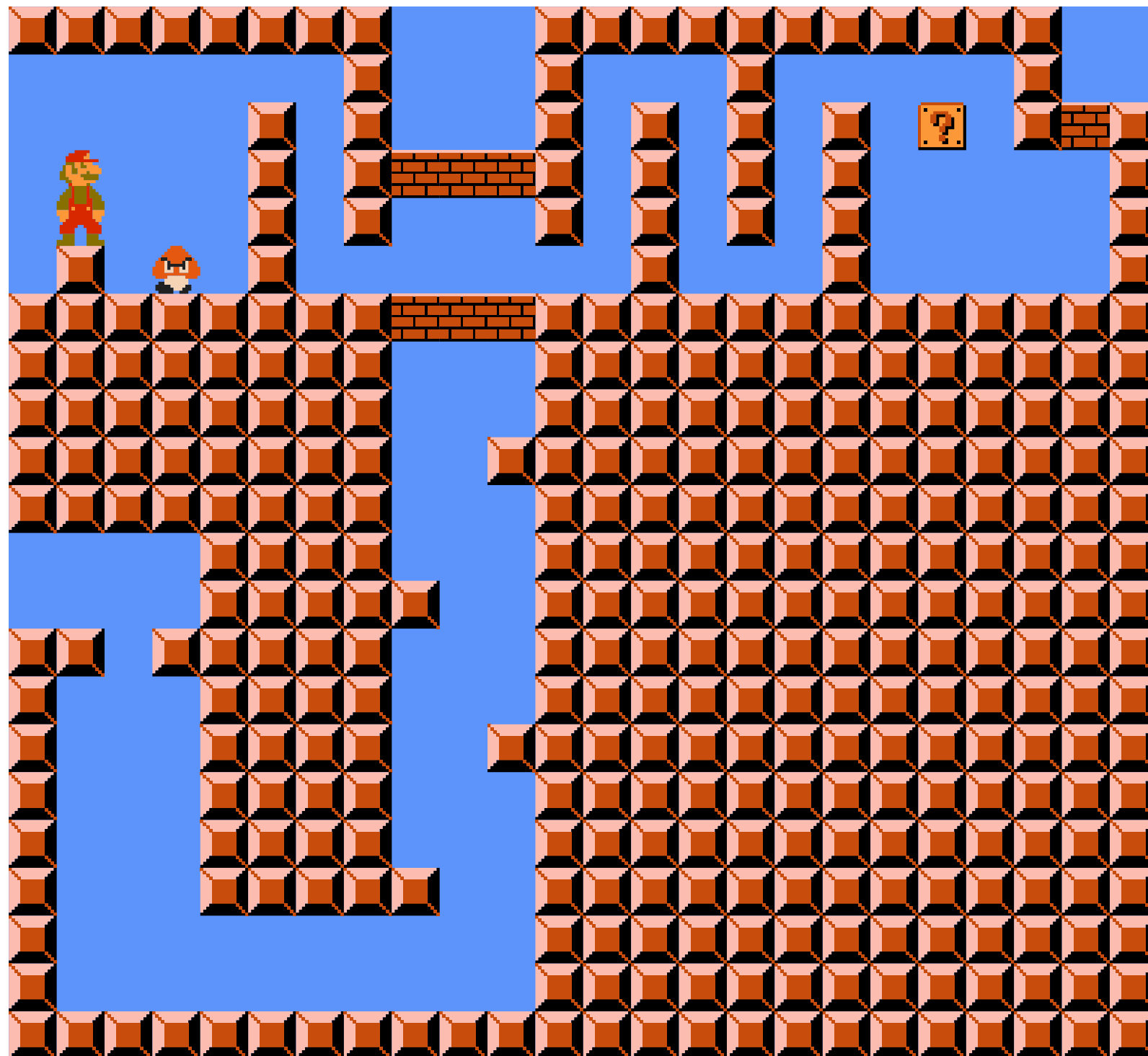
E gli incroci?



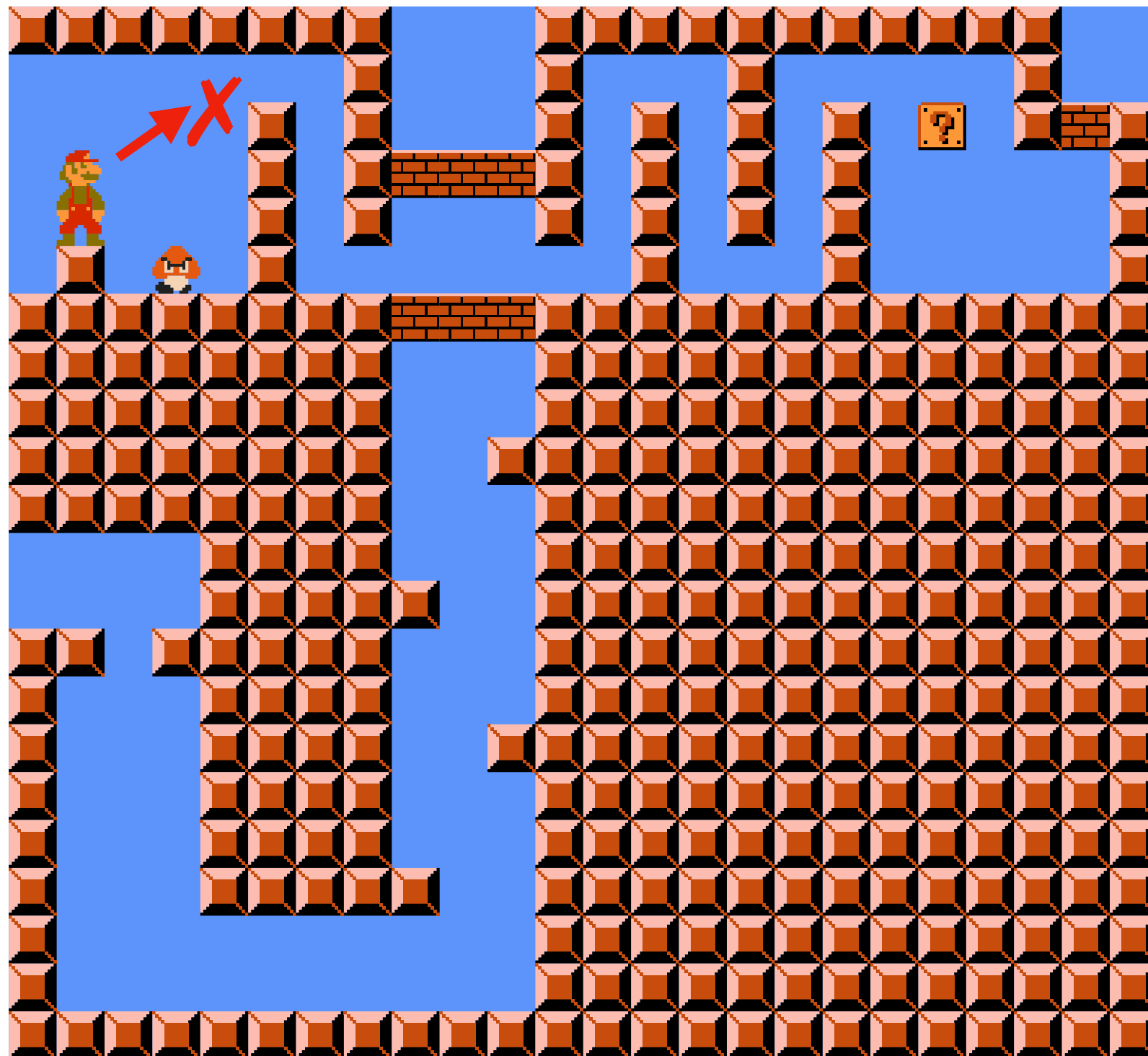
Gadget per gli incroci



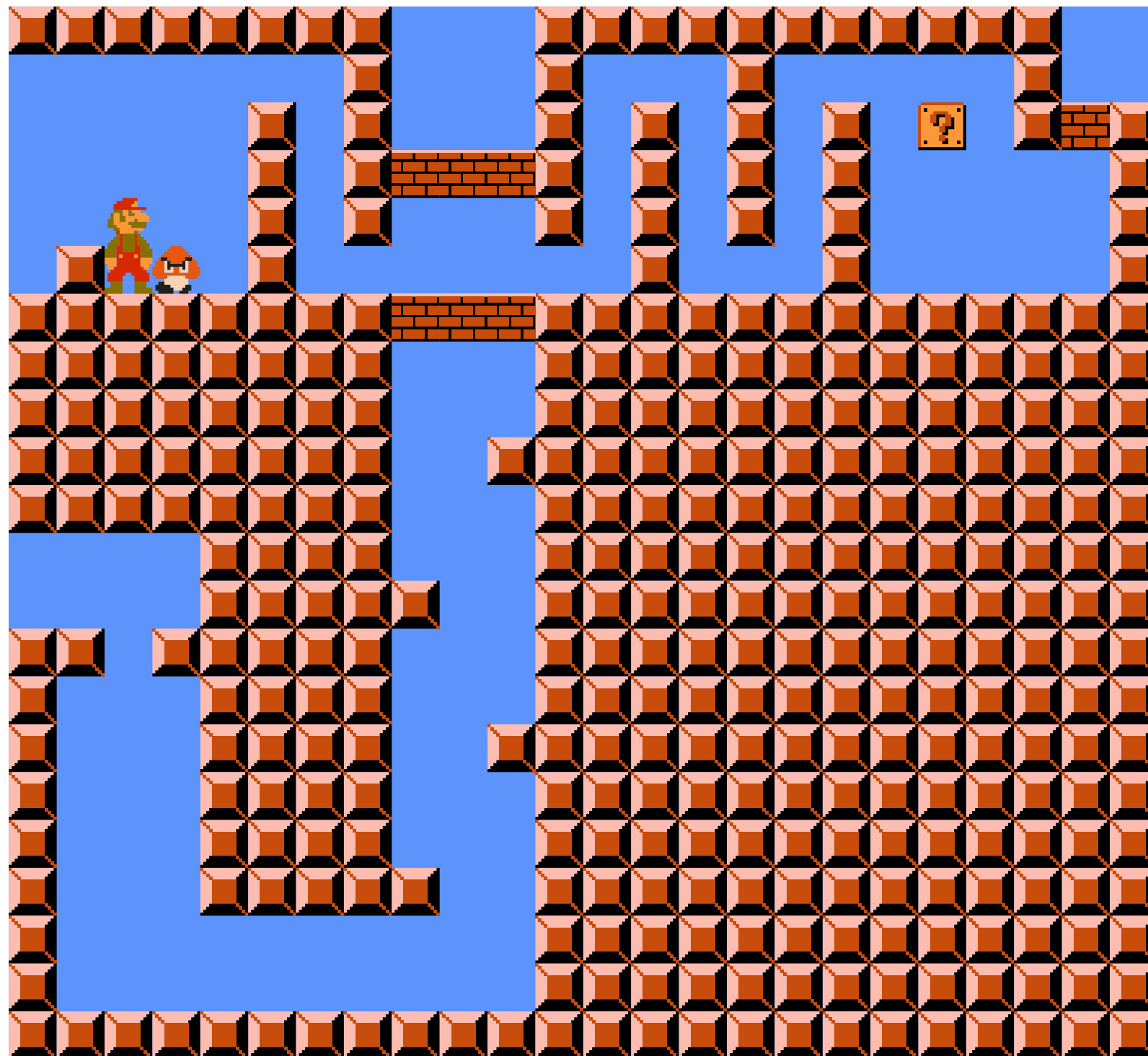
Gadget per gli incroci



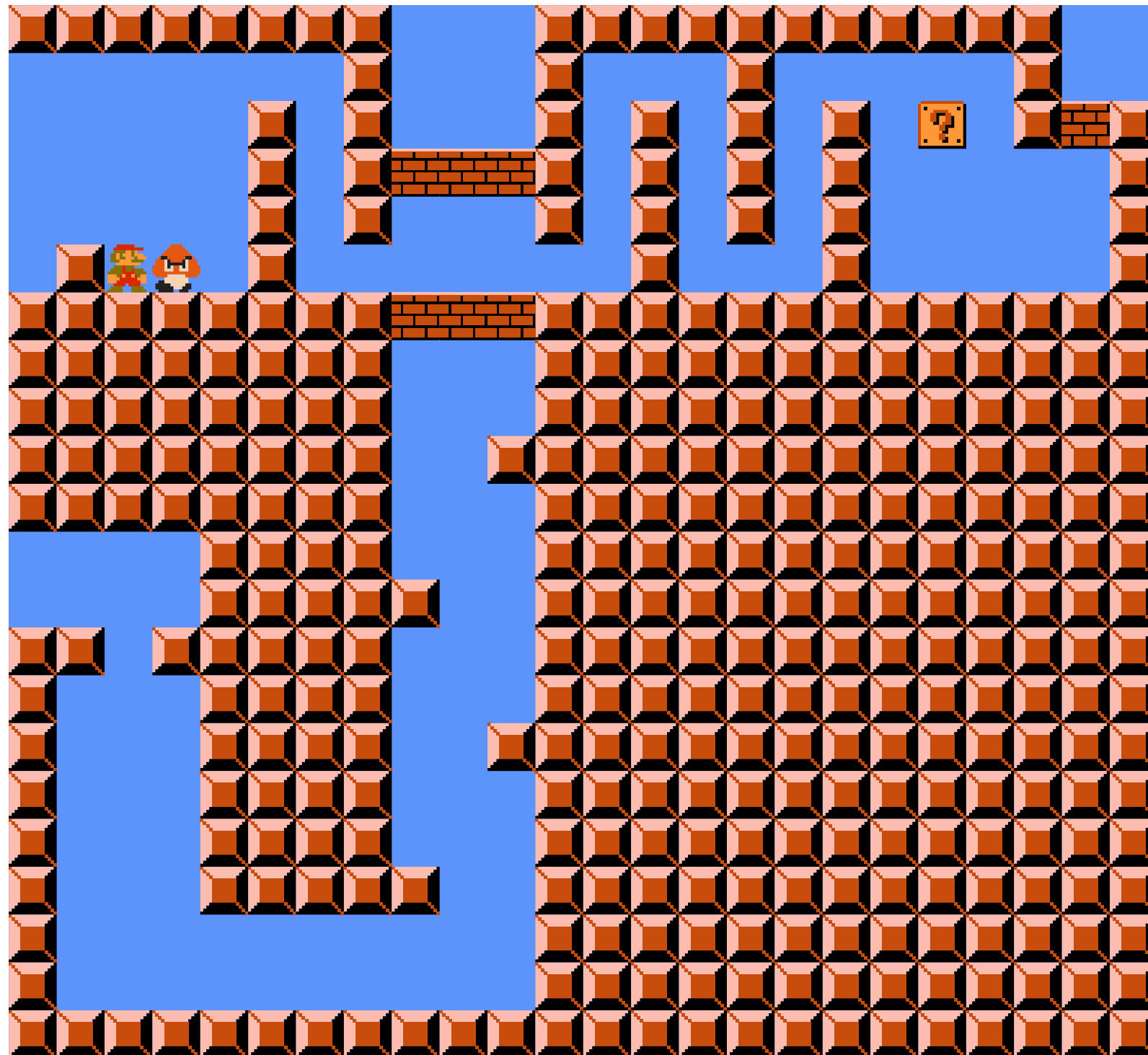
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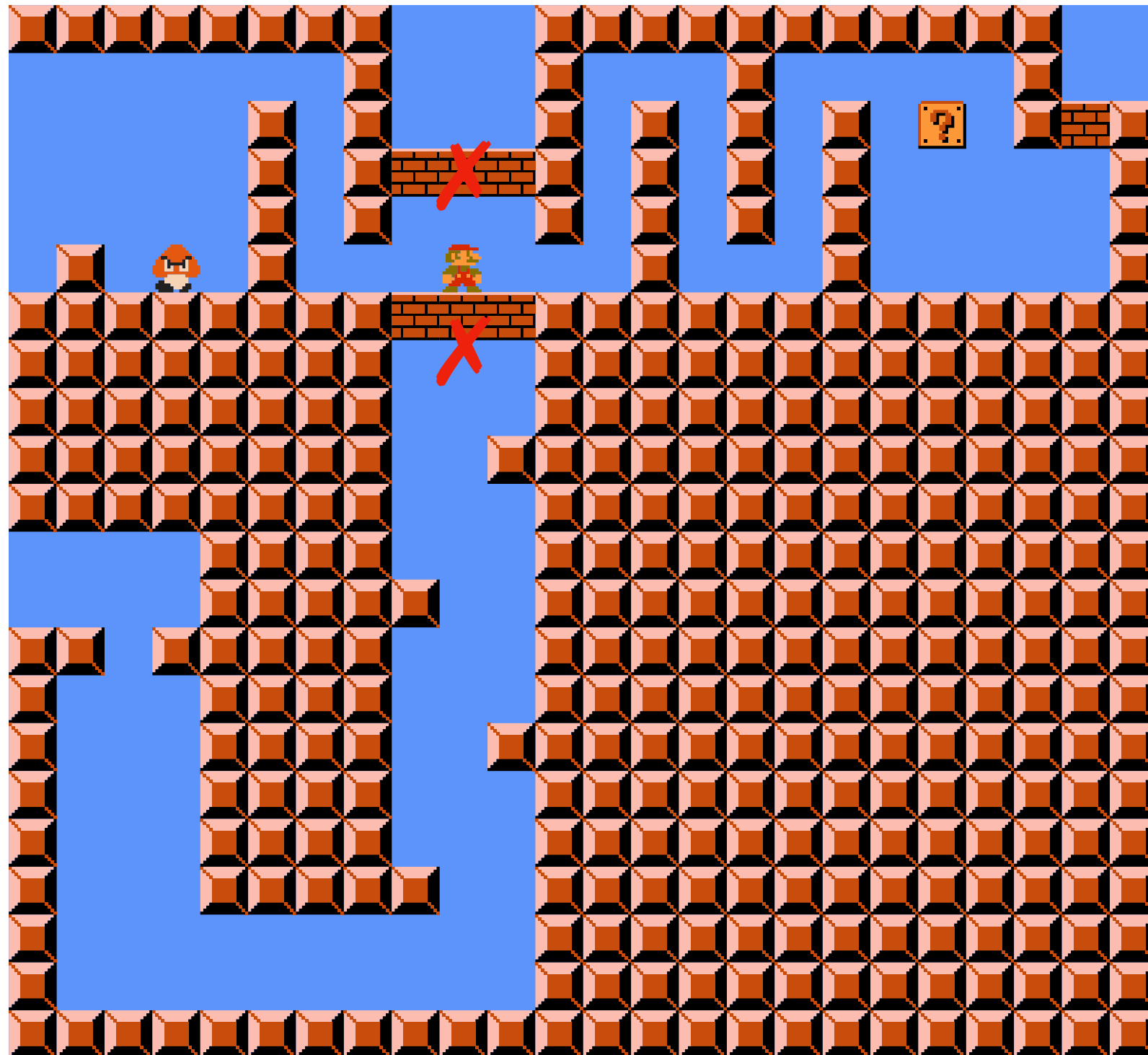
