Since in reality a force can never be applied suddenly, it is of interest to consider a dynamic force that has a finite rise time, t_r, but remains constant thereafter, as shown in Fig. 4.5.1b:

$$p(t) = \begin{cases} p_o(t/t_r) & t \le t_r \\ p_o & t \ge t_r \end{cases} \tag{4.5.1}$$

The excitation has two phases: ramp or rise phase and constant phase.

For a system without damping starting from rest, the response during the ramp phase is given by Eq. (4.4.2), repeated here for convenience:

$$u(t) = (u_{\text{st}})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad t \le t_r \tag{4.5.2}$$

The response during the constant phase can be determined by evaluating Duhamel's integral after substituting Eq. (4.5.1) in Eq. (4.2.4). Alternatively, existing solutions for free vibration and step force could be utilized to express this response as

$$u(t) = u(t_r) \cos \omega_n (t - t_r) + \frac{\dot{u}(t_r)}{\omega_n} \sin \omega_n (t - t_r) + (u_{\text{st}})_o [1 - \cos \omega_n (t - t_r)] \qquad t \ge t_r \tag{4.5.3}$$

The third term is the solution for a system at rest subjected to a step force starting at $t = t_r$; it is obtained from Eq. (4.3.2). The first two terms in Eq. (4.5.3) account for free vibration of the system resulting from its displacement $u(t_r)$ and velocity $\dot{u}(t_r)$ at the end of the ramp phase. Determined from Eq. (4.5.2), $u(t_r)$ and $\dot{u}(t_r)$ are substituted in Eq. (4.5.3) to obtain

$$u(t) = (u_{\text{st}})_o \left\{ 1 + \frac{1}{\omega_n t_r} \left[(1 - \cos \omega_n t_r) \sin \omega_n (t - t_r) - \sin \omega_n t_r \cos \omega_n (t - t_r) \right] \right\} \qquad t \ge t_r \tag{4.5.4a}$$

This equation can be simplified, using a trigonometric identity, to

$$u(t) = (u_{\text{st}})_o \left\{ 1 - \frac{1}{\omega_n t_r} \left[\sin \omega_n t - \sin \omega_n (t - t_r) \right] \right\} \qquad t \ge t_r \tag{4.5.4b}$$

The normalized deformation, $u(t)/(u_{\text{st}})_o$, is a function of the normalized time, t/T_n, because $\omega_n t = 2\pi(t/T_n)$. This function depends only on the ratio t_r/T_n because $\omega_n t_r = 2\pi(t_r/T_n)$, not

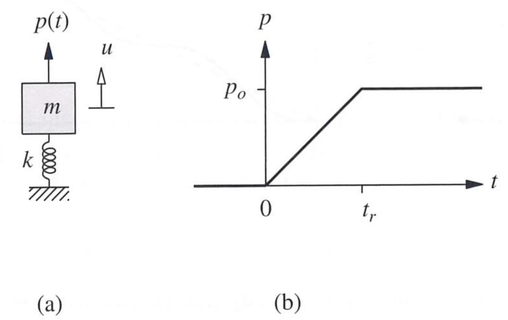

Figure 4.5.1 (a) SDF system; (b) step force with finite rise time.

... static ratio t_r/T_n. Also plotted is $u_{st}(t) = p(t)/k$, the static deformation at each time instant. These results permit several observations:

1. During the force-rise phase the system oscillates at the natural period T_n about the static solution.
2. During the constant-force phase the system oscillates also at the natural period T_n about the static solution.
3. If the velocity $\dot{u}(t_r)$ is zero at the end of the ramp, the system does not vibrate during the constant-force phase.

