Contributo di Δ_{gauge}

$$\Delta^{\mu\nu}_{gauge} = - D^2 \delta^{\mu\nu} - 2i\, F^{a\mu\nu} t^a_{Ad} = \underbrace{\Delta_{gh}}\, \delta^{\mu\nu} - 2i\, F^{a\mu\nu} t^a_{Ad}$$

quando tracciamo sugli indici di Lorentz, otteniamo un fattore 4. ← op. come nei ghost ma ora abbiamo $\delta^{\mu\nu}$

$$\mathrm{Tr} \log \Delta_{gauge} = \mathrm{Tr} \log \Big(-\partial^2 \delta^{\mu\nu} + \overbrace{(\Delta_1 + \Delta_2)\delta^{\mu\nu} - 2i F^{\mu\nu a} t^a_{Ad}}^{(\dots)} \Big) =$$

$$= \mathrm{Tr} \log (-\partial^2 \delta^{\mu\nu}) + \mathrm{Tr} \log \big(1 + (-\partial^2)^{-1} (\dots)\big) \simeq$$

cost. trascurabile

termini con Δ_1, Δ_2 ricostruiscono $\mathrm{Tr} \log \Delta_{gh} \delta^{\mu\nu}$

$$\simeq \mathrm{Tr}\big[(-\partial^2)^{-1} (\dots)\big] - \frac{1}{2} \mathrm{Tr}\Big[\big((-\partial^2)^{-1}(\dots)\big)^2\Big] =$$

$F_{\mu\nu} t^a_{Ad}$ non contrib.

termini misti $F_{\mu\nu} \Delta_1 \delta^{\mu\nu}$ sono zero perché $F_{\mu\nu}\delta^{\mu\nu} = 0$

$$= 4\, \mathrm{Tr} \log \Delta_{gh} + F^{\mu\nu}\text{-terms}$$ ← quadratici in F

dove

$$F^{\mu\nu}\text{-terms} = \overbrace{-\frac{1}{2} (-2i)^2}^{+2} \mathrm{Tr}\Big((-\partial^2)^{-1} F^{a\mu\nu} t^a_{Ad} (-\partial^2)^{-1} F^b_{\nu\mu} t^b_{Ad} \Big)$$

$$= -2 \int d^d y \, \langle y | (-\partial^2)^{-1} F^{\mu\nu a} (-\partial^2)^{-} F^b_{\mu\nu} (-\partial^2)^{-1} | y \rangle \, \mathrm{tr}(t^a_{Ad} t^b_{Ad})$$

$\int d^d x \, |x\rangle\langle x|$

$$= -2 \int d^d y \, d^d x \, \langle y | (-\partial^2)^{-1} | x \rangle \langle x | (-\partial^2)^{-1} | y \rangle F^{\mu\nu a}(x) F^b_{\mu\nu}(y) \, \mathrm{tr}(t^a_{Ad} t^b_{Ad})$$

$$= -2 \int d^d y \, d^d x \, \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k'^2} F^{\mu\nu a}(x) F^b_{\mu\nu}(y) \, \mathrm{tr}(t^a t^b) e^{ik(y-x)} e^{ik'(x-y)}$$

trasf. Fourier

$$= -2 \int d^d y \, d^d x \, \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k'^2} \mathrm{tr}(t^a t^b) e^{-ipx} e^{-iqy} e^{i(k-k')y} e^{i(k'-k)x}$$

$$\cdot (-i)^2 \big(p^\rho \tilde{A}^{a\sigma}(p) - p^\sigma \tilde{A}^{a\rho}(p)\big)\big(q_\rho \tilde{A}^b_\sigma(q) - q_\sigma \tilde{A}^b_\rho(q)\big)$$

$$\delta(k' - k - p) \qquad \delta(k - k' - q) \rightarrow \begin{array}{l} q = -p \\ k' = k + p \end{array}$$

$$= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \tilde{A}^a_\mu(p) \tilde{A}^b_\nu(-p) \, \mathrm{tr}(t^a t^b) \cdot \int \frac{d^d k}{(2\pi)^d} \frac{4 (p^\rho \delta^{\mu\sigma} - p^\sigma \delta^{\mu\rho})(p_\rho \delta^\nu_\sigma - p_\sigma \delta^\nu_\rho)}{k^2 (k+p)^2}$$

$$= 4\int \frac{d^dp}{(2\pi)^d}\, \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(-p)\, \mathrm{tr}(t^a t^b)\, (p^\mu p^\nu - p^2 \delta^{\mu\nu}) \int_0^1 d\xi\, \frac{\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-\frac{d}{2}}(4\pi)^{d/2}}$$

$$\underset{\text{div.}}{\sim}\quad \frac{4\; C(\mathrm{Adj})\,\delta^{ab}}{(4\pi)^2}\,\frac{1}{2-\omega} \int \frac{d^dp}{(2\pi)^d}\, \hat{A}^a_\mu(p)\, \hat{A}^b_\nu(-p)\, (p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\Downarrow$$