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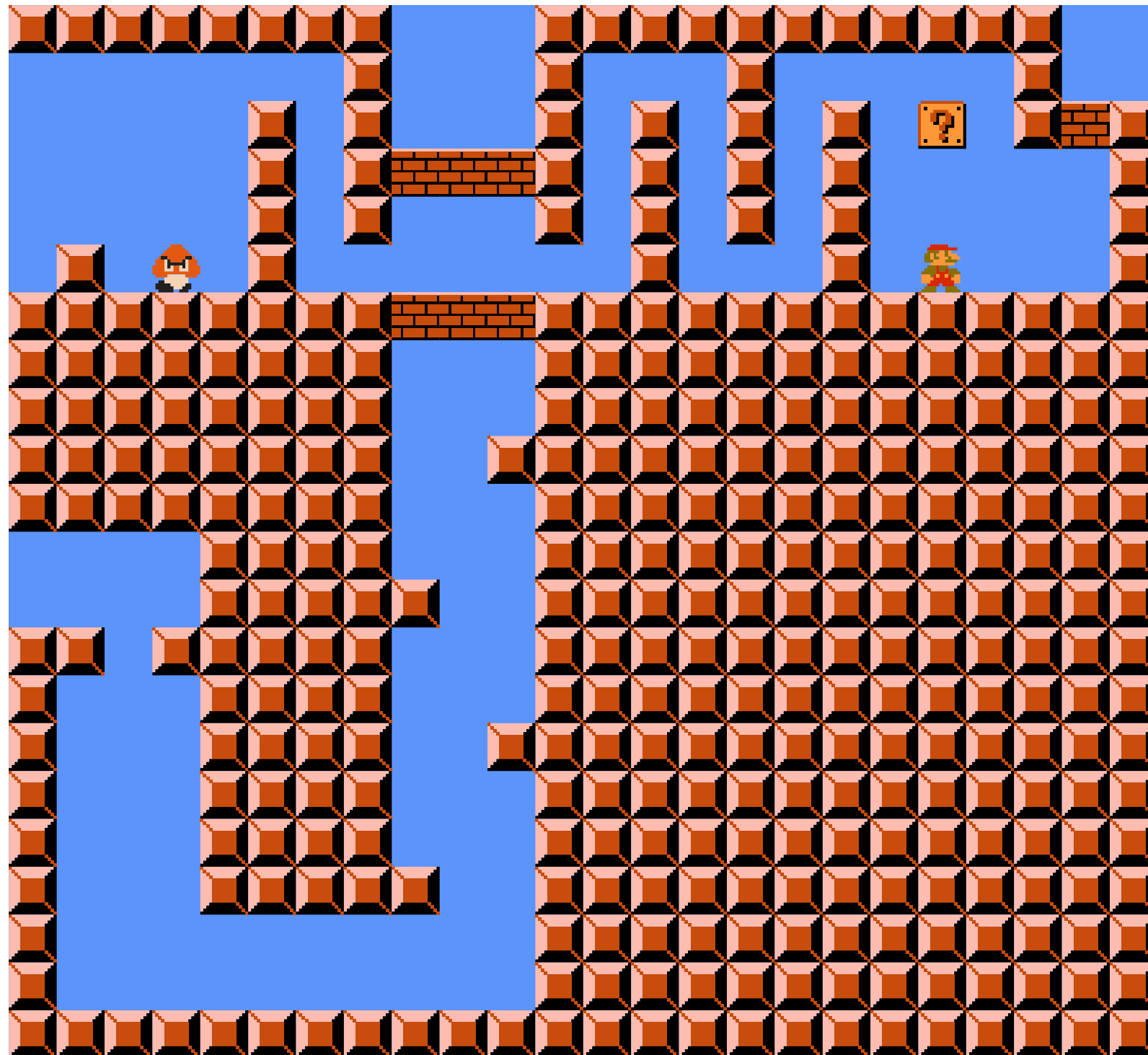
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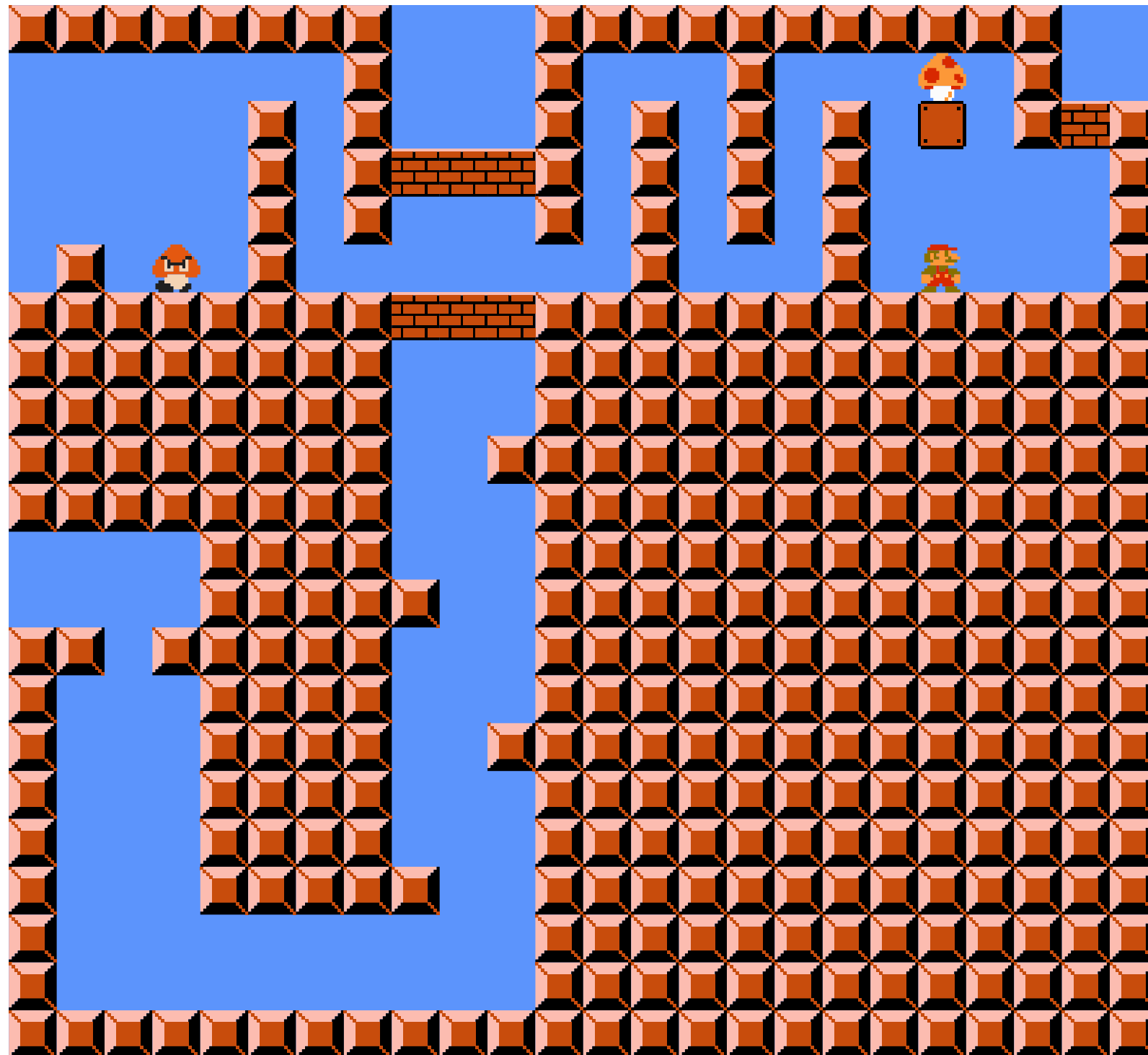
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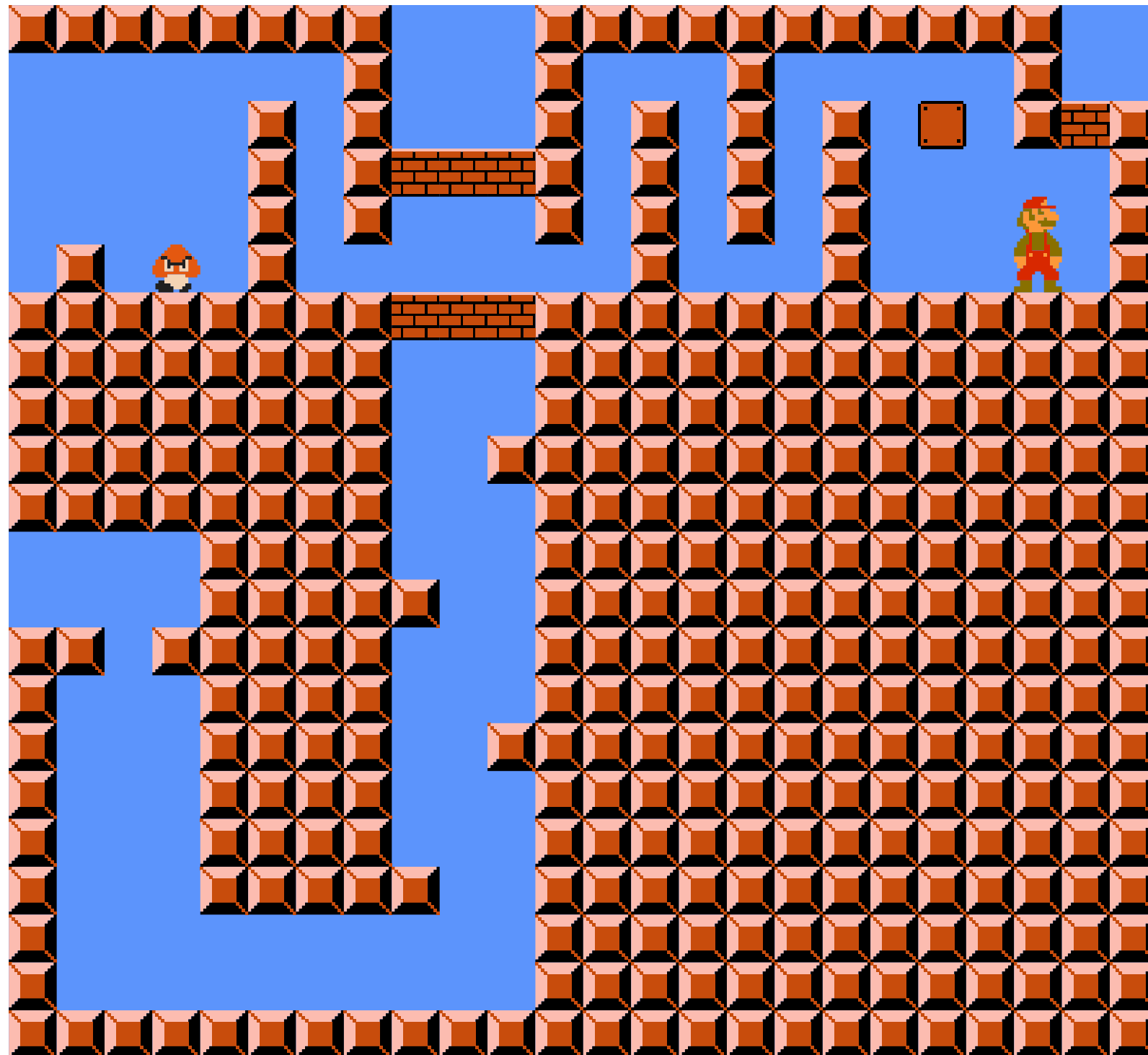
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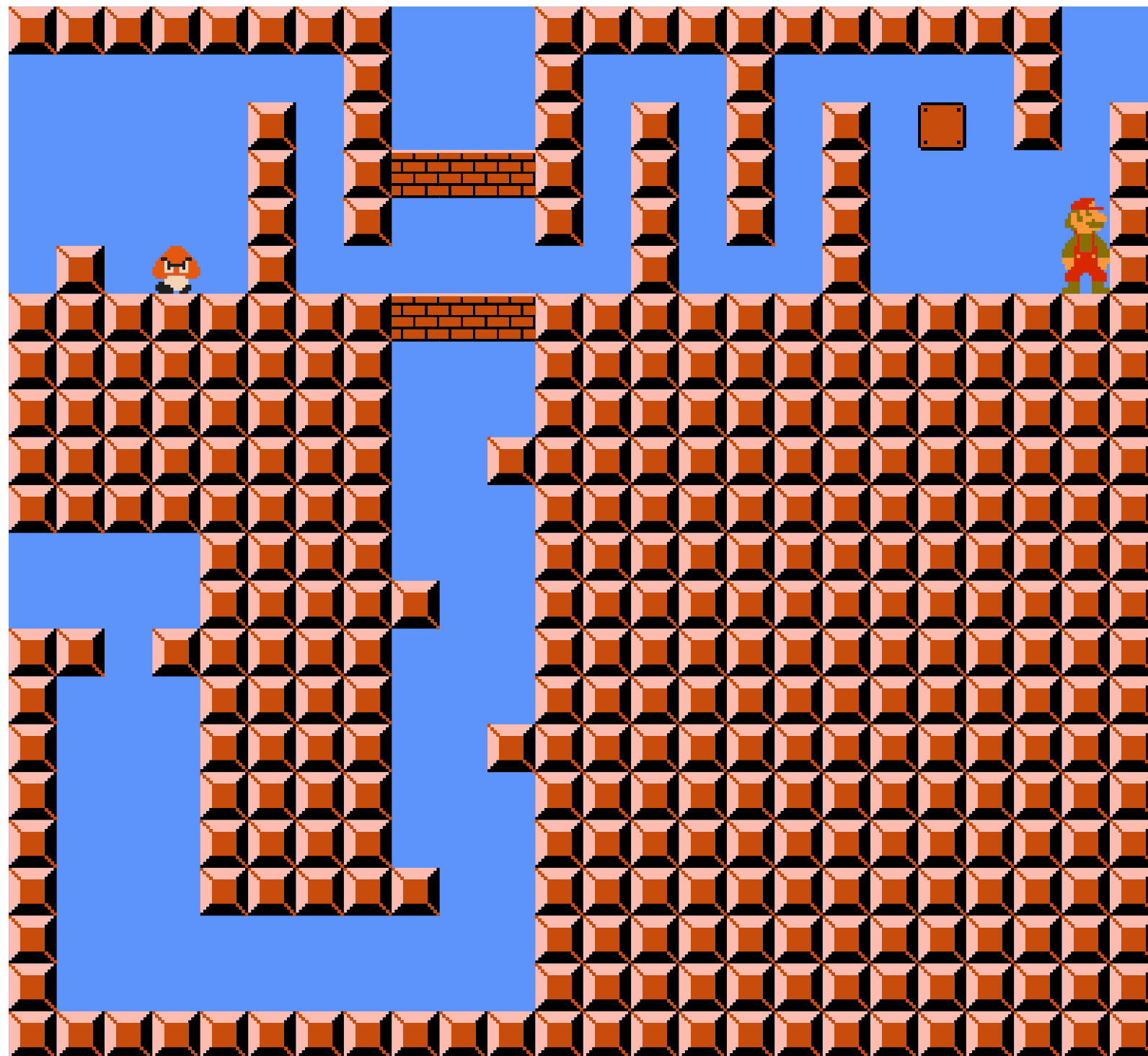
