15 Novembre

<u>Teor</u> Sia $f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$, $f\in C^0(a,b)$ e t.c. $f^{(k)}(x_0)$ esistono $\forall\, k=0,\dots,n$. Allora posto $p_n(x)$ il polinomio di Taylor di f di ordine n rispetto a x_0, ho

$$f(x) = p_n(x) + R_n(x)$$

dove

$$R_n(x) = o\left((x-x_0)^n\right)$$

$$\Updownarrow$$

$$\lim_{x\to x_0} \frac{R_n(x)}{(x-x_0)^n} = 0 \, .$$

$$P_1(x) = f(x_0) + f'(x_0)(x-x_0)$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + R_1(x)$$

Verifichiamo che $\lim_{x \to x_0} \frac{R_1(x)}{x-x_0} = 0$

$$R_1(x) = f(x) - f(x_0) - f'(x_0)(x-x_0)$$

$$\frac{R_1(x)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)$$

$$\lim_{x \to x_0} \frac{R_1(x)}{x-x_0} = \lim_{x \to x_0} \left[\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0) \right] = 0 \iff \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$$

Lemma Sia $p(x)$ un polinomio di grado $\leq n$ e sia $x_0 \in \mathbb{R}$.

Supponiamo $p(x) = o\left((x-x_0)^n\right)$ $\left(\text{cioè } \lim_{x \to x_0} \frac{p(x)}{(x-x_0)^n} = 0\right)$

Allora $p(x) \equiv 0$.

Dim Sappiamo che nel punto x_0, $p(x)$ è il polinomio di Taylor di se stesso di ordine n.

$$p(x) = \sum_{k=0}^{n} \frac{p^{(k)}(x_0)}{k!}(x-x_0)^k$$

Se per assurdo $p(x) \not\equiv 0$ allora esiste un primo indice $k_0 \in \{0, \ldots, n\}$ t.c. $p^{(k_0)}(x_0) \neq 0$, ~~e per~~ cioè t.c. per $j < k_0$ ho $p^{(j)}(x_0) = 0$

Risulta allora

$$\lim_{x\to x_0} \frac{P(x)}{(x-x_0)^{k_0}} = \lim_{x\to x_0} \frac{\sum_{k=k_0}^{n} \frac{P^{(k)}(x_0)}{k!}(x-x_0)^k}{(x-x_0)^k} = \lim_{x\to x_0} \frac{\frac{P^{(k_0)}(x_0)}{k_0!}\cancel{(x-x_0)^{k_0}}}{\cancel{(x-x_0)^{k_0}}}$$

$$= \frac{P^{(k_0)}(x_0)}{k_0!} \neq 0$$

D'altra parte $k_0 \leq n$ e quindi

$$\lim_{x\to x_0} \frac{P(x)}{(x-x_0)^{k_0}} \frac{(x-x_0)^n}{(x-x_0)^n} = \lim_{x\to x_0} \frac{P(x)}{(x-x_0)^n} \frac{(x-x_0)^n}{(x-x_0)^{k_0}} =$$

$$= \lim_{x\to x_0} \frac{P(x)}{(x-x_0)^n} (x-x_0)^{\underbrace{n-k_0}_{\geq 0}} = 0 .$$

Si ottiene un assurdo.

$$\Rightarrow P(x) \equiv 0$$

Corollario Sia f come nel Teorema di Peano e sia $P(x)$ un polinomio di grado $\leq n$ t.c.

$$f(x) = P(x) + o((x-x_0)^n) \quad \star$$

Allora $P(x) \equiv P_n(x)$, cioè P è polinomio di Taylor di ordine n di f nel punto x_0.

Dim So che esiste anche il polinomio di Taylor $P_n(x)$ e si ha

$$f(x) = P_n(x) + o((x-x_0)^n) \leftarrow \text{Teor Peano}$$

Ho anche

$$f(x) = P(x) + o((x-x_0)^n) \leftarrow \star$$

$$0 = P_n(x) - P(x) + o((x-x_0)^n) \Rightarrow P(x) - P_n(x) = o((x-x_0)^n)$$

Abbiamo $p(x) - P_n(x) = o\left((x-x_0)^n\right)$ e, siccome $p(x) - P_n(x)$ ha grado $\leq n$, dal lemma precedente, $p(x) - P_n(x) \equiv 0$

$$\Longleftrightarrow \quad p(x) \equiv P_n(x).$$

<u>Esempio</u>: Calcolare tutti i polinomi di McLaurin di $f(x) = x^2 \sin(x^3)$.

$$\forall n \qquad P_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j .$$

$$\sin(y) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} y^{2k+1} + o\left(y^{2n+1}\right)$$

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$$y = x^3$$

$$\sin(x^3) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{6k+3} + o\left(x^{6n+3}\right)$$

$$f(x) = x^2 \sin(x^3) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{6k+5} + \underbrace{x^2 o\left(x^{6n+3}\right)}_{= o\left(x^{6n+5}\right)}$$

$$\lim_{x \to 0} \frac{x^2 o\left(x^{6n+3}\right)}{x^{6n+5}} = \lim_{x \to 0} \frac{o\left(x^{6n+3}\right)}{x^{6n+3}} = 0$$

$$\Longrightarrow \quad x^2 o\left(x^{6n+3}\right) = o\left(x^{6n+5}\right)$$

$$f(x) = x^2 \sin(x^3) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{6k+5} + o(x^{6n+5})$$

$$\Rightarrow P_{6n+5}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!} x^{6k+5}.$$

$$P_5(x) = x^5$$

$$P_0(x),\ P_1(x),\ P_2(x),\ P_3(x),\ P_4(x)\ ?$$

$$f(x) = \underset{\substack{\| \\ o(x^4)}}{x^5} + \underset{\substack{\| \\ o(x^4)}}{o(x^5)} = o(x^4) = 0 + o(x^4) = 0 + o(x^3)$$

$$\Rightarrow P_4(x) \equiv 0 \qquad \Rightarrow P_3(x) \equiv 0$$

sono tutti uguali a 0.

