eccentricity is $e = 1.5$ ft, and the height of the building is 12 ft. Determine the natural periods and modes of vibration of the structure.

Solution

$$\text{Weight of roof slab: } w = 30 \times 20 \times 100 \text{ lb} = 60 \text{ kips}$$

$$\text{Mass: } m = w/g = 1.863 \text{ kips-sec}^2/\text{ft}$$

$$\text{Moment of inertia: } I_O = \frac{m(b^2 + d^2)}{12} = 201.863 \text{ kips-ft-sec}^2$$

Lateral motion of the roof diaphragm in the x-direction is governed by Eq. (9.5.18):

$$m\ddot{u}_x + 2k_x u_x = 0 \qquad \text{(a)}$$

Thus the natural frequency of x-lateral vibration is

$$\omega_x = \sqrt{\frac{2k_x}{m}} = \sqrt{\frac{2(40)}{1.863}} = 6.553 \text{ rad/sec}$$

The corresponding natural mode is shown in Fig. E10.6c.

The coupled lateral (u_y)-torsional (u_θ) motion of the roof diaphragm is governed by Eq. (9.5.19). Substituting for m and I_O gives

$$\mathbf{m} = \begin{bmatrix} 1.863 & \\ & 201.863 \end{bmatrix}$$

From Eqs. (9.5.16) and (9.5.19) the stiffness matrix has four elements:

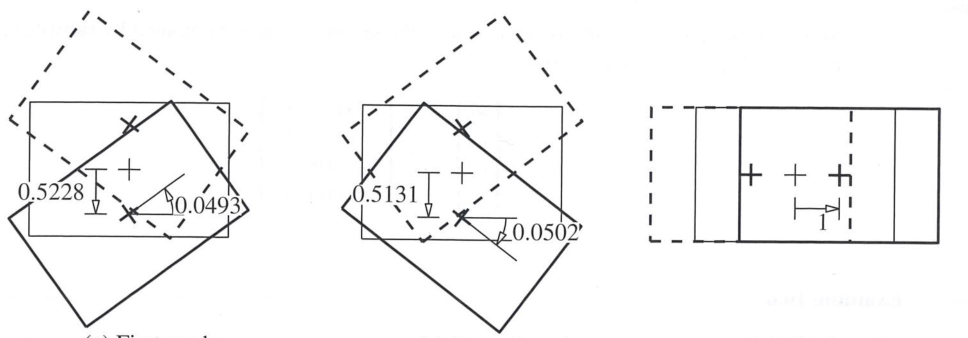
$$k_{yy} = k_y = 75 \text{ kips/ft}$$

$$k_{y\theta} = k_{\theta y} = ek_y = 1.5 \times 75 = 112.5 \text{ kips}$$

$$k_{\theta\theta} = e^2 k_y + \frac{d^2}{2} k_x = 8168.75 \text{ kips-ft}$$

Hence,

$$\mathbf{k} = \begin{bmatrix} 75.00 & 112.50 \\ 112.50 & 8168.75 \end{bmatrix}$$



(a) First mode
$\omega_1 = 5.878$ rad/sec

(b) Second mode
$\omega_2 = 6.794$ rad/sec

(c) Third mode
$\omega_x = 6.553$ rad/sec

Figure E10.6

With **k** and **m** known, the eigenvalue problem for this two-DOF system is solved by standard procedures to obtain:

$$\text{Natural frequencies (rad/sec): } \omega_1 = 5.878;\ \omega_2 = 6.794$$

$$\text{Natural modes: } \phi_1 = \begin{Bmatrix} -0.5228 \\ 0.0493 \end{Bmatrix};\ \phi_2 = \begin{Bmatrix} -0.5131 \\ -0.0502 \end{Bmatrix}$$

These mode shapes are plotted in Fig. E10.6a and b. The motion of the structure in each mode consists of translation of the rigid diaphragm coupled with torsion about the vertical axis through the center of mass.

