$$e^{-S_{eff}[A]} = e^{-\frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F^{\mu\nu}F_{\mu\nu}} \left(\det \Delta_{gauge}\right)^{-1/2} \left(\det \Delta_{gh}\right)$$

$$\Delta^{\mu\nu}_{gauge} = -D^2\delta^{\mu\nu} - 2i[F^{\mu\nu}, \cdot\,] \qquad \Delta_{gh} = -D^2$$

metrica che agisce su indice ν

$$\left(\det \Delta = e^{\log \det \Delta} = e^{\mathrm{tr} \log \Delta} \qquad \log \det \Delta = \mathrm{tr} \log \Delta \right)$$

$$S_{eff}[A] = \frac{1}{2g^2}\int d^4x \, \mathrm{tr}\, F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\,\mathrm{Tr} \log \Delta_{gauge} - \mathrm{Tr} \log \Delta_{gh}$$

traccia su group indices, lorentz indices, sp. funz.

S_{eff} ha termini che divergono a 1-loop. Per cancellarli poi bisogna RINORMALIZZARE le cost. di accoppiamento

$$S_{eff} = S_{YM}(A) + \text{1-loop}$$

$$\frac{1}{g_B^2}(\dots) = \frac{1}{g_r^2}(\dots) - \cancel{div.} \qquad \cancel{div.} + \dots$$

↳ questo darà la relazione tra g_B e g_r (che poi ci permette di calcolare β)

S_{eff} è inv. sotto trasf. di gauge di A → questa è simmetria che vale per $S_{YM}(A)$ e deve valere anche per i termini divergenti ⇒ termini divergenti saranno del tipo

$$\frac{1}{2-\omega}\,(\dots)$$

funzionale in A invariante sotto trasf. di gauge di A
QUADRATICO

⇓ cioè $(\dots) \propto S_{YM}(A)$

Δ_{gh}

$$\Delta_{gh} = -D^2 = -\partial^2 + \Delta_1 + \Delta_2$$

$\Delta_2 \longrightarrow [A^\mu, [A_\mu, \cdot]] = A^{\mu a} A^b_\mu \cdot T^a_{Adj} \cdot T^b_{Adj}$

$$\Delta_1 = -i\,\partial^\mu A_\mu - i A_\mu \partial^\mu = -i\,\{\partial^\mu, A_\mu\} = -i\{\partial^\mu, A^a_\mu\} T^a_{Adj}$$

$$\mathrm{Tr}\log\Delta_{gh} = \mathrm{Tr}\log(\underbrace{-\partial^2 + \Delta_1 + \Delta_2}_{(-\partial^2)(1+(-\partial^2)^{-1}(\Delta_1+\Delta_2))}) = \mathrm{Tr}\log(-\partial^2) + \mathrm{Tr}\log(1+(-\partial^2)^{-1}(\Delta_1+\Delta_2))$$

$$= \mathrm{Tr}\log(-\partial^2) + \mathrm{Tr}[(-\partial^2)^{-1}(\Delta_1+\Delta_2)] - \frac{1}{2}\mathrm{Tr}[((-\partial^2)^{-1}(\Delta_1+\Delta_2))^2] + \dots$$

termine cost che trascuriamo

lineare in A_μ e $\mathrm{tr}\, t^a = 0$

termini che contengono potenze di A^μ superiori a 2 (Infiniti vengono cancellati dalla parte cubica e quartica di S_{YM})

$|x\rangle \qquad |k\rangle$

$$f(x) = \langle x|f\rangle \rightsquigarrow \Delta\cdot f(x) \equiv \langle x|\Delta|f\rangle$$

$$\int d^dx\, |x\rangle\langle x| = \mathbb{1} \qquad \int \frac{d^dk}{(2\pi)^d} |k\rangle\langle k| = \mathbb{1}$$

$$\langle x|k\rangle = e^{ikx}$$

$\rightsquigarrow$ In qsto formalismo, facile passare a trasf. di Fourier $\Delta\cdot\tilde{f}(k) = \langle k|\Delta|f\rangle$

$$\langle x|(-\partial^2)^{-1} \overset{\int \frac{d^dk}{(2\pi)^d}|k\rangle\langle k|}{\,} |y\rangle = \frac{1}{(2\pi)^d}\int d^dk\, \frac{1}{k^2} e^{ik(x-y)}$$