$$f(x) = x^2 \sin(x^3)$$

$$P_0 \equiv P_1 \equiv P_2 \equiv P_3 \equiv P_4 \equiv 0$$

$$P_{6n+5}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

Verifichiamo che se $6n+5 \leq m < 6(n+1)+5 = 6n+11$

$$P_m = P_{6n+5}$$

$$f(x) = \sum_{k=0}^{n+1} (-1)^k \frac{x^{6k+5}}{(2k+1)!} + o\left(x^{6n+11}\right) =$$

$$f(x) = P_{6n+5}(x) + (-1)^{n+1} \frac{x^{6n+11}}{(2n+3)!} + o\left(x^{6n+11}\right)$$

$$f(x) = P_{6n+5} + (-1)^{n+1} \frac{x^{6n+11}}{(2n+3)!} + o(x^{6n+11})$$

$$6n+5 < m < 6n+11$$

$$= o(x^m)$$

$$f(x) = P_{6n+5}(x) + o(x^m) \quad \forall \quad 6n+5 \le m < 6n+11$$

$$\Longrightarrow \quad P_{6n+5}(x) \equiv P_m(x)$$

Quali sono i ~~fo~~ valori di $f^{(i)}(0) = ?$

$$f(x) = x^2 \sin(x^3)$$

$$P_{6n+5}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{6k+5}}{(2k+1)!}$$

$$= \sum_{j=0}^{6n+5} \frac{f^{(j)}(0)}{j!} x^j .$$

Se j non è della forma $j = 6k+5$

$$f^{(j)}(0) = 0$$

Se invece ho $j = 6k+5$

$$\Longrightarrow f^{(6k+5)}(0) = (-1)^k \frac{(6k+5)!}{(2k+1)!}$$

$$\frac{f^{(6k+5)}(0)}{(6k+5)!} = \frac{(-1)^k}{(2k+1)!}$$

$$f(x) = \frac{1}{1-x}$$

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$f(x) = \frac{1}{1-x} = \sum_{j=0}^{n} x^j + \frac{x^{n+1}}{1-x}$$ dove $$\frac{x^{n+1}}{1-x} = o(x^n)$$

$$\lim_{x\to 0} \frac{\frac{x^{\cancel{n}+1}}{1-x}}{\cancel{x^n}} = \lim_{x\to 0} \frac{x}{1-x} = \frac{0}{1} = 0$$

$$\Rightarrow f(x) = \frac{1}{1-x} = \sum_{j=0}^{n} x^j + o(x^n) \Rightarrow P_n(x) = \sum_{j=0}^{n} x^j$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + o(x^n)$$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-x)^j + o(x^n) = \underbrace{\sum_{j=0}^{n} (-1)^j x^j}_{P_n(x)} + o(x^n)$$

$$(1+x)^{-1} = \sum_{j=0}^{n} \binom{-1}{j} x^j + o(x^n)$$

$$\binom{-1}{j} = (-1)^j$$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n)$$

$$\frac{1}{1+x^2} = \underbrace{\sum_{j=0}^{n} (-1)^j x^{2j}}_{P_{2n}(x)} + o(x^{2n})$$

Esame 13/1/2014

Calcolare $\lim_{x \to 0^+} \frac{\sqrt{1+2x^a} - 1 - x^2}{\lg(1+x+x^2) - x}$ al variare di $a \in \mathbb{R}_+$

$\lg(1+x+x^2)$

$$\left(\lg(1+y)\right)^{(n)} = \left((1+y)^{-1}\right)^{(n-1)} = \prod_{j=1}^{n-1}(-\cancel{1}-j+\cancel{1})\,(1+y)^{-n} =$$

$$= (-1)^{n-1}(n-1)!\,(1+y)^{-n}$$

$$\left(\lg(1+y)\right)^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$$\lg(1+y) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} y^k + o(y^n)$$

$$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} y^k + o(y^n)$$

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\lg(1+\underbrace{x+x^2}_{y}) = x+x^2 - \frac{(x+x^2)^2}{2} + \underbrace{o\left((x+x^2)^2\right)}_{o(x^2)}$$

$$(x+x^2)^2 = \left(x(1+x)\right)^2 = x^2\,(1+x)^2$$

$$o\left((x+x^2)^2\right) = o\left(x^2\,(1+x)^2\right) = o(x^2)$$

$$\lg(1+x+x^2) = x+x^2 - \frac{x^2+2x^3+x^4}{2} + o(x^2) =$$

$$= x + x^2 - \frac{x^2}{2} + o(x^2) = x + \frac{1}{2}x^2 + o(x^2)$$

$$\lg(1+x+x^2)-x = \cancel{x}+\frac{x^2}{2}-\cancel{x}+o(x^2) = \frac{x^2}{2}+o(x^2)$$

$$\lim_{x\to 0}\frac{\sqrt{1+2x^a}-1-x^2}{\lg(1+x+x^2)-x} = \lim_{x\to 0}\frac{\sqrt{1+2x^a}-1-x^2}{\frac{x^2}{2}}$$