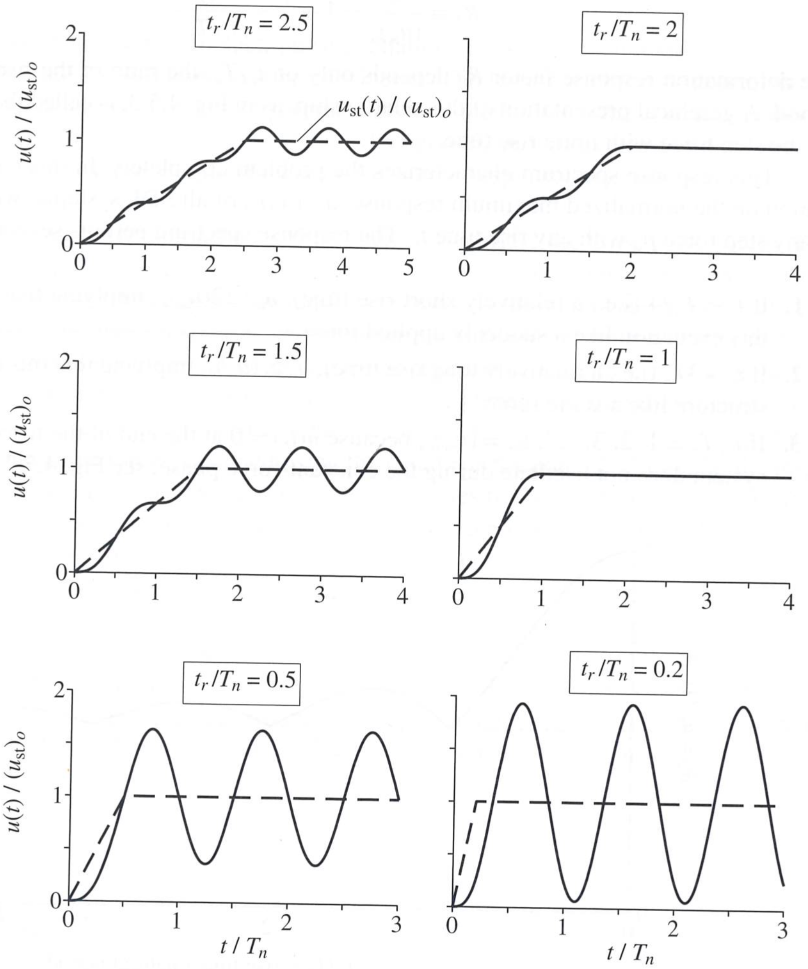

Figure 4.5.2 Dynamic response of undamped SDF system to step force with finite rise time; static solution is shown by dashed lines.

4. For smaller values of t_r/T_n (i.e., short rise time), the response is similar to that due to a sudden step force; see Fig. 4.3.1c.
5. For larger values of t_r/T_n, the dynamic displacement oscillates close to the static solution, implying that the dynamic effects are small (i.e., a force increasing slowly—relative to T_n—from 0 to p_o affects the system like a static force).

The deformation attains its maximum value during the constant-force phase of the response. From Eq. (4.5.4a) the maximum value of $u(t)$ is

$$u_o = (u_{st})_o \left[1 + \frac{1}{\omega_n t_r} \sqrt{(1 - \cos \omega_n t_r)^2 + (\sin \omega_n t_r)^2} \right] \tag{4.5.5}$$

Using trigonometric identities and $T_n = 2\pi/\omega_n$, Eq. (4.5.5) can be simplified to

$$R_d \equiv \frac{u_o}{(u_{st})_o} = 1 + \frac{|\sin(\pi t_r/T_n)|}{\pi t_r/T_n} \tag{4.5.6}$$

The deformation response factor R_d depends only on t_r/T_n, the ratio of the rise time to the natural period. A graphical presentation of this relationship, as in Fig. 4.5.3, is called the *response spectrum* for the step force with finite rise time.