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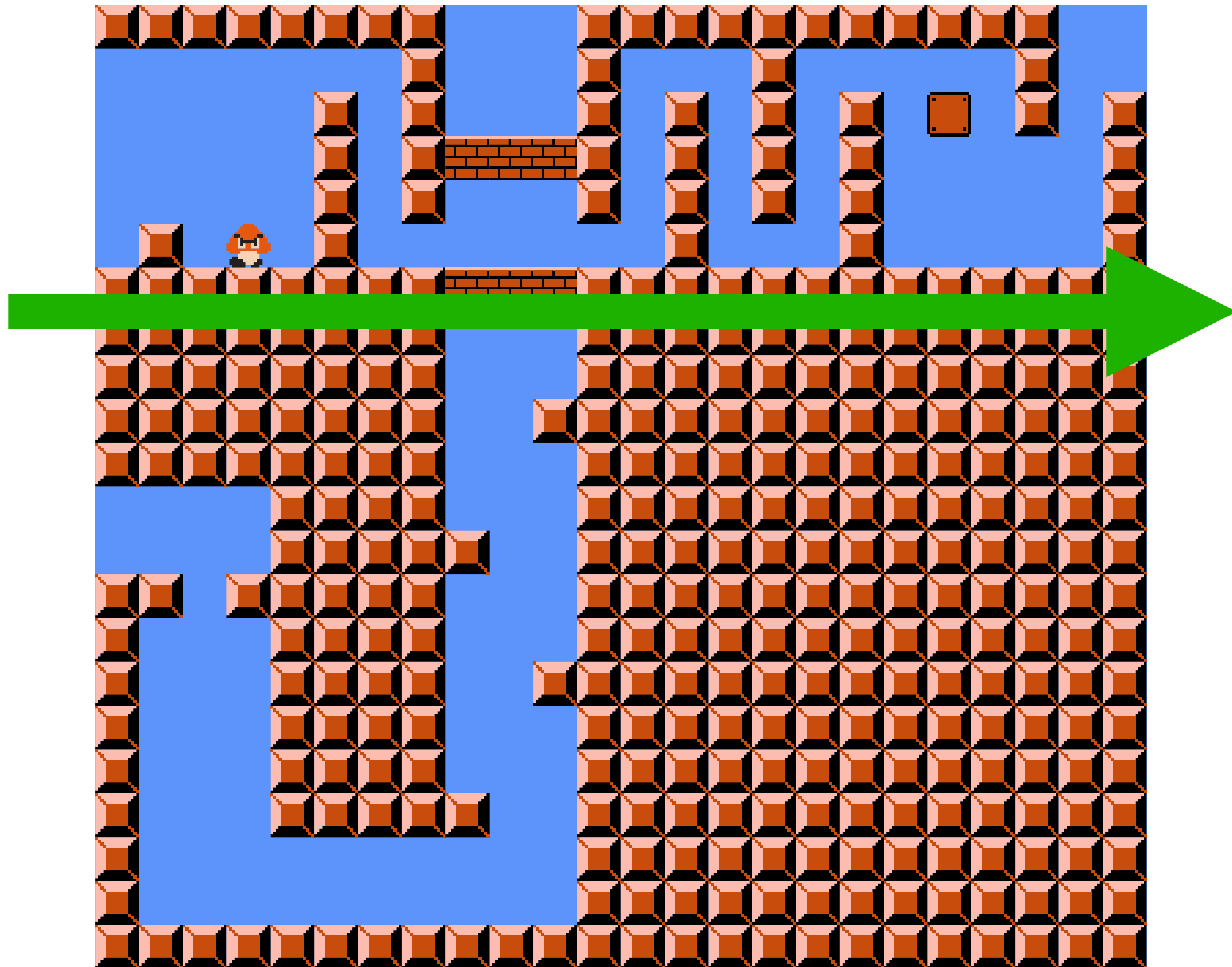
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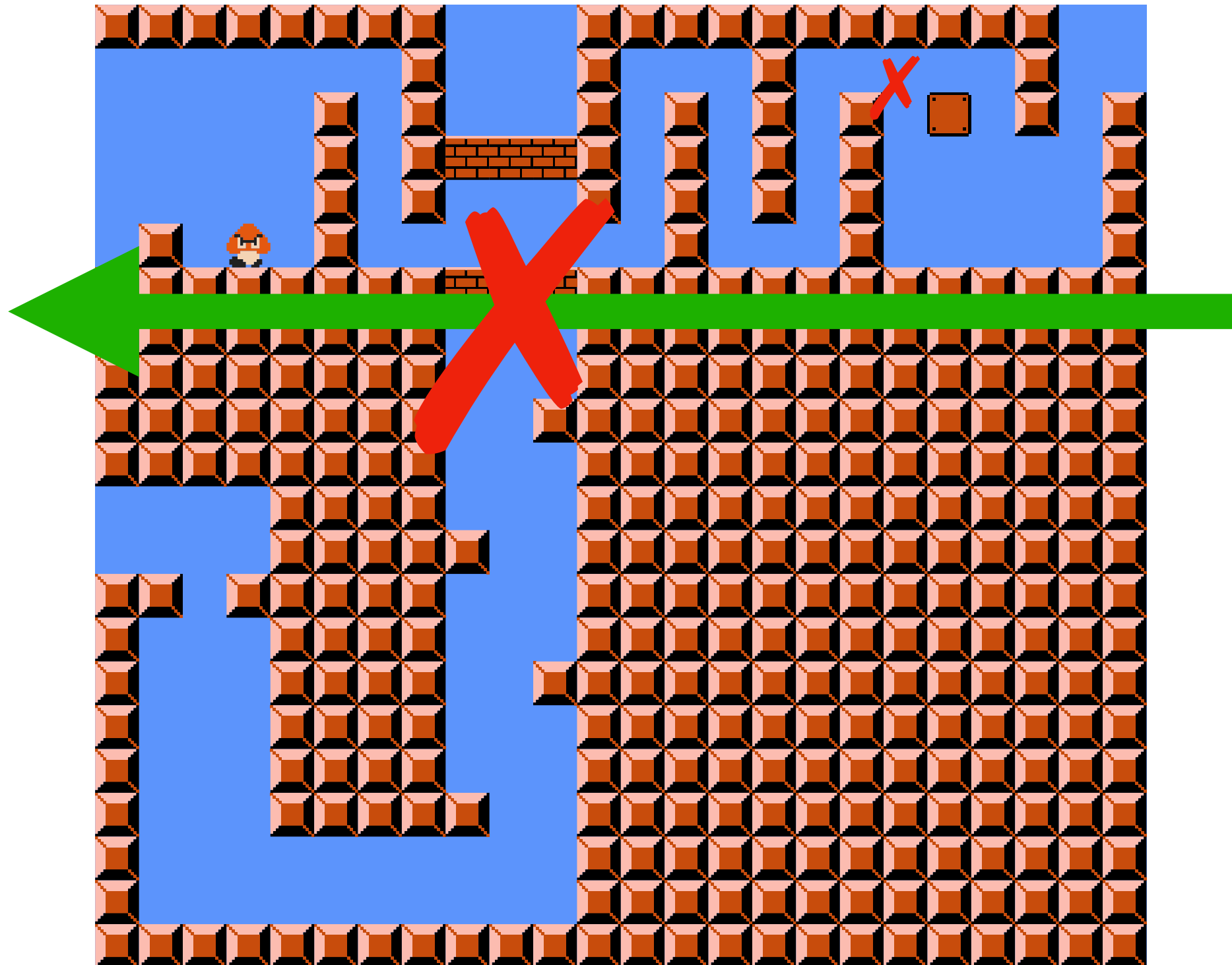
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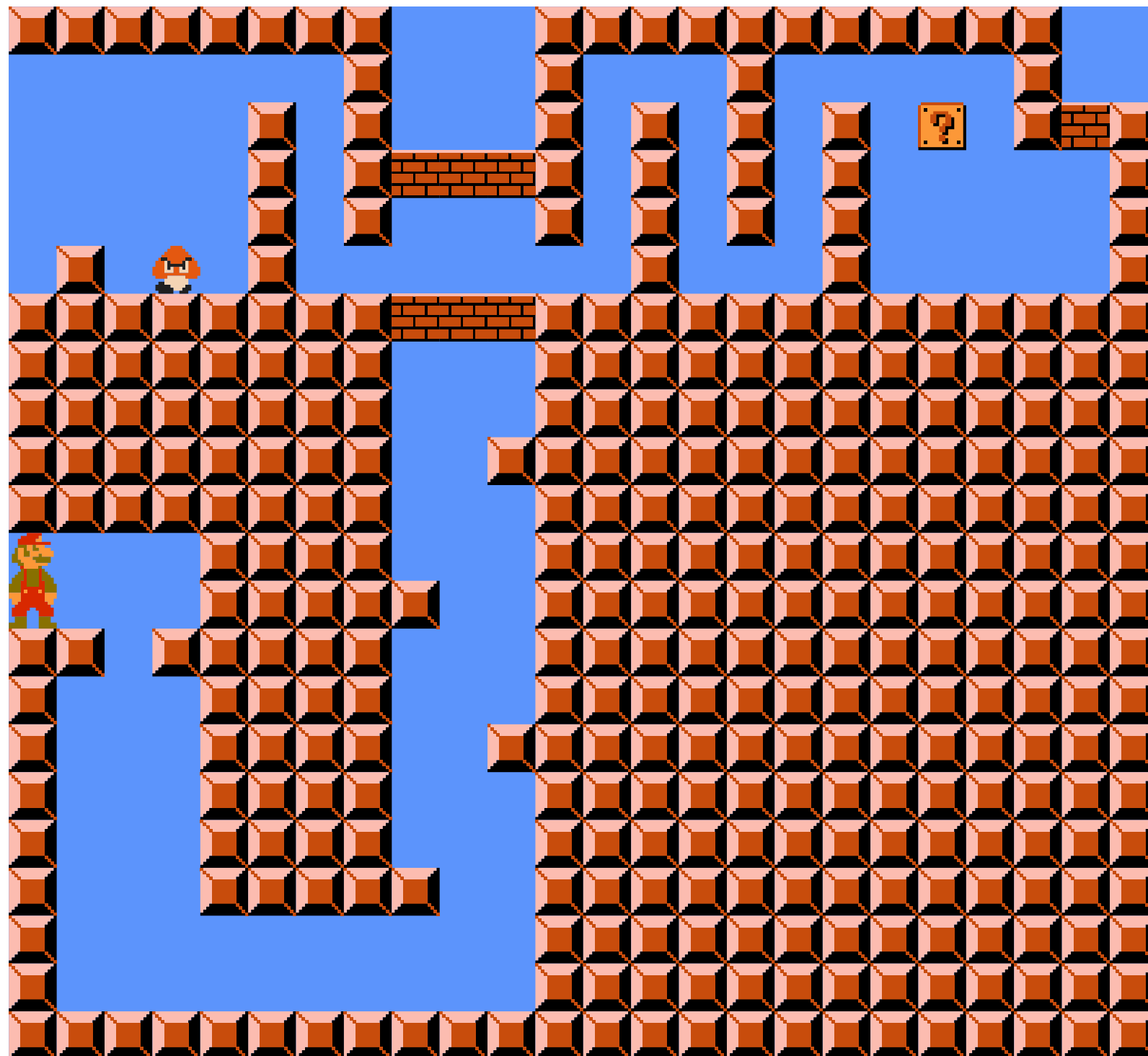
