$$f(x) = P_n(x) + E_n(x) \qquad E_n(x) = o\left(|x-x_0|^n\right) \qquad \lim_{x \to x_0} \frac{E_n(x)}{|x-x_0|^n} = 0$$

Resto nella forma di Peano

$I \subseteq \mathbb{R}$ intervallo $x_0 \in I$ $f: I \to \mathbb{R}$ $n+1$ volte derivabile su I; allora esiste una funzione $\varphi(x)$ infinitesima per $x \to x_0$ tale che

$$E_n(x) = \left[\frac{f^{(n+1)}(x_0)}{(n+1)!} + \varphi(x) \right] (x-x_0)^{n+1}$$

Dim utilizzo la formula di Taylor di ordine $n+1$

$$f(x) = P_n(x) + \underbrace{\left[\frac{1}{(n+1)!} f^{(n+1)}(x_0)(x-x_0)^{n+1} + E_{n+1}(x) \right]}_{= E_n(x)}$$

$$E_n(x) = \left[\frac{f^{(n+1)}(x_0)}{(n+1)!} + \varphi(x) \right] (x-x_0)^{n+1}$$

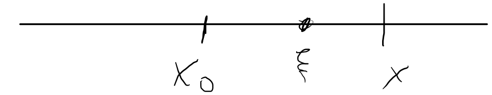
$$\varphi(x) = \frac{E_{n+1}(x)}{(x-x_0)^{n+1}}$$

è infinitesima per $x \to x_0$

Resto nella forma di Lagrange

I intervallo $\quad x_0 \in I \quad f: I \to \mathbb{R} \quad n+1$ volte derivabile in I. Allora

$\forall x \in I \quad \exists \; \xi \in I$ tale che $\quad |x_0 - \xi| < |x - x_0|$



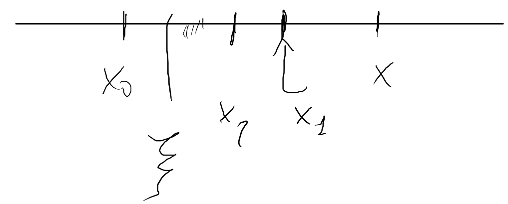
e

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

OSS: se $n = 0 \qquad f(x) - f(x_0) = f'(\xi)(x - x_0) \qquad$ Lagrange!

<u>Dim</u> usiamo ripetutamente il teorema di Cauchy

$x_0 < x$



considero le funzioni

$f(x) - P_n(x)$ e $\underbrace{(x-x_0)^{n+1}}$

$$\frac{[f(x)-P_n(x)] - [\underbrace{f(x_0) - P_n(x_0)}_{=0}]}{(x-x_0)^{n+1} - [\underbrace{(x_0-x_0)^{n+1}}_{=0}]} = \frac{f'(x_1)-P_n'(x_1)}{(n+1)(x_1-x_0)^n} = \frac{[f'(x_1)-P_n'(x_1)] - [\underbrace{f'(x_0)-P_n'(x_0)}_{=0}]}{(n+1)[(x_1-x_0)^n - \underbrace{(x_0-x_0)^n}_{=0}]}$$

$$= \frac{1}{(n+1)}\cdot\frac{f''(x_2)-P_n''(x_2)}{n(x_2-x_0)^{n-1}} = \dots = \frac{1}{(n+1)!}\,\frac{f^{(n)}(x_n)-P_n^{(n)}(x_n)}{x-x_0} = \frac{1}{(n+1)!}\, f^{(n+1)}(\xi)$$

Cauchy

Def: si chiama polinomio di Mac Laurin di f il polinomio di Taylor di f con $x_0 = 0$.

$$f(x) = e^x \qquad f^{(k)}(x) = e^x \qquad f^{(k)}(0) = 1 \quad \forall k$$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots + \frac{1}{n!}x^n$$

Es: stimare e con un errore $< 10^{-4}$

$e^x = P_n(x) + E_n(x)$ $\qquad f^{(n+1)}(x) = e^x$

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^{n+1}$$

$f(x) = e^x \qquad f(1) = e$

$e = f(1) = P_n(1) + E_n(1)$

$720 \cdot 7$

chiedo $|E_n(1)| < 10^{-4}$

$e < 3$

$$|E_n(1)| = \left| \frac{e^\xi}{(n+1)!} \cdot 1^{n+1} \right| = \frac{e^\xi}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!} \overset{?}{<} 10^{-4}$$

$0 < \xi < 1$

$(n+1)! > 30'000$

$8! > 30'000$ $\qquad n = 7$

$k!$ $\quad 3! = 6 \quad 4! = 24 \quad 5! = 120$

$6! = 720 \quad 7! = 5040 \quad 8! = 40320$

$n = 7$ $\quad e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} \pm 10^{-4}$

errore

$\simeq \underbrace{2,7182}4 2$

$e = 2,71828\ldots$

Pendolo piccole oscillazioni $\quad \operatorname{sen} x \simeq x$

$f(x) = \operatorname{sen} x \quad x_0 = 0 \quad P_n(x) = f(0) + f'(0)x + \frac{1}{2} f''(0) x^2 + \cdots + \frac{1}{n!} f^{(n)}(0) x^n$