This response spectrum characterizes the problem completely. In this case it contains information on the normalized maximum response, $u_o/(u_{st})_o$, of all SDF systems (without damping) due to any step force p_o with any rise time t_r. The response spectrum permits several observations:

1. If $t_r < T_n/4$ (i.e., a relatively short rise time), $u_o \simeq 2(u_{st})_o$, implying that the structure "sees" this excitation like a suddenly applied force.
2. If $t_r > 3T_n$ (i.e., a relatively long rise time), $u_o \simeq (u_{st})_o$, implying that this excitation affects the structure like a static force.
3. If $t_r/T_n = 1, 2, 3, \ldots, u_o = (u_{st})_o$, because $\dot{u}(t_r) = 0$ at the end of the force-rise phase, and the system does not oscillate during the constant-force phase; see Fig. 4.5.2.

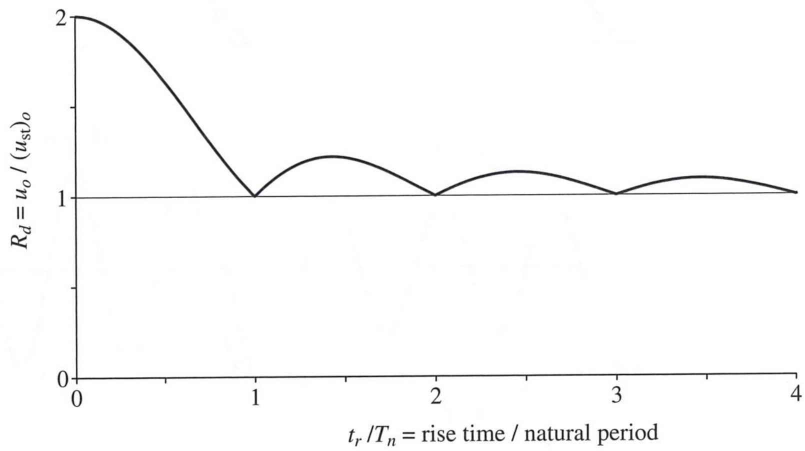

Figure 4.5.3 Response spectrum for step force with finite rise time.

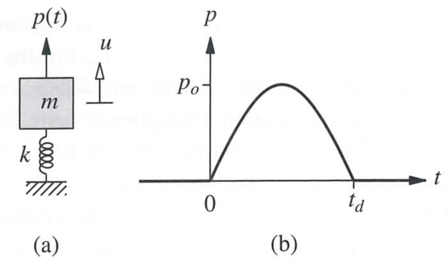

Figure 4.8.1 (a) SDF system; (b) half-cycle sine pulse force.

Case 1: $t_d/T_n \neq \frac{1}{2}$

Forced Vibration Phase. The force is the same as the harmonic force $p(t) = p_o \sin \omega t$ considered earlier with frequency $\omega = \pi/t_d$. The response of an undamped SDF system to such a force is given by Eq. (3.1.6b) in terms of ω and ω_n, the excitation and natural frequencies. The excitation frequency ω is not the most meaningful way of characterizing the pulse because, unlike a harmonic force, it is not a periodic function. A better characterization is the pulse duration t_d, which will be emphasized here. Using the relations $\omega = \pi/t_d$ and $\omega_n = 2\pi/T_n$, and defining $(u_{\text{st}})_o = p_o/k$, as before, Eq. (3.1.6b) becomes

$$\frac{u(t)}{(u_{\text{st}})_o} = \frac{1}{1-(T_n/2t_d)^2}\left[\sin\left(\pi\frac{t}{t_d}\right) - \frac{T_n}{2t_d}\sin\left(2\pi\frac{t}{T_n}\right)\right] \qquad t \leq t_d \tag{4.8.2}$$

Free Vibration Phase. After the force pulse ends, the system vibrates freely with its motion described by Eq. (4.7.3). The displacement $u(t_d)$ and velocity $\dot{u}(t_d)$ at the end of the pulse are determined from Eq. (4.8.2). Substituting these in Eq. (4.7.3), using trigonometric identities and manipulating the mathematical quantities, we obtain

$$\frac{u(t)}{(u_{\text{st}})_o} = \frac{(T_n/t_d)\cos(\pi t_d/T_n)}{(T_n/2t_d)^2-1}\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \qquad t \geq t_d \tag{4.8.3}$$