Example 10.7

Consider a special case of the system of Example 10.6 in which frame A is located at the center of mass (i.e., $e = 0$). Determine the natural frequencies and modes of this system.

Solution Equation (9.5.20) specialized for free vibration of this system gives three equations of motion:

$$m\ddot{u}_x + 2k_x u_x = 0 \qquad m\ddot{u}_y + k_y u_y = 0 \qquad I_O\ddot{u}_\theta + \frac{d^2}{2} k_x u_\theta = 0 \qquad \text{(a)}$$

The first equation of motion indicates that translational motion in the x-direction would occur at the natural frequency

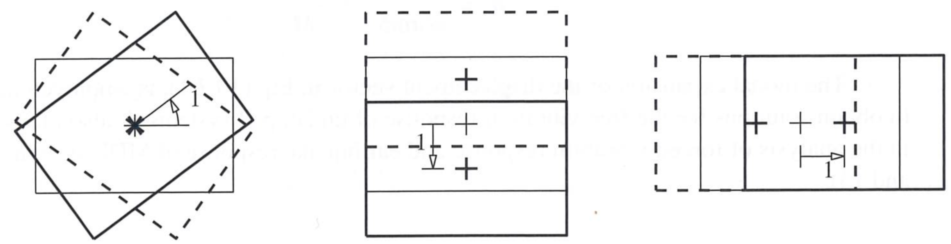
$$\omega_x = \sqrt{\frac{2k_x}{m}} = \sqrt{\frac{2(40)}{1.863}} = 6.553 \text{ rad/sec}$$

This motion is independent of lateral motion u_y or torsional motion u_θ (Fig. E10.7c). The second equation of motion indicates that translational motion in the y-direction would occur at the natural frequency

$$\omega_y = \sqrt{\frac{k_y}{m}} = \sqrt{\frac{75}{1.863}} = 6.344 \text{ rad/sec}$$

This motion is independent of the lateral motion u_x or torsional motion u_θ (Fig. E10.7b). The third equation of motion indicates that torsional motion would occur at the natural frequency

$$\omega_\theta = \sqrt{\frac{d^2 k_x}{2I_O}} = \sqrt{\frac{(20)^2 40}{2(201.863)}} = 6.295 \text{ rad/sec}$$



(a) First mode
$\omega_\theta = 6.295$ rad/sec

(b) Second mode
$\omega_y = 6.344$ rad/sec

(c) Third mode
$\omega_x = 6.553$ rad/sec

Figure E10.7

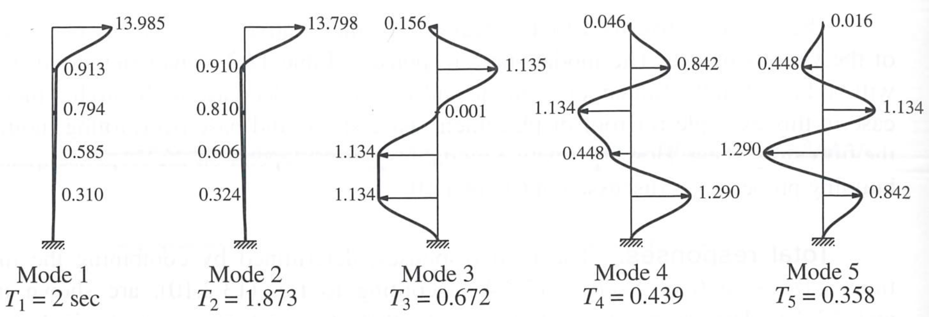


Figure 13.2.9 Natural periods and modes of vibration of building with appendage.

large deformations in the appendage. Table 13.2.4 gives the modal static responses for the base shear V_b and appendage shear V_5. Observe that V_{bn}^{st} for the first two modes are similar in magnitude and of the same algebraic sign; V_{5n}^{st} for the first two modes are also of similar magnitude but of opposite signs. The responses $D_n(t)$ and $A_n(t)$ of the SDF systems corresponding to the five modes of the system are shown in Fig. 13.2.10. Note that $D_n(t)$—also $A_n(t)$—for the first two modes are essentially in phase because the two natural periods are close; the peak values are similar because of similar periods and identical damping in the two modes.