$\mathrm{Tr}M = \sum_i e_i \cdot M e_i$

$$\mathrm{Tr}[(-\partial^2)^{-1}\Delta_2] = \mathrm{Tr}[(-\partial^2)^{-1} A^{\mu a} t^a_{Ad} A^b_\mu t^b_{Ad}] =$$

traccia sulle funz. e sullo sp. di rap. Adj

$$= \int d^dx\, \langle x|\,\mathrm{tr}[(-\partial^2)^{-1} \overset{\int d^dy\, |y\rangle\langle y|}{\,} A^{\mu a} t^a_{Ad} A^b_\mu t^b_{Ad}]\,|x\rangle =$$

traccia su sp. rap. Adj

funz. di operatore $\hat{X}$

$$= \int d^dx\, d^dy\, \langle x|(-\partial^2)^{-1}|y\rangle \langle y|A^{\mu a} A^b_\mu|x\rangle\, \mathrm{tr}(t^a_{Ad} t^b_{Ad})$$

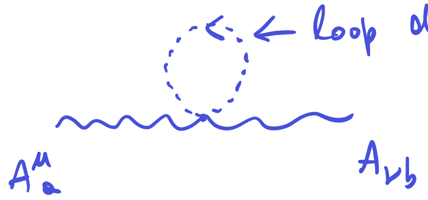
$$= \int d^dx\, d^dy\, \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2}\, e^{ik(x-y)}\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) A^{\mu a}(y) A^b_\mu(y)\, \delta(y-x)$$

$$= \int d^dx\, A^{\mu a}(x) A^b_\mu(x)\, \underbrace{\mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right)} \int \frac{d^dk}{(2\pi)^d}\, \frac{1}{k^2}$$

← DIVERGENZA in $d=4$

termine in $S_{eff}(A)$ che contribuisce alla funz. a 2pt. (tree-level)

Derivando $\mathcal{L}_{eff}(A)$ per A due volte, otteniamo correz. a propagatore di teoria eff.



← bkg gauge field

← qui c'è interaz. perchè abbiamo $G(A)$ diverso da quello di gauge di Lorentz

$$\mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) = C(Adj)\, \delta^{ab} = c_2(G)\, \delta^{ab}$$

$\Delta_1 = -i\,\partial^\mu A_\mu - i A_\mu \partial^\mu$

$$-\frac{1}{2}\, \mathrm{Tr}\left((-\partial^2)^{-1}\Delta_1 (-\partial^2)^{-1}\Delta_1\right) = -\frac{1}{2}\int d^dy\, \langle y|\, \mathrm{tr}\left((-\partial^2)^{-1}\Delta_1(-\partial^2)^{-1}\Delta_1\right)|y\rangle =$$

$$= +\frac{1}{2}\int d^dy\, \langle y| \Big[\mathrm{tr}\left((-\partial^2)^{-1} \overset{\int d^dx |x\rangle\langle x|}{\vee} \partial^\mu A^a_\mu\, t^a_{Ad}\, (-\partial^2)^{-1}\, \partial^\nu A^b_\nu\, t^b_{Ad}\right) +$$

?