Case 2: $t_d/T_n = \frac{1}{2}$

Forced Vibration Phase. The forced response is now given by Eq. (3.1.13b), repeated here for convenience:

$$\frac{u(t)}{(u_{\text{st}})_o} = \frac{1}{2}\left(\sin\frac{2\pi t}{T_n} - \frac{2\pi t}{T_n}\cos\frac{2\pi t}{T_n}\right) \qquad t \leq t_d \tag{4.8.4}$$

Free Vibration Phase. After the force pulse ends at $t = t_d$, free vibration of the system is initiated by the displacement $u(t_d)$ and velocity $\dot{u}(t_d)$ at the end of the force pulse. Determined from Eq. (4.8.4), these are

$$\frac{u(t_d)}{(u_{\text{st}})_o} = \frac{\pi}{2} \qquad \dot{u}(t_d) = 0 \tag{4.8.5}$$

The second equation implies that the displacement in the forced vibration phase reaches its maximum at the end of this phase. Substituting Eq. (4.8.5) in Eq. (4.7.3) gives the response of the system after the pulse has ended:

$$\frac{u(t)}{(u_{\text{st}})_o} = \frac{\pi}{2}\cos 2\pi\left(\frac{t}{T_n} - \frac{1}{2}\right) \qquad t \geq t_d \tag{4.8.6}$$

Response history. The time variation of the normalized deformation, $u(t)/(u_{st})_o$, given by Eqs. (4.8.2) and (4.8.3) is plotted in Fig. 4.8.2 for several values of t_d/T_n. For the special case of $t_d/T_n = \frac{1}{2}$, Eqs. (4.8.4) and (4.8.6) describe the response of the system, and these are also plotted in Fig. 4.8.2. The nature of the response is seen to vary greatly by changing just the duration t_d of the pulse. Also plotted in Fig. 4.8.2 is $u_{st}(t) = p(t)/k$, the static solution. The difference between the two curves is an indication of the dynamic effects, which are seen to be small for $t_d = 3T_n$ because this implies that the force is varying slowly relative to the natural period T_n of the system.

The response during the force pulse contains both frequencies ω and ω_n and it is positive throughout. After the force pulse has ended, the system oscillates freely about its undeformed configuration with constant amplitude for lack of damping. If $t_d/T_n = 1.5, 2.5, \ldots$, the mass stays still after the force pulse ends because both the displacement and velocity of the mass are zero when the force pulse ends.