TABLE 13.2.4 MODAL STATIC RESPONSES

	Mode				
	1	2	3	4	5
V_{bn}^{st}/m	1.951	1.633	0.333	0.078	0.015
V_{5n}^{st}/m_5	9.938	−8.979	0.046	−0.007	0.0001

The modal contributions to the base shear and to the appendage shear together with the total response are presented in Fig. 13.2.11. Observe that the response contributions of the first two modes are similar in magnitude because the modal static responses are about the same and the $A_n(t)$ are similar. In the case of base shear, the two modal static responses are of the same algebraic sign, implying that the two modal contributions are essentially in phase [because so are $A_1(t)$ and $A_2(t)$], and hence the combined base shear is much larger than the individual modal responses. In contrast, the modal static responses for the appendage shear are of opposite algebraic sign, indicating that the two modal contributions are essentially out of phase, and hence the combined appendage shear is much smaller than the individual modal responses. However, it is very large, being almost equal to its own weight. As a result, significant damage to appendages at the top of essentially undamaged structures has been observed during earthquakes. Two examples are shown in Figures 13.2.12 and 13.2.13.

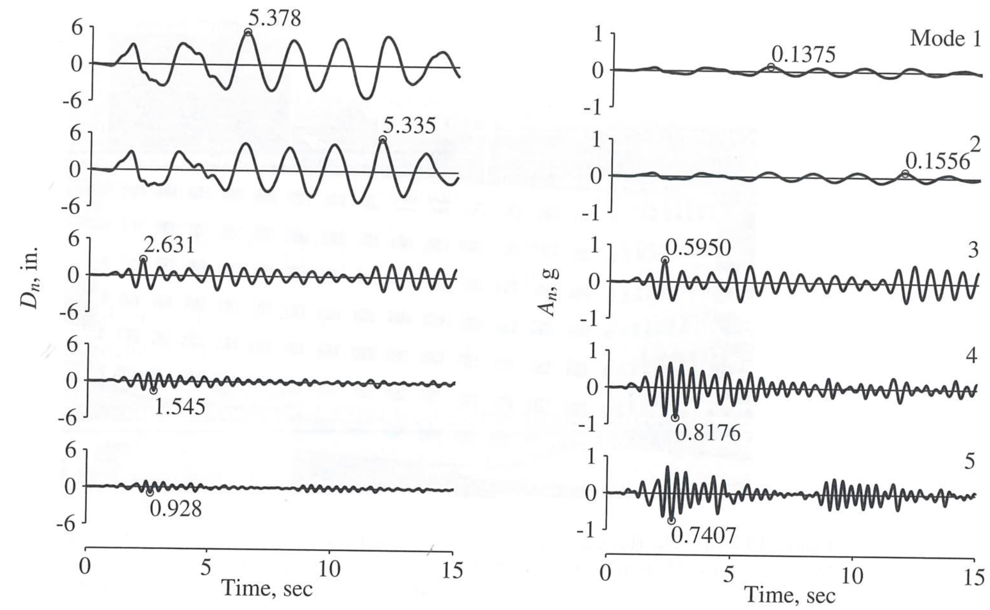


Figure 13.2.10 Displacement $D_n(t)$ and pseudo-acceleration $A_n(t)$ responses of modal SDF systems.