$$+ \mathrm{tr}\left((-\partial^2)^{-1} A^a_\mu\, t^a_{Ad} \overset{\int d^dx |x\rangle\langle x|}{\vee} \partial^\mu (-\partial^2)^{-1}\, \partial^\nu A_\nu^b\, t^b_{Ad}\right) +$$

$$+ \mathrm{tr}\left((-\partial^2)^{-1} \overset{\int d^dx |x\rangle\langle x|}{\vee} \partial^\mu A^a_\mu\, t^a_{Ad}\, (-\partial^2)^{-1} A^b_\nu\, \partial^\nu t^b_{Ad}\right) +$$

ciclicità traccia

$$+ \mathrm{tr}\left((-\partial^2)^{-1} A^a_\mu\, t^a_{Ad} \overset{\int d^dx |x\rangle\langle x|}{\vee} \partial^\mu (-\partial^2)^{-1} A^b_\nu\, T^b\, \partial^\nu\right) \Big] |y\rangle$$

Consideriamo il primo termine

$\langle y|k\rangle = e^{iky}$

$$1^{st}\ \text{term} = \frac{1}{2}\int d^dx\, d^dy\ \mathrm{tr}\left[A_\mu^a(x)\, t^a_{Ad}\, A_\nu^b(y)\, t^b_{Ad} \right] \cdot$$

$\int \frac{d^dk}{(2\pi)^d} |k\rangle\langle k|$ $\quad$ k' – completezza

$$\langle y| (-\partial^2)^{-1} \partial^\mu |x\rangle \langle x| (-\partial^2)^{-1} \partial^\nu |y\rangle$$

$\langle y|k\rangle = e^{iky}$

$$\frac{1}{-(ik)^2}\, ik^\mu\, e^{ik(y-x)} \qquad \frac{1}{-(ik')^2}\, ik'^\nu\, e^{-ik'(y-x)}$$

$$-\frac{k^\mu}{k^2}\frac{k'^\nu}{k'^2}\, e^{i(k-k')(y-x)}$$

2^{nd} term $=$ $\quad\ldots$ ← stessa prima riga del 1^{st} term

$$\langle y| (-\partial^2)^{-1} |x\rangle \cdot \langle x| \partial^\mu (-\partial^2)^{-1} \partial^\nu |y\rangle$$

$$+\frac{1}{k^2}\cdot \frac{(-)k'^\mu k'^\nu}{k'^2}\, e^{i(k-k')(y-x)}$$

3^{rd} term

$$-\frac{k^\mu k^\nu}{k^2}\cdot\frac{1}{k^2}\, e^{i(k-k')(y-x)}$$

4^{th} term

$$-\frac{k^\nu}{k^2}\,\frac{k'^\mu}{k^2}\, e^{i(k-k')(y-x)}$$

$$-\frac{e^{i(k-k')(y-x)}}{k^2 k'^2}(k+k')^\mu (k+k')^\nu$$

$$-\frac{1}{2}\int d^dx\, d^dy \int \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\, e^{i(k-k')(y-x)}\, \frac{(k+k')^\mu (k+k')^\nu}{k^2 k'^2}\, \mathrm{tr}\left(A_\mu^a(x)\, t^a_{Ad}\, A_\nu^b(y)\, t^b_{Ad}\right)$$

$$\tilde{A}(k) = \int d^dx\, e^{ikx} A(x) \;\rightarrow\; A(x) = \int \frac{d^dp}{(2\pi)^d}\, e^{-ipx}\, \tilde{A}(p)$$

$y \qquad q$

$$-\frac{1}{2}\int d^dx\, d^dy\, \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\; \underbrace{e^{i(k-k')y}\, e^{-iqy}}_{\delta(k-k'-q)}\; \underbrace{e^{-i(k-k')x}\, e^{-ipx}}_{\delta(k-k'+p)} \,\cdot$$

$$\cdot\frac{(k+k')^\mu (k+k')^\nu}{k^2\, k'^2}\, \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(q)\, \mathrm{tr}\,(t^a_{Ad}\, t^b_{Ad})$$

$$\begin{cases} k-k'-q=0 \\ k-k'+p=0 \end{cases} \Rightarrow \begin{cases} q=-p \\ k'=p+k \end{cases}$$

$$= -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}(t^a_{Ad}\, t^b_{Ad})\, \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(-p)\int \frac{d^dk}{(2\pi)^d}\frac{(p+2k)^\mu(p+2k)^\nu}{k^2\,(k+p)^2}$$