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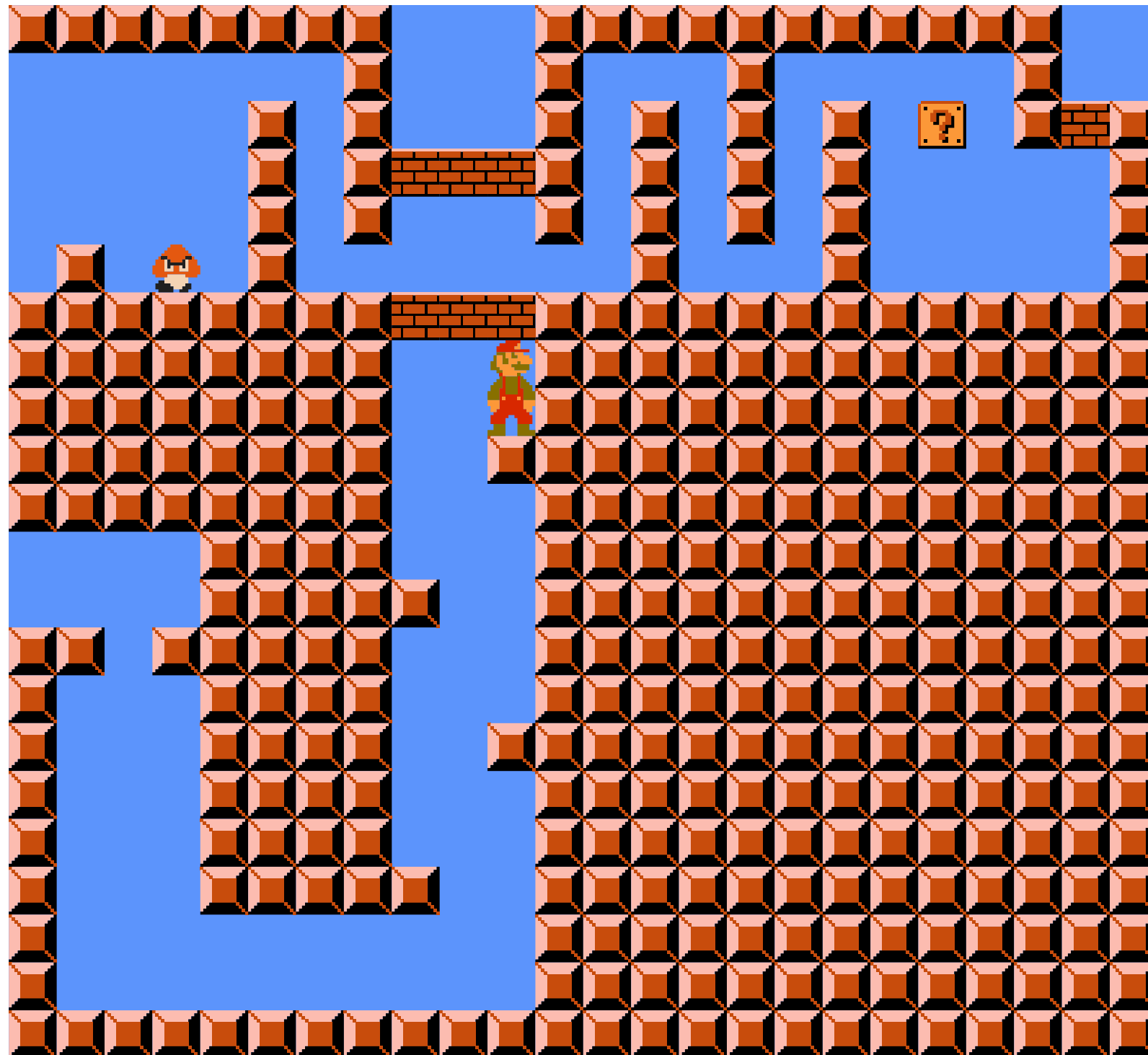
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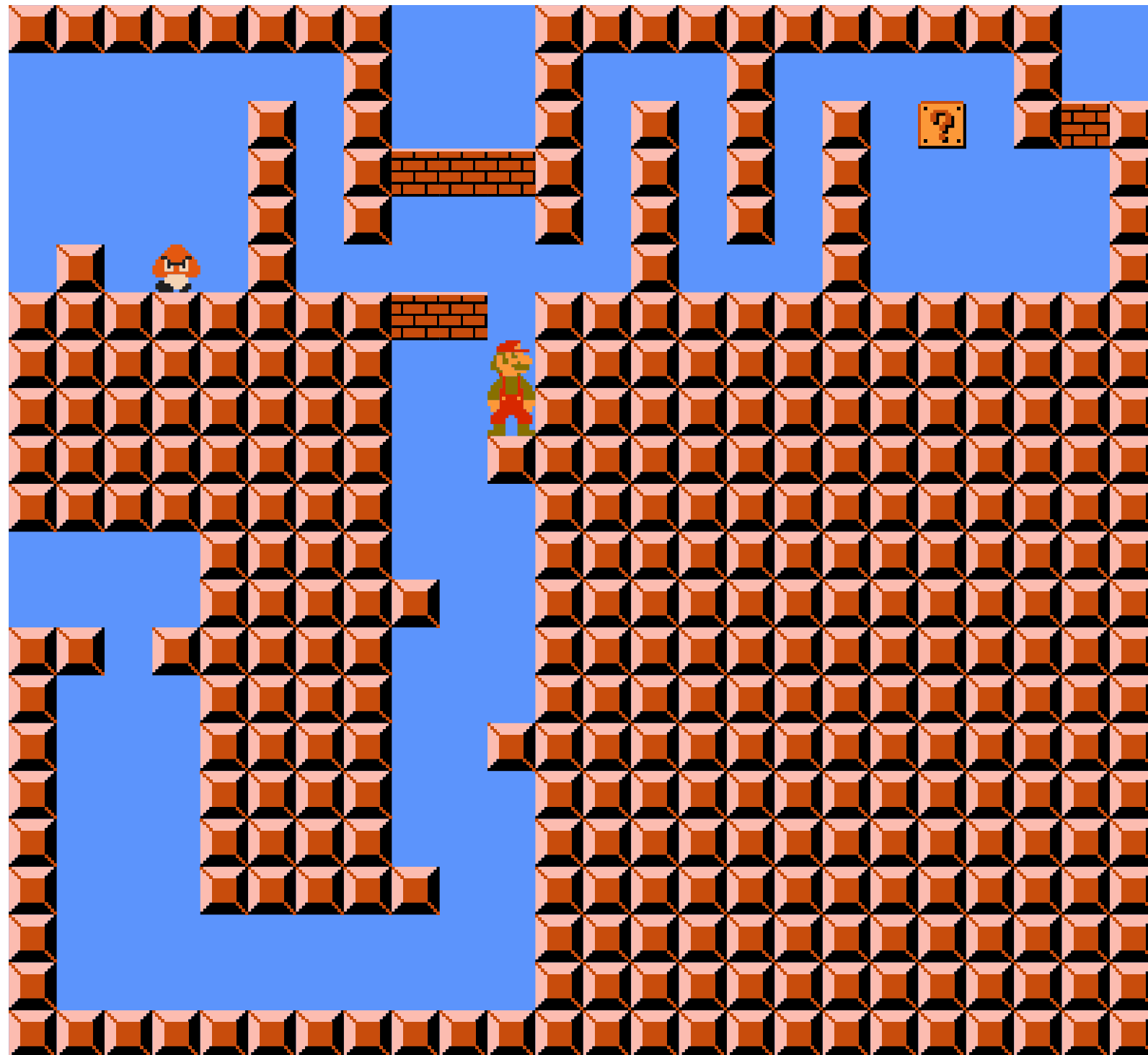
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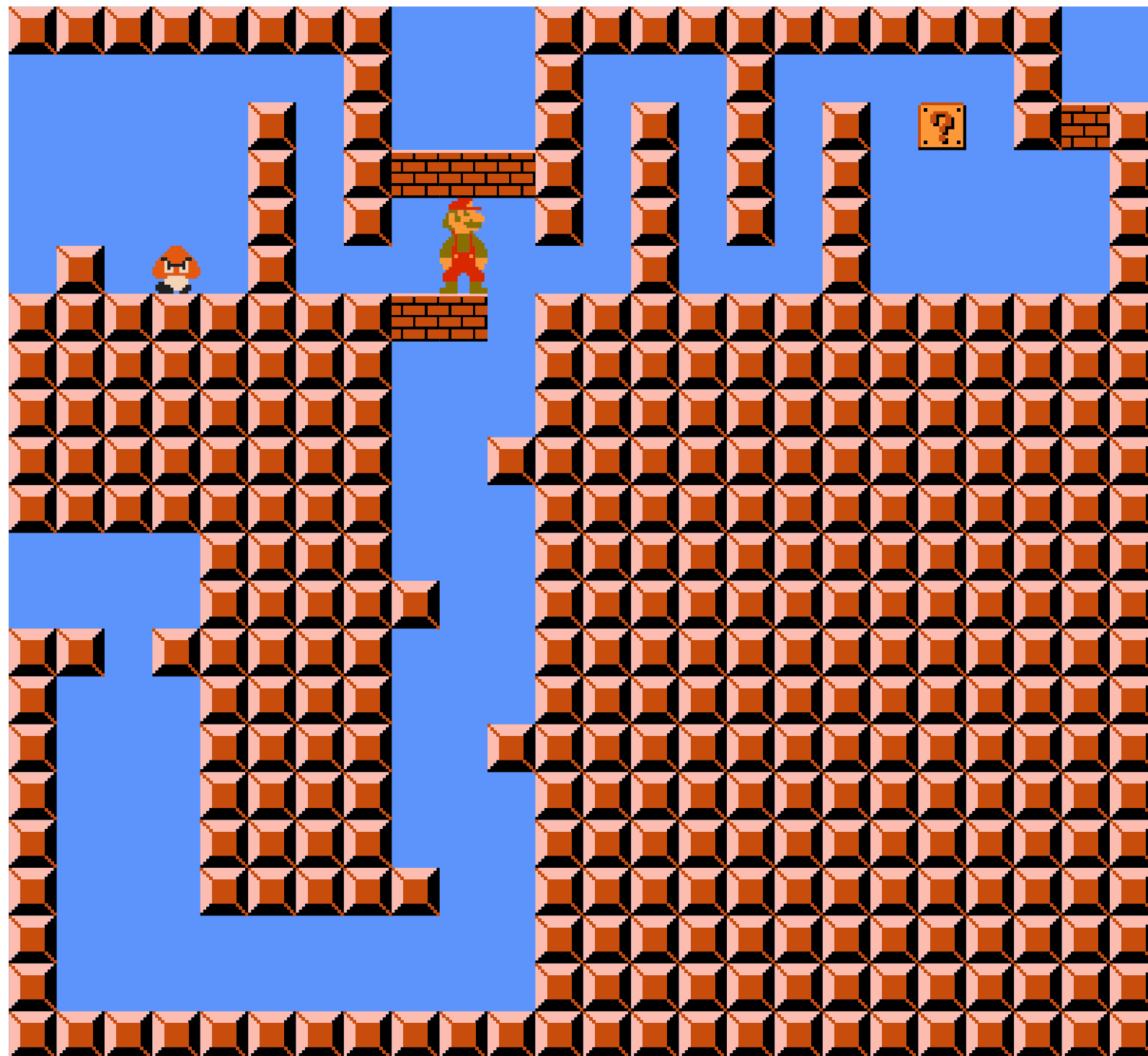
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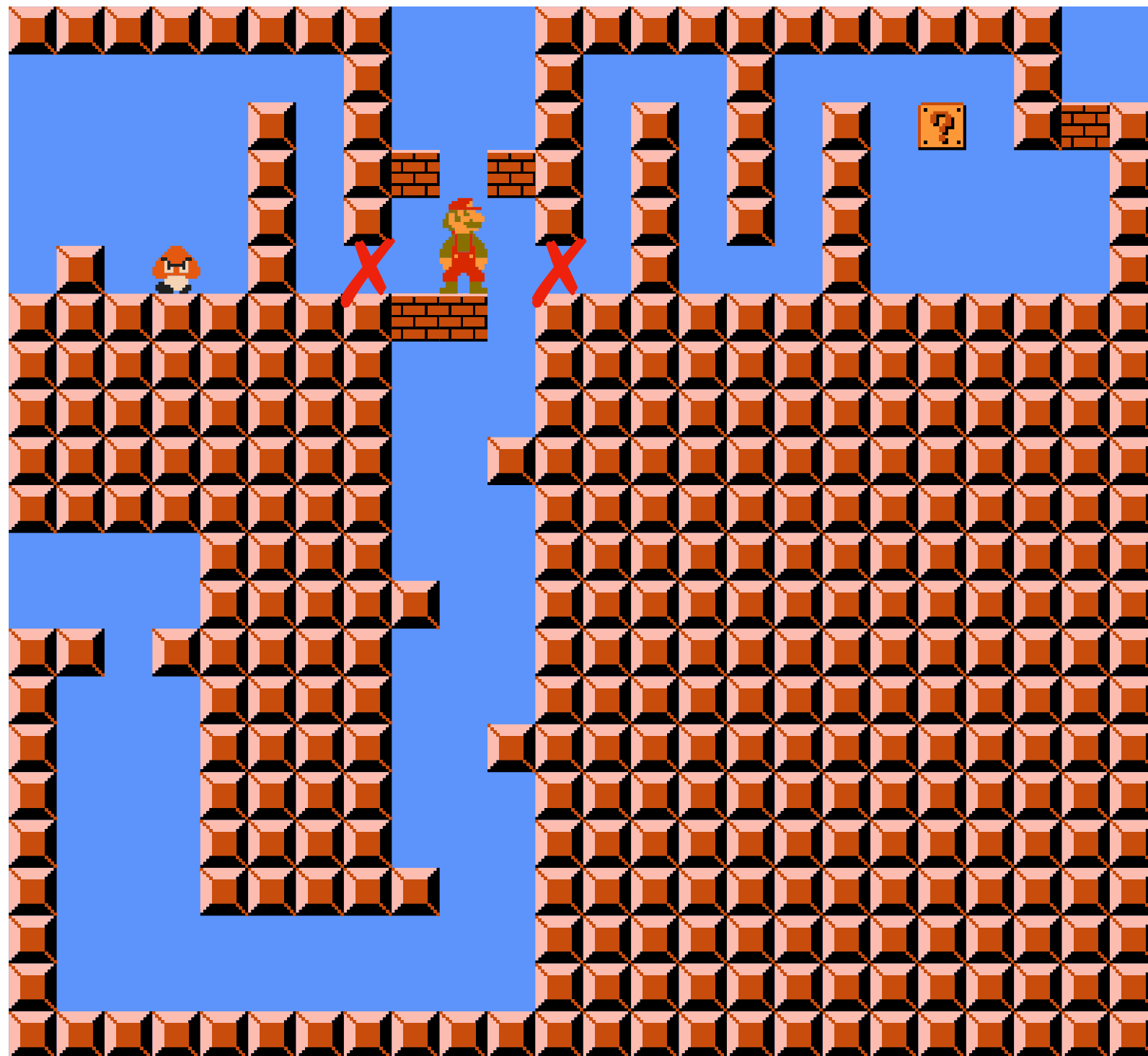