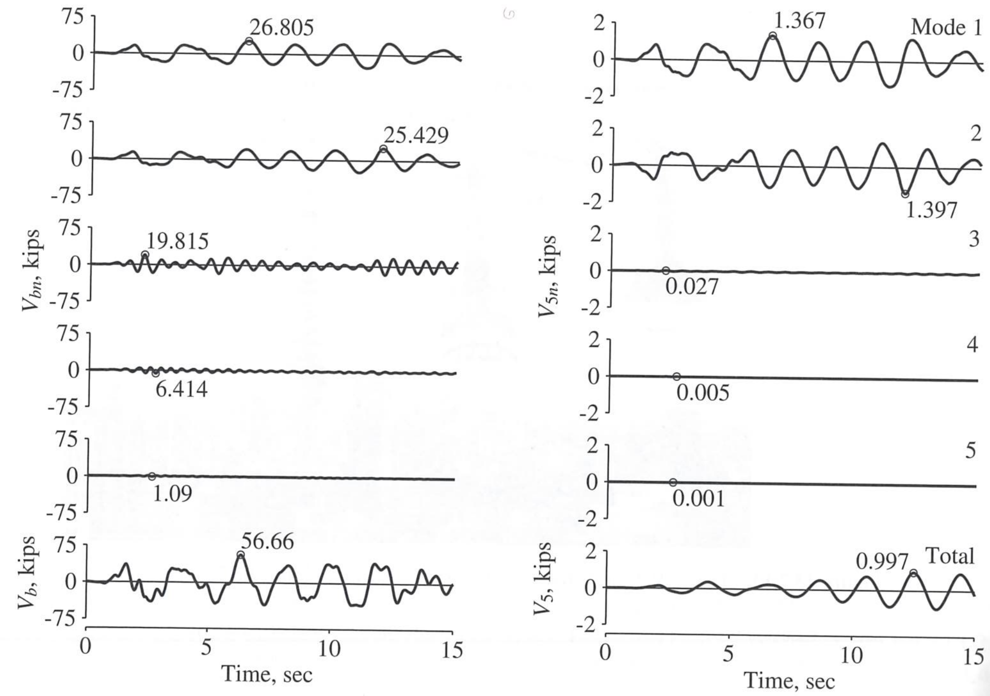


Figure 13.2.11 Base shear and appendage shear: modal contributions, $V_{bn}(t)$ and $V_{5n}(t)$, and total responses, $V_b(t)$ and $V_5(t)$.

TABLE E13.12a PEAK MODAL RESPONSES

Mode	u_y (in.)	b/2 u_θ (in.)	V_b (kips)	T_b (kip-ft)	V_{bA} (kips)	V_{bB} (kips)
1	2.168	−3.065	11.63	−118.8	11.63	6.814
2	2.042	2.999	14.64	155.2	14.64	−6.662

Step 4: For this system with two modes, the ABSSUM, SRSS, and CQC rules, Eqs. (13.7.2)–(13.7.4), specialize to

$$r \le |r_1| + |r_2| \qquad r \simeq (r_1^2 + r_2^2)^{1/2} \qquad r \simeq (r_1^2 + r_2^2 + 2\rho_{12} r_1 r_2)^{1/2} \tag{l}$$

For this system, $\beta_{12} = \omega_1/\omega_2 = 5.878/6.794 = 0.865$. For this value of β_{12} and $\zeta = 0.05$, Eq. (13.7.10) gives $\rho_{12} = 0.322$. The results from Eq. (l) are summarized in Table E13.12b, wherein the peak values of total responses determined by RHA are also included. These were computed using the results of Example 13.8, where $D_n(t)$ and $A_n(t)$ were computed by dynamic analysis of the nth-mode SDF system.

TABLE E13.12b RSA AND RHA VALUES OF PEAK RESPONSE

	u_y (in.)	(b/2) u_θ (in.)	V_b (kips)	T_b (kip-ft)	V_{bA} (kips)	V_{bB} (kips)
ABSSUM	4.210	6.064	26.27	274.0	26.27	13.48
SRSS	2.978	4.289	18.70	195.5	18.70	9.530
CQC	3.423	3.532	21.43	162.3	21.43	7.848
RHA	3.349	3.724	20.63	174.3	20.63	8.275

As expected, the ABSSUM estimate is always larger than the RHA value. The SRSS estimate is better, but the CQC estimate is the best because it accounts for the cross-correlation term in the modal combination, which is significant in this example because the natural frequencies are close, a situation common for unsymmetric-plan systems.