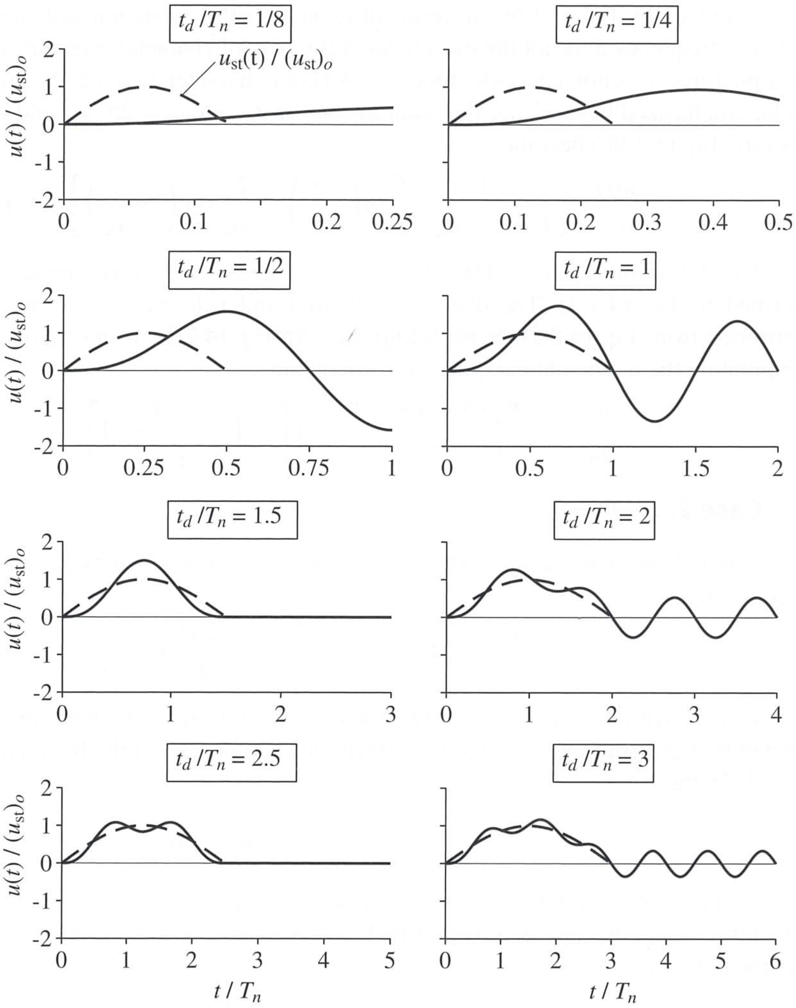

Figure 4.8.2 Dynamic response of undamped SDF system to half-cycle sine pulse force; static solution is shown by dashed lines.

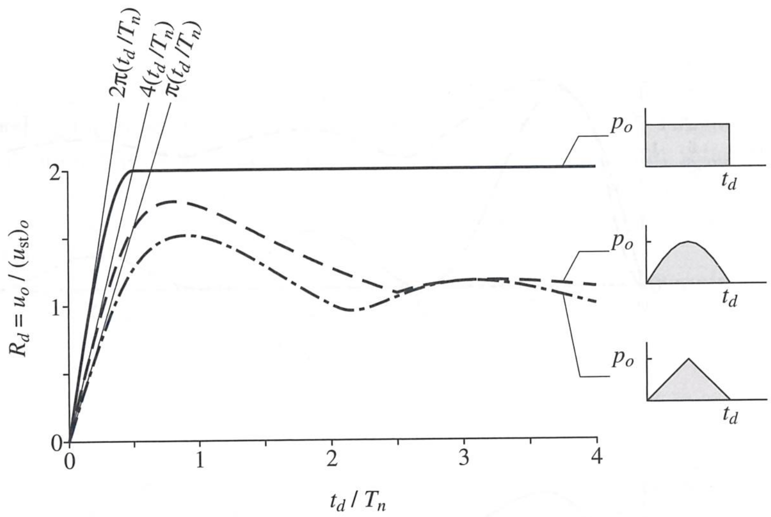

Figure 4.10.1 Shock spectra for three force pulses of equal amplitude.

If the pulse duration t_d is shorter than $T_n/2$, the overall maximum response of the system occurs during its free vibration phase and is controlled by the time integral of the pulse. This can be demonstrated by considering the limiting case as t_d/T_n approaches zero. As the pulse duration becomes extremely short compared to the natural period of the system, it becomes a pure impulse of magnitude

$$\mathcal{I} = \int_0^{t_d} p(t)\, dt \tag{4.10.1}$$

The response of the system to this impulsive force is the unit impulse response of Eq. (4.1.6) times $\mathcal{I}$:

$$u(t) = \mathcal{I}\left(\frac{1}{m\omega_n}\sin\omega_n t\right) \tag{4.10.2}$$

The maximum deformation,

$$u_o = \frac{\mathcal{I}}{m\omega_n} = \frac{\mathcal{I}}{k}\frac{2\pi}{T_n} \tag{4.10.3}$$

is proportional to the magnitude of the impulse.