$f'(x) = \cos x \quad f^{(k+4)}(x) = f^{(k)}(x) \quad \operatorname{sen}(0) = 0$

$f''(x) = -\operatorname{sen} x \quad \cos(0) = 1$

$f'''(x) = -\cos x \quad f^{(k+2)}(x) = -f^{(k)}(x)$

$f^{IV}(x) = \operatorname{sen} x$

$\operatorname{sen} x$

$$f^{(2k)}(0) = 0 \qquad\qquad f'(0) = 1$$

$$f^{(2k+1)}(0) = \pm 1 \qquad\qquad f^{(4k+1)}(0) = 1$$

$$f^{(4k+3)}(0) = -1$$

$$P_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots +$$

$P_2(x) = x$ è il polinomio di Taylor di ordine 2

Quanto devono essere "piccole" le oscillazioni nel modello lineare del pendolo affinché l'errore commesso sia $< \frac{1}{10}$?

$|x| < \delta \rightsquigarrow \underbrace{|P_2(x) - \sin x|} < \frac{1}{10} \quad \forall x$

$$|E_2(x)| = \left| \frac{f^{(3)}(\xi)}{3!} x^3 \right| \overset{?}{<} \frac{1}{10}$$

$$\left| \frac{\cos(\xi)}{3!} x^3 \right| \leq \left| \frac{1}{3!} x^3 \right| \overset{?}{<} \frac{1}{10}$$

$|\cos x| \leq 1$

$$|x|^3 \overset{?}{<} \frac{6}{10}$$

$$|x| < \sqrt[3]{6/10} \simeq \underline{0{,}843}$$

$$P_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

Polinomio di ordine $2n+1$

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \frac{1}{24}$$

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4}{x^4} = 0$$

$f(x) = \sin x$

$f(x) = \cos x$ $\quad 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$P_{2n}(x) = \sum_{k=0}^{n} (-1)^k \frac{1}{(2k)!} x^{2k}$$

$$P_2(x) = 1 - \frac{1}{2}x^2$$

$$E_2(x) = \cos x - 1 + \frac{1}{2}x^2$$

$$\lim_{x \to 0} \frac{E_2(x)}{x^2} = 0$$

$$0 = \lim_{x \to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^2} = \lim_{x \to 0} \left[\frac{\cos x - 1}{x^2} + \frac{1}{2}\right]$$

$$\operatorname{senh} x = \frac{1}{2}\left(e^{x} - e^{-x}\right) \qquad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$D \operatorname{senh} x = \cosh x$$

$$D \cosh x = \operatorname{senh} x$$

$$f(x) = \operatorname{senh} x \qquad f^{(2k)}(x) = \operatorname{senh} x \qquad f^{(2k)}(0) = $$

$$f^{(2k+1)}(x) = \cosh x \qquad f^{(2k+1)} = 1$$

$\operatorname{senh} x$

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$$

$\cosh x$

$$P_{2n}(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$$

$\log x$

$\log(1+x) = f(x) \qquad x_0 = 0$

$P_n(x) = x - \frac{1}{2}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 \cdots$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 \cdots +$

$\sum_{k=0}^{n} (-1)^{k+1} \frac{1}{k} x^k$

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots + E_n(x)$

$f(0) = 0$

$f'(x) = (1+x)^{-1} \qquad f'(0) = 1$

$f''(x) = (-1)(1+x)^{-2} \qquad f''(0) = -1$

$f'''(x) = (-1)(-2)(1+x)^{-3} \qquad f'''(0) = 2 \qquad f^{IV}(0) = -3!$

$\vdots$

$f^{(k)}(x) = (-1)(-2)\cdots(-k+1)(1+x)^{-k} = (-1)^{k+1}(k-1)!(1+x)^{-k}$

$k \geq 1$

$f^{(k)}(0) = (-1)^{k+1}(k-1)!$

$$\log(1+x) \simeq x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$(1+x)^{-1} \simeq 1 - x + x^2 - x^3 + x^4 \cdots$$

$(1+x)^\alpha \simeq ?$ potenza di un binomio $\alpha \in \mathbb{R}$

$$f(x) = (1+x)^\alpha \qquad f'(x) = \alpha(1+x)^{\alpha-1} \qquad f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2} \ \ldots \ f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)\cdot(1+x)^{\alpha-n}$$

$$\frac{f^{(k)}(0)}{k!} = \boxed{\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}} = \binom{\alpha}{k}$$

coefficiente binomiale generalizzato

definito $\forall \alpha \in \mathbb{R}$

$\forall k \in \mathbb{N}$

$$(1+x)^{\alpha} \approx \sum_{k=0}^{n} \binom{\alpha}{k} x^{k}$$

$\alpha = -\frac{1}{2}$ $\quad \frac{1}{\sqrt{1+x}}$

$\arcsin x$

$\arccos x$

$\operatorname{arctg} x$

$$\alpha = -1 \qquad \binom{-1}{3} = \frac{-1\cdot(-2)}{3!} = \frac{1}{3}$$

$$\alpha = \frac{1}{2} \qquad (1+x)^{1/2} = \sqrt{1+x} \approx \sum_{k=0}^{n} \binom{1/2}{k} x^{k}$$

$$n = 3 \qquad \underset{\sqrt{1+x}}{P_3(x)} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$