Example 13.13

Figure E13.13a–c shows a two-story building consisting of rigid diaphragms supported by three frames, A, B, and C. The lumped weights at the first and second floor levels are 120 and 60 kips, respectively. The lateral stiffness matrices of these frames, each idealized as a shear frame, are

$$\mathbf{k}_{yA} = \mathbf{k}_y = \begin{bmatrix} 225 & -75 \\ -75 & 75 \end{bmatrix} \qquad \mathbf{k}_{xB} = \mathbf{k}_{xC} = \mathbf{k}_x = \begin{bmatrix} 120 & -40 \\ -40 & 40 \end{bmatrix}$$

The design spectrum for $\zeta_n = 5\%$ is given by Fig. 6.9.5 scaled to 0.5g peak ground acceleration. Determine the peak value of the base shear in frame A.

Solution This system has four DOFs: u_{yj} and $u_{\theta j}$ (Fig. E13.13a); $j = 1$ and 2. The stiffness matrix of Eqs. (9.5.25) and (9.5.26) is specialized for this system with three frames:

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_y & e\mathbf{k}_y \\ e\mathbf{k}_y & e^2\mathbf{k}_y + (d^2/2)\mathbf{k}_x \end{bmatrix}$$

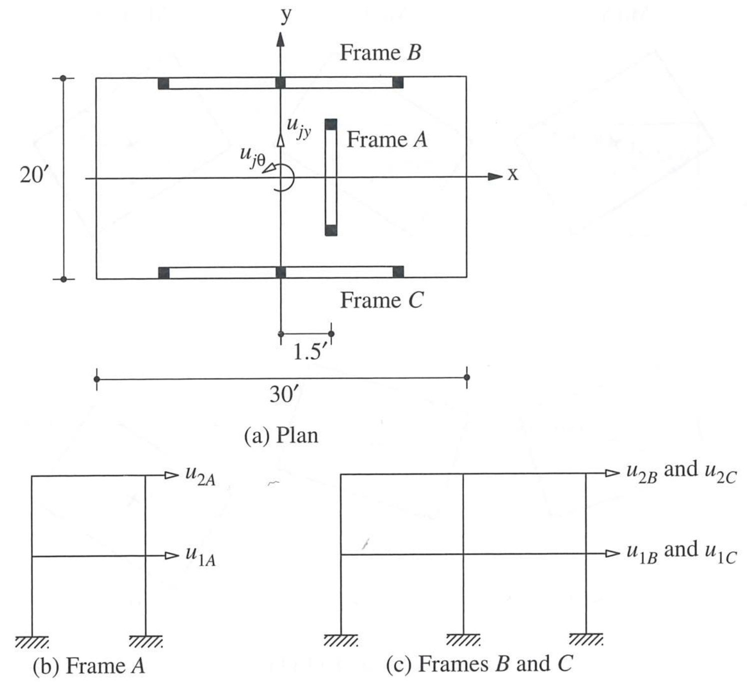


Figure E13.13a–c (continued)

Substituting for $\mathbf{k}_x$, $\mathbf{k}_y$, $e = 1.5$ ft, $d = 20$ ft, gives

$$\mathbf{k} = \begin{bmatrix} 225.0 & -75.00 & 337.5 & -112.5 \\ & 75.00 & -112.5 & 112.5 \\ & (\text{sym}) & 24{,}506 & -8169 \\ & & & 8169 \end{bmatrix}$$

The floor masses are $m_1 = 120/g = 3.727$ kip-sec²/ft and $m_2 = 60/g = 1.863$ kip-sec²/ft, and the floor moments of inertia are $I_{Oj} = m_j(b^2 + d^2)/12 = m_j(30^2 + 20^2)/12 = 1300m_j/12$. Substituting these data in the mass matrix of Eq. (9.5.27) gives

$$\mathbf{m} = \begin{bmatrix} 3.727 & & & \\ & 1.863 & & \\ & & 403.7 & \\ & & & 201.9 \end{bmatrix}$$