Riscriviamo anche il primo termine $\int d^dx\, A^{\mu a}(x)\, A^b_\mu(x)\, \mathrm{tr}(t^a_{Ad}\, t^b_{Ad}) \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$ nello sp. dei momenti:

$$\mathrm{Tr}\,((-\partial^2)\Delta_2) = \int d^dx \int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{d^dk}{(2\pi)^d}\, \mathrm{tr}(t^a_{Ad}\, t^b_{Ad})\, \tilde{A}^{\mu a}(p)\, \tilde{A}^b_\mu(q)\frac{1}{k^2}\, \underbrace{e^{-ipx}\, e^{-iqx}}_{\delta(p+q)}$$

$$= \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}(t^a_{Ad}\, t^b_{Ad})\, \tilde{A}_\mu(p)\, \tilde{A}_\nu(-p)\int \frac{d^dk}{(2\pi)^d}\frac{\delta^{\mu\nu}}{k^2}$$

siamo su sp. Euclideo

$$\mathrm{Tr}\log\Delta_{gh} = \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}(t^a_{Ad}\, t^b_{Ad})\, \tilde{A}_\mu(p)\, \tilde{A}_\nu(-p)\cdot\left\{ \int \frac{d^dk}{(2\pi)^d}\frac{\delta_{\mu\nu}}{k^2} + \right.$$

$$\left. -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{(p+2k)^\mu(p+2k)^\nu}{k^2(k+p)^2} \right\}$$

Per calcolare $\{\ldots\}$ usiamo gli integrali di Feynman

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+a^2)^A} = \frac{\Gamma(A-d/2)}{(a^2)^{A-d/2}\,(4\pi)^{d/2}\,\Gamma(A)}$$

$$\Rightarrow \lim_{\epsilon\to 0} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\epsilon^2} = \lim_{\epsilon\to 0} \frac{\Gamma(1-d/2)}{\epsilon^{1-d/2}\,(4\pi)^{d/2}} = 0 \qquad \mu\ d>2$$

$A=1 \quad a=\epsilon$

$$\frac{1}{k^2(k+p)^2} = \int_0^1 d\xi \frac{1}{\left[k^2(1-\cancel{\xi})+(\cancel{k^2}+2kp+p^2)\xi\right]^2} = \int_0^1 d\xi \frac{1}{(k^2+2\xi p\cdot k+\xi p^2)^2}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2kp+b^2)^A} = \frac{\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}(4\pi)^{d/2}\Gamma(A)} \qquad \Gamma(n+1)=n!$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{p^\mu p^\nu}{(k^2+2\xi pk+\xi p^2)^2} = \frac{p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2+2kP+b^2)^A} = \frac{-p^\mu\,\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}(4\pi)^{d/2}\Gamma(A)}$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{2k^\mu p^\nu+2k^\nu p^\mu}{(k^2+2\xi pk+\xi p^2)^2} = -\frac{4\xi\,p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2+2kP+b^2)^A} = \frac{1}{(4\pi)^{d/2}\Gamma(A)}\left[\frac{p^\mu p^\nu\,\Gamma(A-\frac{d}{2})}{(b^2-p^2)^{A-d/2}} + \frac{1}{2}\,\delta^{\mu\nu}\,\frac{\Gamma(A-1-d/2)}{(b^2-p^2)^{A-1-d/2}}\right]$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{4k^\mu k^\nu}{(k^2+2\xi kp+\xi p^2)^2} = \frac{4\xi^2\,p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}} + \frac{1}{2}\,\frac{4\,\delta^{\mu\nu}\,\Gamma(1-d/2)}{(\xi(1-\xi)p^2)^{1-d/2}(4\pi)^{d/2}}$$