$$\frac{1}{2}\left(4\,\mathrm{Tr}\log\Delta_{gh} + F^{\mu\nu}\text{-terms}\right)$$

$$S_{eff}(A) = \underbrace{\frac{1}{2g^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})}_{\frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^{a\mu\nu}} + \overbrace{\frac{1}{2}\mathrm{Tr}\log\Delta_{gauge}}^{=} - \mathrm{Tr}\log\Delta_{gh}$$

$$\underbrace{\frac{1}{2}\mathrm{Tr}\log\Delta_{gauge} - \mathrm{Tr}\log\Delta_{gh}}_{\frac{C(\mathrm{Adj})}{(4\pi)^2}\left[\frac{1}{2}\cdot 4\left(-\frac{1}{6}\right) + \frac{1}{2}\cdot 4 + \frac{1}{6}\right]\frac{1}{2-\omega}\int \frac{d^4p}{(2\pi)^4}\tilde{A}^a_\mu(p)\tilde{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})}$$

$$\underbrace{\frac{1}{2}\cdot 4\left(-\frac{1}{6}\right) + \frac{1}{2}\cdot 4 + \frac{1}{6}}_{-\frac{2}{6}+\frac{1}{6}+2 = \frac{11}{6}}$$

$$-\frac{1}{2g^2_{bare}}\int \frac{d^4p}{(2\pi)^4}\, \tilde{A}^a_\mu(p)\tilde{A}^a_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

RINORMALIZZAZIONE

$$\Downarrow$$

$$-\frac{\mu^{4-2\omega}}{2g^2_{bare}} + \frac{1}{2}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{11}{3}\,\frac{1}{2-\omega} = -\frac{1}{2g_r^2(\mu)}$$

$$\frac{(\mu^{2-\omega})^2}{g^2_{bare}} = \frac{1}{g_r^2} + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega} = \frac{1}{g_r^2}\left(1 + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2\right)$$

$$\to\quad g_{bare} = g_r(\mu)\,\mu^{2-\omega}\left(1 - \frac{11}{6}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2(\mu)\right)$$

$$\beta(g) = -\frac{11}{3}\,\frac{C_2(G)}{16\pi^2}\,g^3 \qquad \beta\text{-funct di YM}$$

Contributo fermioni

Se abbiamo un fermione in rep. R di G, abbiamo

$$e^{-S_{eff}(A)} = \int e^{-S(A,c,\bar{c},a)} \mathcal{D}a\mathcal{D}c\mathcal{D}\bar{c}\ \underbrace{e^{-S_f(\psi,A,a)}\mathcal{D}\psi\mathcal{D}\bar{\psi}}_{\det i\not{D}}$$

$$\not{D} = \gamma^\mu D_\mu$$

→ contributo a S_{eff} pari a

$$-\log\det(i\not{D}) \qquad \left(\det M = \sqrt{\det(M^2)}\right)$$

$$\det(i\not{D}) = \det{}^{1/2}(-\gamma^\mu\gamma^\nu D_\mu D_\nu) =$$

$$= \det{}^{1/2}\Big(-\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2\delta^{\mu\nu}\mathbb{1}_4}D_\mu D_\nu - \frac{1}{2}[\gamma^\mu,\gamma^\nu]\overbrace{D_\mu D_\nu}^{\frac{1}{2}[D_\mu,D_\nu] = -\frac{i}{2}F_{\mu\nu}}\Big) =$$

$$= \det{}^{1/2}\left(-D^2\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)$$

Se ho N_f fermioni devo elevare questo det alla N_f

anche su indici spinoriali

$$\Rightarrow -\log\det(i\not{D}) = -\frac{1}{2}\mathrm{Tr}\log\Big(\underbrace{-D^2\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}\Big)$$

$(\dots)$

N_f compare come fattore

$$= -\frac{1}{2}\mathrm{Tr}\log\Big(-\partial^2\mathbb{1}_4 + \overbrace{(\Delta_1+\Delta_2)\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}^{(\dots)}\Big) =$$

espando log

$$= -\frac{1}{2}\mathrm{Tr}\cancel{\log}(-\partial^2\mathbb{1}_4) - \frac{1}{2}\mathrm{Tr}\log\left(1 + (-\partial^2)^{-1}(\dots)\right) \overset{\simeq}{=}$$

cost. che trascuriamo

$$\simeq -\frac{1}{2}\mathrm{Tr}\left((-\partial^2)^{-1}(\dots)\right) - \frac{1}{2}\left(-\frac{1}{2}\mathrm{Tr}\left[\left((-\partial^2)^{-1}(\dots)\right)^2\right]\right)$$

termini Δ_2 e Δ_1 ricostr. $\mathrm{Tr}\log(-D^2)\mathbb{1}_4$

$\mathrm{tr}\, t_R^a = 0 \Rightarrow$
$\Rightarrow F_{\mu\nu}$ non compare

termini misti $F_{\mu\nu}\Delta_1$ fanno zero perché $\mathrm{Tr}[\gamma^\mu,\gamma^\nu] = 0$ (cicl. tr.)