The eigenvalue problem is solved to determine the natural periods T_n and modes ϕ_n shown in Fig. E13.13d. Observe that each mode includes lateral and torsional motion. In the first mode the two floors displace in the same lateral direction and the two floors rotate in the same direction. In the second mode the two floors rotate in the same direction, which is opposite to the first mode. In the third and fourth modes the lateral displacements at the two floors are in opposite directions; the same is true for the rotations of the two floors.

The Γ_n are computed from Eqs. (13.3.4) to (13.3.6): $\Gamma_1 = 1.591$, $\Gamma_2 = 1.561$, $\Gamma_3 = -0.562$, and $\Gamma_4 = 0.552$.

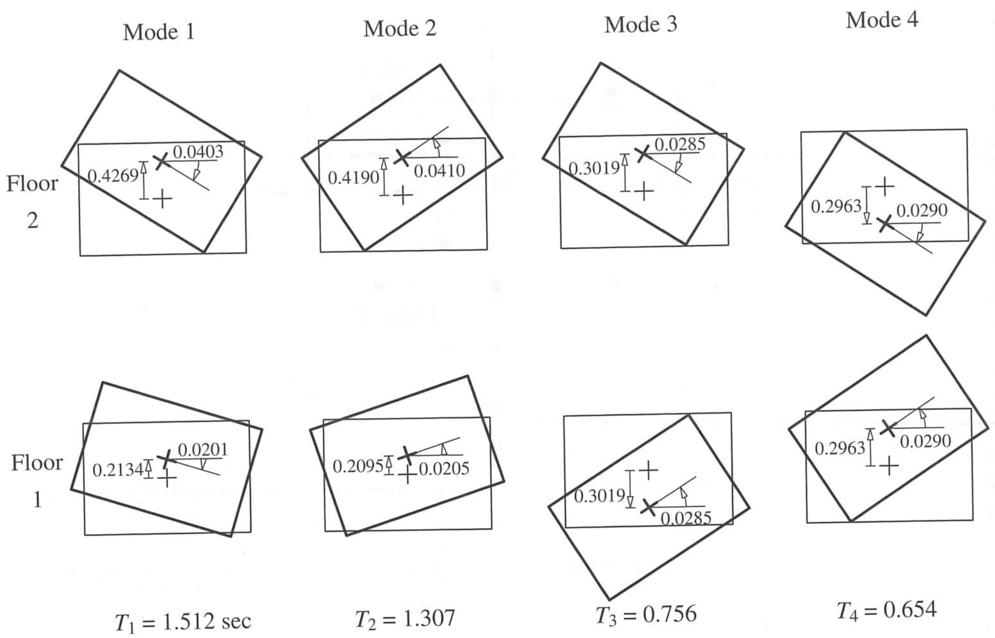


Figure E13.13d (continued)

For T_n = 1.512, 1.307, 0.756, and 0.654 sec, the design spectrum gives $A_1/g = 0.595$, $A_2/g = 0.688$, $A_3/g = 1.191$, and $A_4/g = 1.355$.

The peak values of the equivalent static lateral forces for frame A are [from Eq. (13.9.4b)]

$$\mathbf{f}_{An} = (\Gamma_n/\omega_n^2)\mathbf{k}_y(\phi_{yn} + e\phi_{\theta n})A_n$$

Substituting for Γ_1, $\omega_1(=4.156)$, $\mathbf{k}_y$, A_1, ϕ_{y1}, and $\phi_{\theta 1}$ gives the lateral forces associated with the first mode:

$$\begin{Bmatrix} f_{A1} \\ f_{A2} \end{Bmatrix}_1 = \frac{1.591}{(4.156)^2}(0.595 \times 32.2) \begin{bmatrix} 225 & -75 \\ -75 & -75 \end{bmatrix} \left(\begin{Bmatrix} 0.2134 \\ 0.4269 \end{Bmatrix} + 1.5 \begin{Bmatrix} -0.0201 \\ -0.0403 \end{Bmatrix} \right) = \begin{Bmatrix} 24.2 \\ 24.2 \end{Bmatrix}$$