Thus the maximum deformation due to the rectangular impulse of magnitude $\mathcal{I} = p_o t_d$ is

$$\frac{u_o}{(u_{st})_o} = 2\pi\frac{t_d}{T_n} \tag{4.10.4}$$

that due to the half-cycle sine pulse with $\mathcal{I} = (2/\pi)p_o t_d$ is

$$\frac{u_o}{(u_{st})_o} = 4\frac{t_d}{T_n} \tag{4.10.5}$$

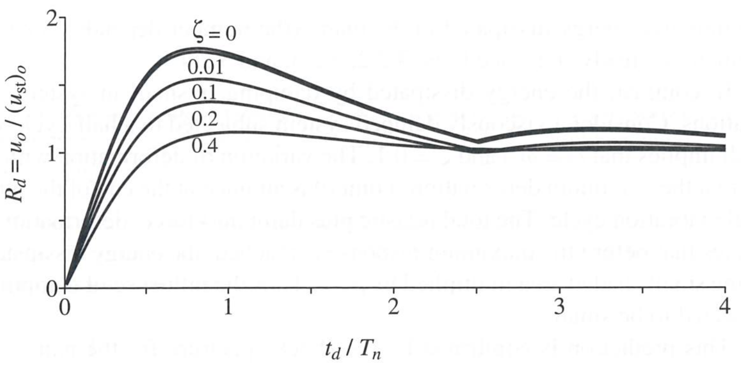

Figure 4.11.2 Shock spectra for a half-cycle sine pulse force for five damping values.

Thus a conservative but not overly conservative estimate of the response of many practical structures with damping to pulse-type excitations may be obtained by neglecting damping and using the earlier results for undamped systems.

4.12 RESPONSE TO GROUND MOTION

The response spectrum characterizing the maximum response of SDF systems to ground motion $\ddot{u}_g(t)$ can be determined from the response spectrum for the applied force $p(t)$ with the same time variation as $\ddot{u}_g(t)$. This is possible because as shown in Eq. (1.7.6), the ground acceleration can be replaced by the effective force, $p_{\text{eff}}(t) = -m\ddot{u}_g(t)$.

The response spectrum for applied force $p(t)$ is a plot of $R_d = u_o/(u_{\text{st}})_o$, where $(u_{\text{st}})_o = p_o/k$, versus the appropriate system and excitation parameters: ω/ω_n for harmonic excitation and t_d/T_n for pulse-type excitation. Replacing p_o by $(p_{\text{eff}})_o$ gives

$$(u_{\text{st}})_o = \frac{(p_{\text{eff}})_o}{k} = \frac{m\ddot{u}_{go}}{k} = \frac{\ddot{u}_{go}}{\omega_n^2} \tag{4.12.1}$$

where $\ddot{u}_{go}$ is the maximum value of $\ddot{u}_g(t)$ and the negative sign in $p_{\text{eff}}(t)$ has been dropped. Thus

$$R_d = \frac{u_o}{(u_{\text{st}})_o} = \frac{\omega_n^2 u_o}{\ddot{u}_{go}} \tag{4.12.2}$$

Therefore, the response spectra presented in Chapters 3 and 4 showing the response $u_o/(u_{\text{st}})_o$ due to applied force also give the response $\omega_n^2 u_o/\ddot{u}_{go}$ to ground motion.

For undamped systems subjected to ground motion, Eqs. (1.7.4) and (1.7.3) indicate that the total acceleration of the mass is related to the deformation through $\ddot{u}^t(t) = -\omega_n^2 u(t)$. Thus the maximum values of the two responses are related by $\ddot{u}_o^t = \omega_n^2 u_o$. Substituting in Eq. (4.12.2) gives

$$R_d = \frac{\ddot{u}_o^t}{\ddot{u}_{go}} \tag{4.12.3}$$

Thus the earlier response spectra showing the response $u_o/(u_{\text{st}})_o$ of undamped systems subjected to applied force also display the response $\ddot{u}_o^t/\ddot{u}_{go}$ to ground motion.