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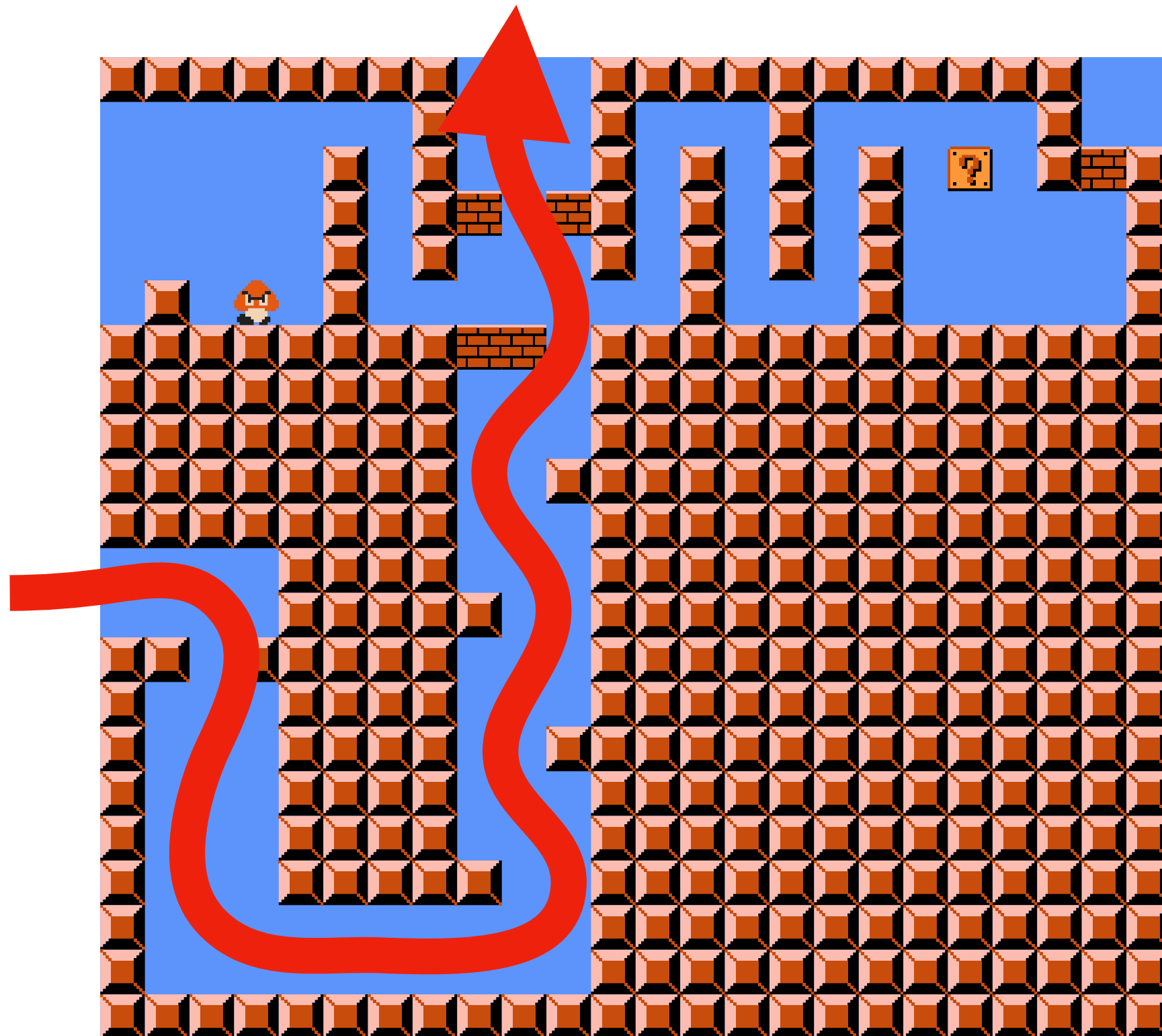
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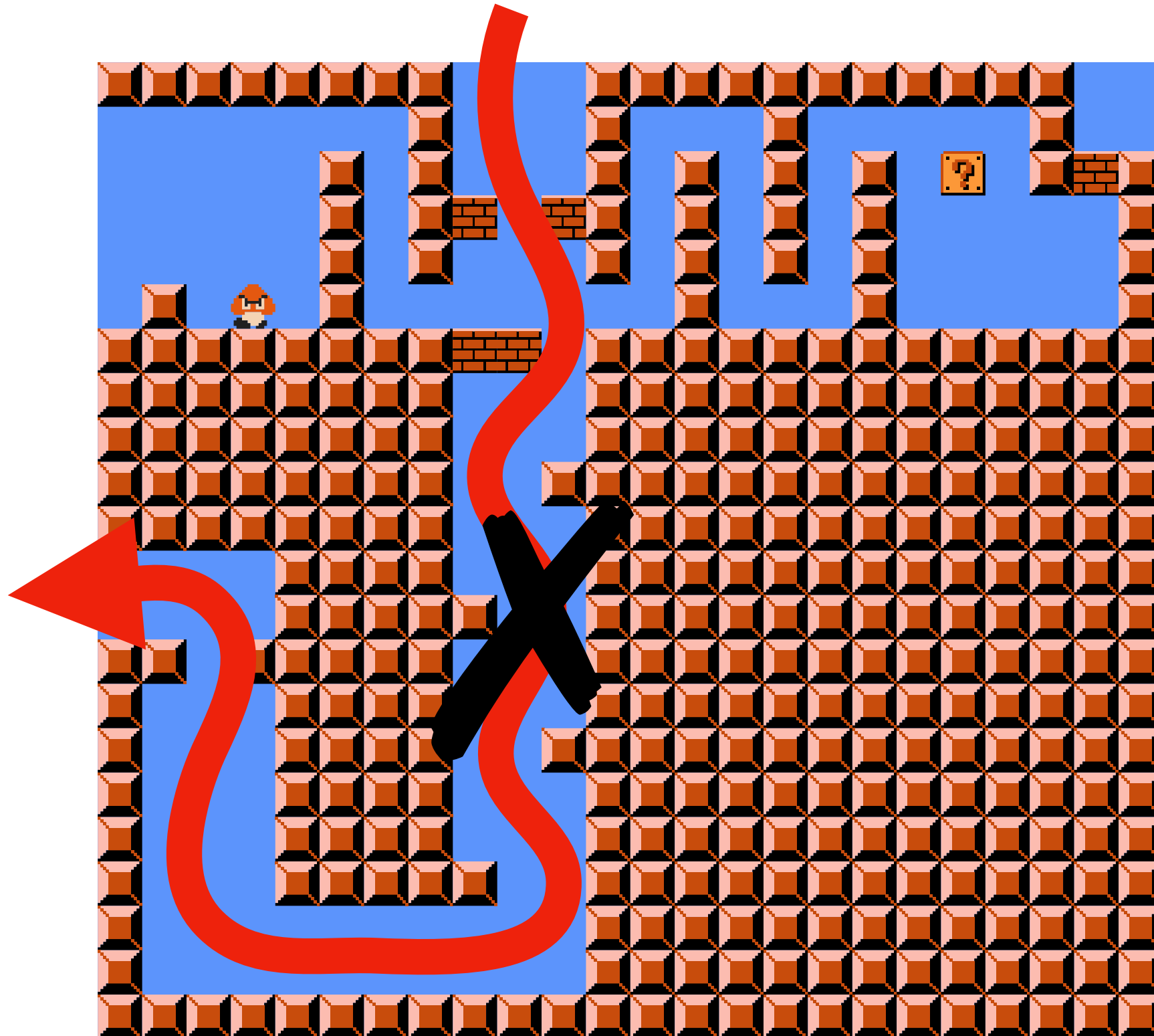
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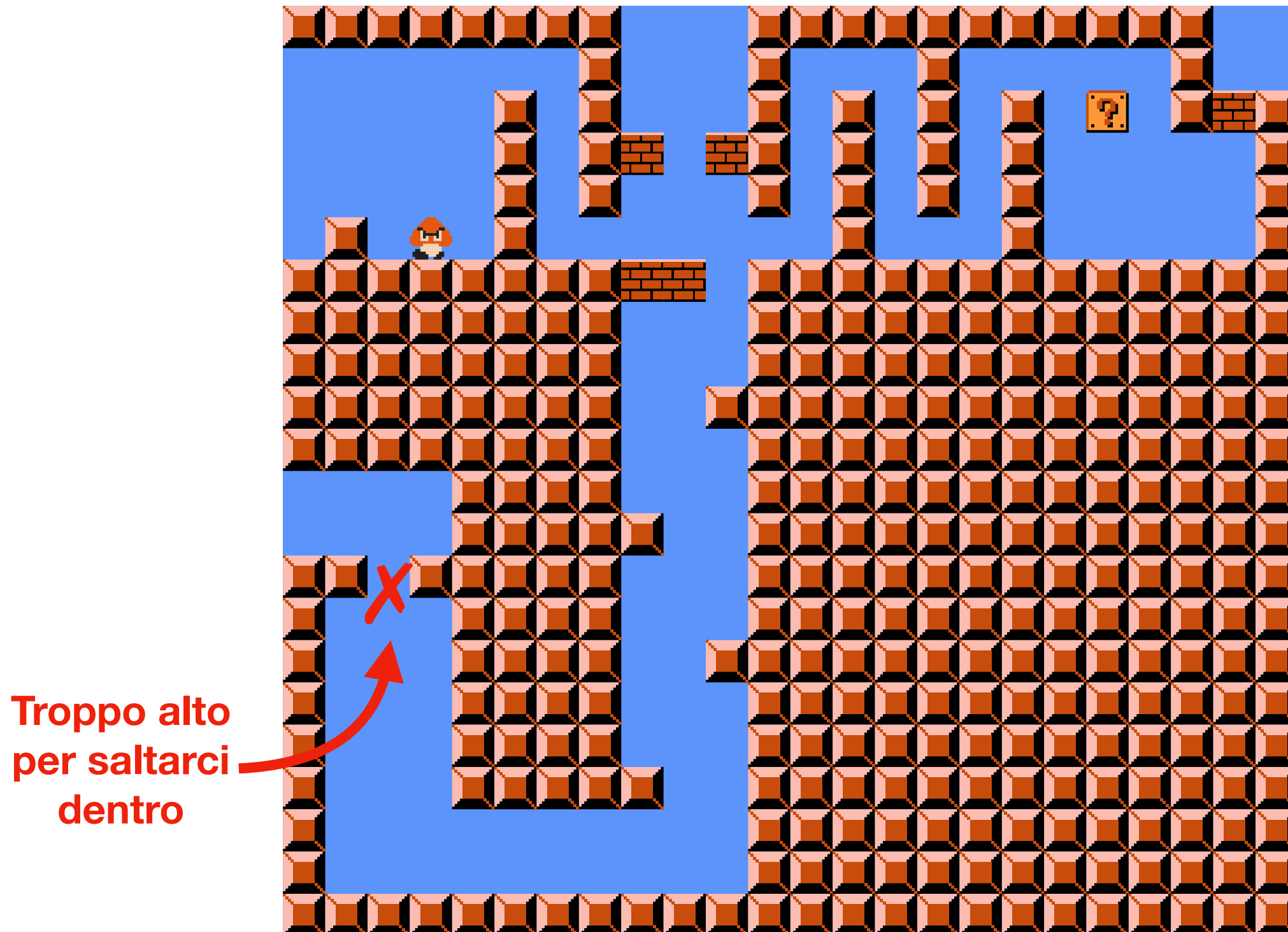