$$\Rightarrow -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{(p+2k)^\mu(p+2k)^\nu}{k^2(k+p)^2} = -\frac{1}{2}\int_0^1 d\xi \left\{ \frac{(1-2\xi)^2 p^\mu p^\nu \Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}} + \frac{2\delta^{\mu\nu}\Gamma(1-d/2)}{(\xi(1-\xi)p^2)^{1-\frac{d}{2}}(4\pi)^{d/2}} \right\}$$

$$4\xi^2 - 4\xi + 1 = (2\xi - 1)^2$$

$d = 2\omega \quad (\omega \to 2)$

espandiamo fto funz. in d attorno a $d=4$

$$\Gamma\left(1-\frac{d}{2}\right) = \frac{1}{1-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)$$

Parte divergente in $d\to 4$:

$$\int_0^1 d\xi \, \frac{(1-2\xi)^2 p^\mu p^\nu}{(4\pi)^2} \frac{1}{2-\omega}$$

$$\int_0^1 d\xi\,(1-4\xi+4\xi^2) = 1 - 2 + \frac{4}{3} = \frac{1}{3}$$

Parte divergente in $d\to 4$:

$$\int_0^1 d\xi \, \xi(1-\xi)\frac{\delta^{\mu\nu}p^2}{(4\pi)^2}\left(-\frac{2}{2-\omega}\right)$$

$$\int_0^1 \xi(1-\xi)\,d\xi = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\rightsquigarrow \text{Parte div.} = -\frac{1}{2}\frac{1}{48\pi^2}\frac{1}{2-\omega}\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

$$\Downarrow$$

$$\operatorname{Tr}\log\Delta_{gh} \approx \int \frac{d^dp}{(2\pi)^p}\operatorname{tr}\left(t^a_{Ad}t^b_{Ad}\right)\tilde{A}^{\mu a}(p)\tilde{A}^{\nu b}(-p)\left(-\frac{1}{2}\frac{1}{3}\frac{1}{16\pi^2}\frac{1}{2-\omega}(p^\mu p^\nu - p^2\delta^{\mu\nu})\right) + \dots$$

$$= -\frac{1}{2}\frac{1}{3}\frac{1}{(4\pi)^2}\underbrace{\operatorname{tr}\left(t^a_{Ad}t^b_{Ad}\right)}_{C(Adj)\,\delta^{ab} = C_2(G)}\underbrace{\int\frac{d^dp}{(2\pi)^d}\tilde{A}^{\mu a}(p)\tilde{A}^{\nu b}(-p)\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)}_{g^2 S[A]}\frac{1}{2-\omega}$$

Parte quadratica di azione S:

$$\frac{1}{4g^2}\int\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)\left(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu}\right)d^dx =$$

$$= \frac{1}{4g^2}\int \underline{\underline{d^dx}}\,\frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\left(-iq_{1\mu}\tilde{A}^a_\nu(q_1) + iq_{1\nu}\tilde{A}^a_\mu(q_1)\right)\left(-iq_2^\mu\tilde{A}^{a\nu}(q_2) + iq_2^\nu\tilde{A}^{a\mu}(q_2)\right)\cdot \underline{e^{-iq_1x - iq_2x}}$$

$$\to \delta(q_1+q_2) \qquad q_1 = -q_2 \equiv p$$

$$= \frac{1}{4g^2} \int \frac{d^d p}{(2\pi)^d} \left(p_\mu \tilde{A}^a_\nu(p) - p_\nu \tilde{A}^a_\mu(p) \right) \left(p^\mu \tilde{A}^{a\nu}(-p) - p^\nu \tilde{A}^{a\mu}(-p) \right) =$$

$$= -\frac{1}{2g^2} \int \frac{d^d p}{(2\pi)^d} \left(p^\mu p^\nu - p^2 \delta^{\mu\nu} \right) \tilde{A}^a_\mu(p) \tilde{A}^a_\nu(-p)$$