$$= \underbrace{-\overset{N_f}{\frac{1}{2}}\mathrm{Tr}\log(-D^2\mathbb{1}_4)}_{-2\mathrm{Tr}\log(-D^2)} - \overset{N_f}{\frac{1}{2}}\Big\{\underbrace{-\frac{1}{2}\left(\frac{i}{4}\right)^2\mathrm{Tr}\left((-\partial^2)^{-1}F^a_{\mu\nu}t^a_R[\gamma^\mu,\gamma^\nu](-\partial^2)^{-1}F^b_{\rho\sigma}t^b_R[\gamma^\rho,\gamma^\sigma]\right)}_{F_{\mu\nu}-\text{terms}}\Big\}$$

$$\{\cdots\} = -\frac{1}{2}\left(-\frac{1}{16}\right)\int d^dy\, d^dx\, \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^a_{\mu\nu}(x)\, F^b_{\rho\sigma}(y)\,\cdot$$

$$\cdot\, \mathrm{tr}\, t^a_R t^b_R \quad \underbrace{\mathrm{tr}_f\left([\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right)}_{-16\cdot 2\,\delta^{\mu\rho}\delta^{\nu\sigma}}$$

$$\mathrm{tr}_f\left([\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right) = \mathrm{tr}_f\, \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + (\mu\leftrightarrow\nu,\rho\leftrightarrow\sigma)$$

$$= 4\left(\delta^{\mu\nu}\delta^{\rho\sigma} \underbrace{- \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}}_{\text{antisim. in } \mu\nu \text{ e } \rho\sigma\text{, ma simm. in } \mu\nu\ \rho\sigma \text{ simultaneamente}}\right) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + (\mu\leftrightarrow\nu,\rho\leftrightarrow\sigma)$$

$$= 4\,\delta^{\mu\nu}\delta^{\rho\sigma}(1-1-1+1) + 4\cdot 4\left(-\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right)$$

$$= 16\left(-\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}\right)$$

$$= -\int d^dy\, d^dx\, \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^a_{\mu\nu}(x) F^{b\,\mu\nu}(y)\, \underbrace{\mathrm{tr}\, t^a_R t^b_R}_{C(R)\,\delta^{ab}}$$

Da calcolo di $\Delta^{\mu\nu}_{gauge}$, sappiamo che

$$-\cancel{2}\int d^dy\, d^dx\, \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^{\mu\nu a}(x)\, F^b_{\mu\nu}(y)\, \cancel{\mathrm{tr}(t^a_{Ad} t^b_{Ad})}$$

$$= \overset{2}{\cancel{4}}\int \frac{d^dp}{(2\pi)^d}\, \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(-p)\, \cancel{\mathrm{tr}(t^a t^b)}\, (p^\mu p^\nu - p^2\delta^{\mu\nu}) \int_0^1 d\xi\, \frac{\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-\frac{d}{2}}(4\pi)^{d/2}}$$

$$\underset{div.}{\sim} \frac{2\cancel{4}\, \cancel{C(Ad)\,\delta^{ab}}}{(4\pi)^2}\, \frac{1}{2-\omega} \int \frac{d^dp}{(2\pi)^d}\, \hat{A}^a_\mu(p)\, \hat{A}^b_\nu(-p)\, (p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\underset{div.}{\sim} 2\, C(R)\, \frac{1}{(4\pi)^2}\, \frac{1}{2-\omega} \int \frac{d^dp}{(2\pi)^d}\, \hat{A}^a_\mu(p)\, \hat{A}^a_\nu(-p)\, (p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\Rightarrow\ -\mathrm{Tr}\log i\not{D} = -2\,\mathrm{Tr}\log(-D^2) - \frac{1}{2}\{\cdots\} =$$

$$= -\cancel{2}\left(-\frac{1}{\cancel{6}\,3}\right)\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$-\frac{1}{\cancel{2}}\left(\overset{1}{\cancel{2}}\right)\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$= -\frac{2}{3}\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\,N_f\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

Quindi ora

$$-\frac{\mu^{4-2\omega}}{2g^2_{bare}} + \frac{1}{2}\left(\frac{C_2(G)}{(4\pi)^2}\frac{11}{3}\frac{1}{2-\omega} - \frac{4}{3}\frac{c(R)}{(4\pi)^2}\frac{N_f}{2-\omega}\right) = -\frac{1}{2g_r^2(\mu)}$$

$$\Rightarrow \boxed{\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}N_f\,c(R)\right)}$$

Funzione β per teorie non-abeliane con gruppo di gauge G e accoppiate a N_f fermioni di Dirac in rep. R di G.