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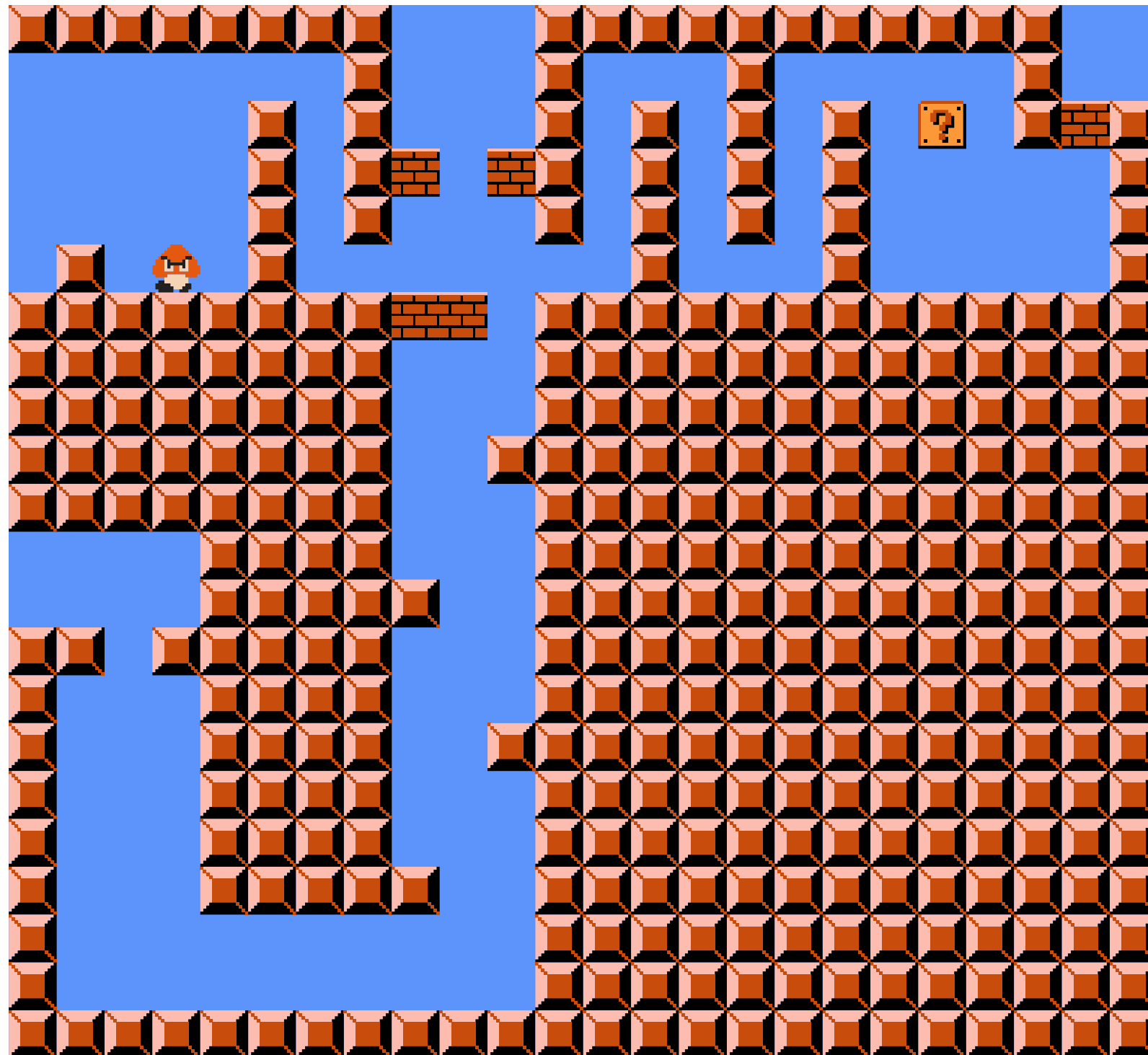
Gadget per gli incroci



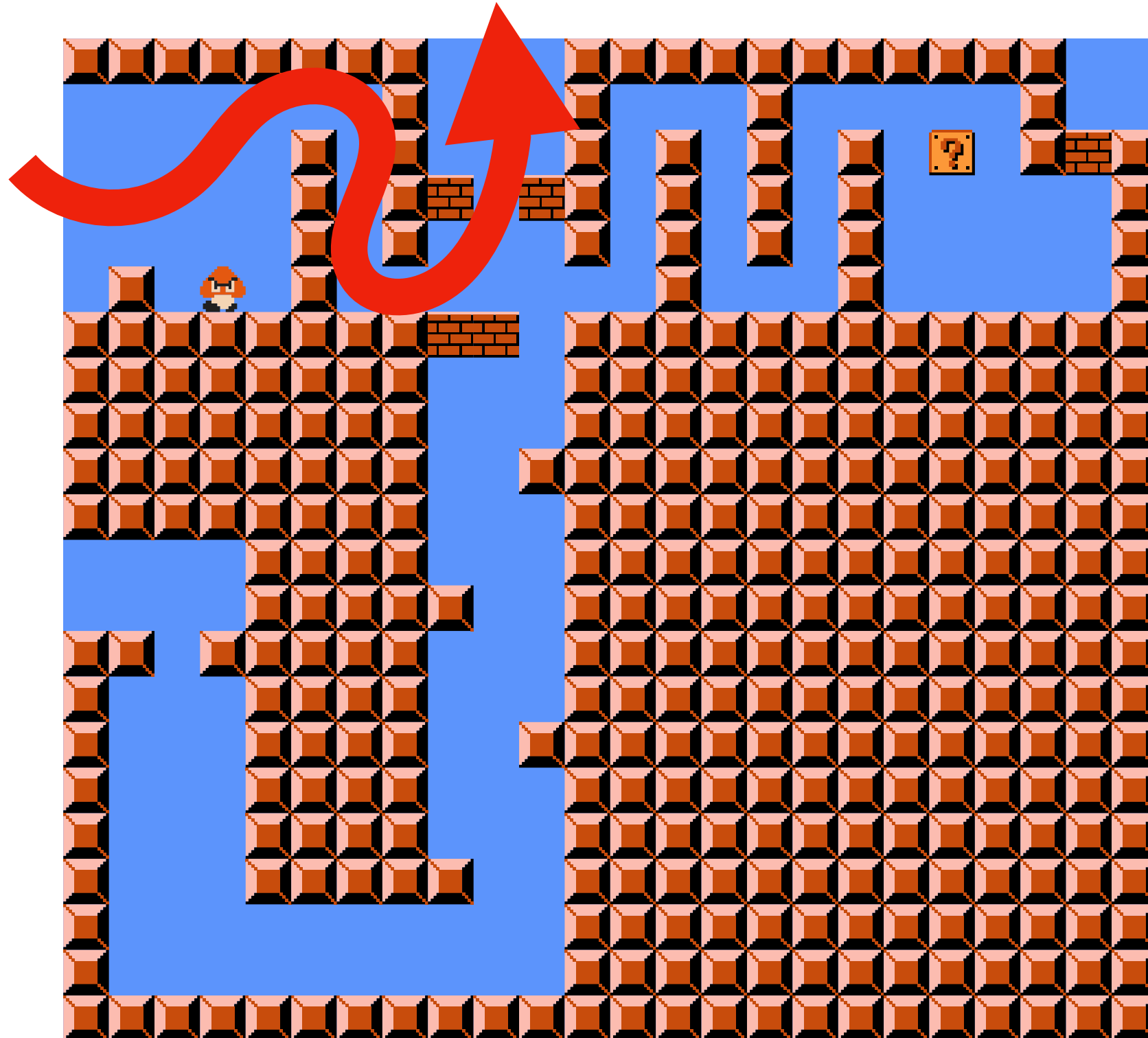
Gadget per gli incroci



Nota Importante

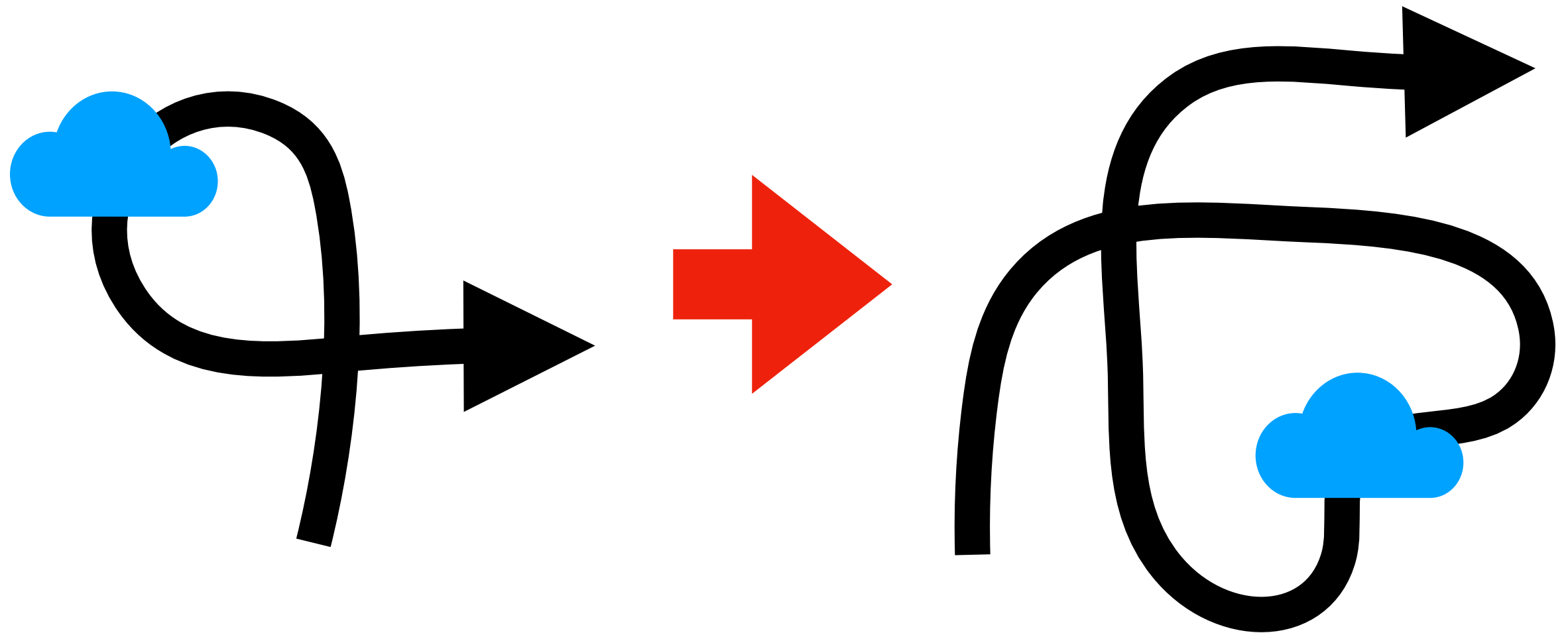


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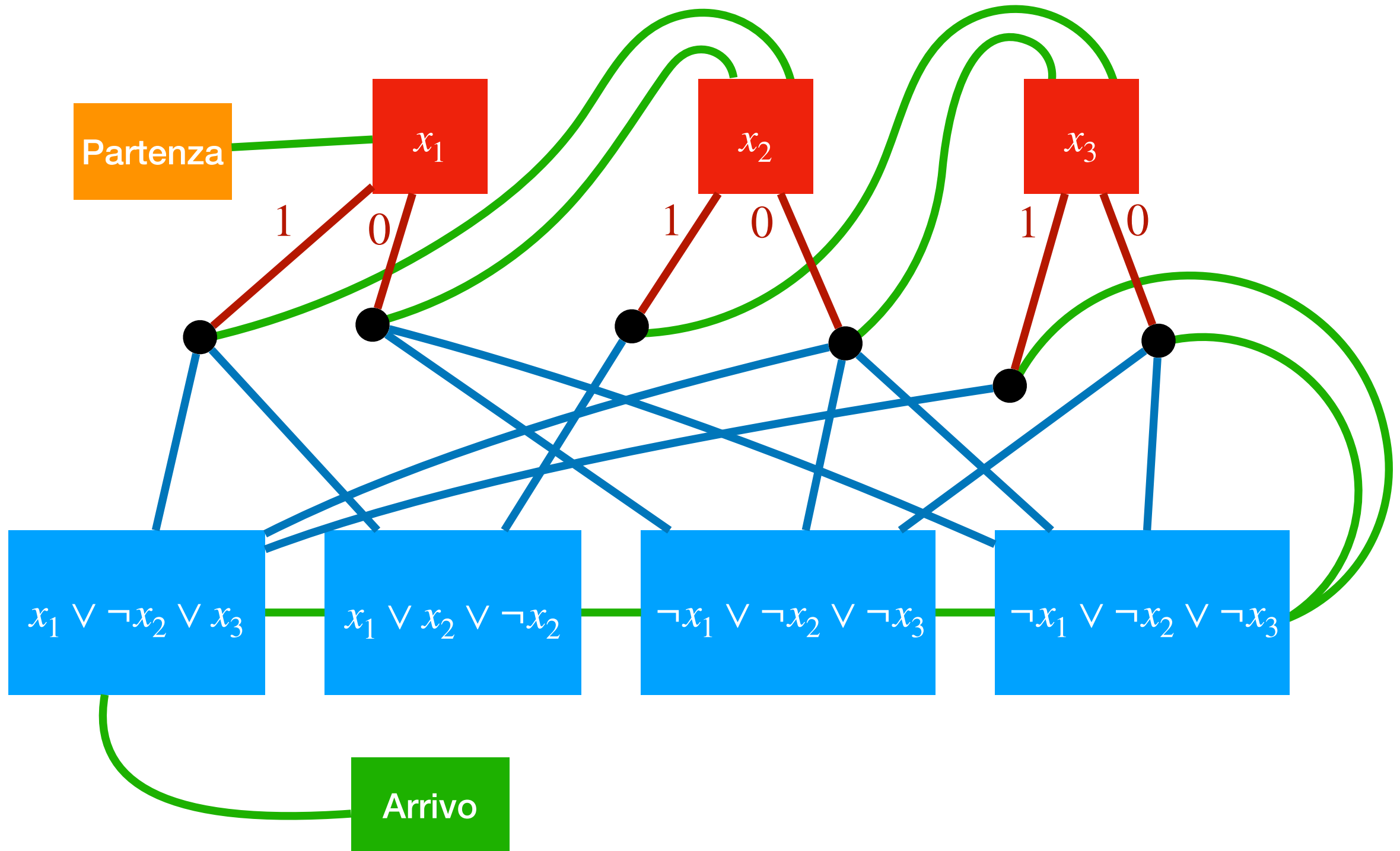


Nota Importante

Questo non è un problema,
si può forzare Mario a passare prima in orizzontale
e solo dopo in verticale



Parte rimanente degli archi = corridoi



Conclusioni

- Mario arriva alla fine se e solo se può attraversare i gadget di tutte le clausole
- Questo richiede di avere almeno una stella per ogni clausola
- Che corrisponde a un letterale vero per ogni clausola
- Ovvero che la clausola sia soddisfatta
- Quindi il problema di Super Mario è NP-completo?

Note

- Abbiamo mostrato che Super Mario è NP-difficile
- Ma in realtà è (assumendo $NP \neq PSPACE$) che Super Mario è fuori da NP
- Infatti, Super Mario è completo per PSPACE