Static analysis of the frame subjected to these lateral forces (Fig. E13.13e) gives the internal forces. In particular, the base shear is $V_{bA1} = f_{11} + f_{21} = 48.4$ kips. Similar computations lead to the peak base shear due to the second, third, and fourth modes: $V_{bA2} = 53.9$, $V_{bA3} = 12.1$, and $V_{bA4} = 13.3$ kips.

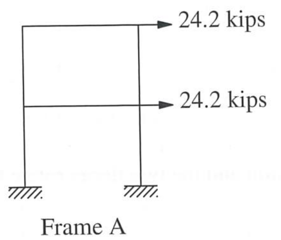


Figure E13.13e

The peak value r of the total response $r(t)$ will be estimated by combining the peak modal responses according to the CQC rule, Eq. (13.7.4). For this purpose it is necessary to determine the

frequency ratios $\beta_{in} = \omega_i/\omega_n$; these are given in Table E13.13a. For each of the β_{in} values the correlation coefficient ρ_{in} is computed from Eq. (13.7.10) with $\zeta = 0.05$ and presented in Table E13.13b.

TABLE E13.13a NATURAL FREQUENCY RATIOS β_{in}

Mode, i	$n=1$	$n=2$	$n=3$	$n=4$	ω_i(rad/sec)
1	1.000	0.865	0.500	0.433	4.157
2	1.156	1.000	0.578	0.500	4.804
3	2.000	1.730	1.000	0.865	8.313
4	2.312	2.000	1.156	1.000	9.608

TABLE E13.13b CORRELATION COEFFICIENTS ρ_{in}

Mode, i	$n=1$	$n=2$	$n=3$	$n=4$
1	1.000	0.322	0.018	0.012
2	0.322	1.000	0.030	0.018
3	0.018	0.030	1.000	0.322
4	0.012	0.018	0.322	1.000

Substituting the peak modal values V_{bAn} and the correlation coefficients ρ_{in} in the CQC rule, we obtain the 16 terms in the double summation of Eq. (13.7.4) (Table E13.13c). Adding the 16 terms and taking the square root gives $V_{bA} = 86.4$ kips. Table E13.13c shows that the terms with significant values are the $i = n$ terms, and the cross terms between modes 1 and 2 and between modes 3 and 4. The cross terms between modes 1 and 3, 1 and 4, 2 and 3, or 2 and 4 are small because those frequencies are well separated. The square root of the sum of the four $i = n$ terms in Table E13.13c gives the SRSS estimate: $V_{bA} = 74.7$ kips. This is less accurate.

TABLE E13.13c INDIVIDUAL TERMS IN CQC RULE: BASE SHEAR V_{bA} IN FRAME A

Mode, i	$n=1$	$n=2$	$n=3$	$n=4$
1	2344.039	839.912	10.833	7.839
2	839.913	2905.669	19.748	13.250
3	10.833	19.748	146.502	51.797
4	7.839	13.250	51.797	176.807

13.10 A RESPONSE-SPECTRUM-BASED ENVELOPE FOR SIMULTANEOUS RESPONSES

The seismic design of a structural element may be controlled by the simultaneous action of two or more responses. For example, a column in a three-dimensional frame must be designed to resist an axial force and bending moments about two axes that act concurrently and vary in time. We limit this section to two response quantities: $r_a(t)$ and $r_b(t)$; for a column $r_a(t)$ represents the bending moment $M(t)$ about a cross-sectional axis and $r_b(t)$ represents the axial force $P(t)$. The values of r_a and r_b at a time instant t are denoted by one point in the two-dimensional response space (Fig. 13.10.1a),