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FOGLIO 3

① UNA MATRICE QUADRATA $A=(a_{ij})_{i,j=1}^{n} \in M(n\times n,\mathbb{R})$ È:

- SIMMETRICA se $\forall i,j \quad a_{i,j}=a_{j,i}$
- ANTISIMMETRICA se $\forall i,j \quad a_{i,j}=-a_{j,i}$

(i) $Sym(n\times n,\mathbb{R})$ SOTTOSPAZIO VETTORIALE? BASE? DIMENSIONE?

- SOTTOSPAZIO?

A, B simmetriche, $\lambda,\mu\in\mathbb{R} \Rightarrow \lambda A+\mu B$ simmetrica?

$$(\lambda A+\mu B)_{ij}=(\lambda a_{ij}+\mu b_{ij})=\lambda a_{ji}+\mu b_{ji}=(\lambda A+\mu B)_{ji} \checkmark.$$

- BASE?

CASO 3x3

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33}\end{pmatrix}=a_{11}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+a_{12}\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}+$$

$$+a_{13}\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}+a_{22}\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}+a_{23}\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}+$$

$$+a_{33}\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}.$$

$\rightarrow$ QUESTE MATRICI SONO LA BASE DI $Sym(3\times 3,\mathbb{R})$.

$\rightarrow$ $n\times n$

BASE: $\{V^{i,j}\}_{\substack{i\leq j\\ i,j=1,\dots,n}}$ le matrici con 1 ai posti i,j e j,i e 0 in tutte le altre posizioni.

se $\{E^{i,j}\}$ è la base canonica di $M(n\times n, \mathbb{R})$,

$V^{i,j} = E^{i,j} = E^{j,i} \quad \forall i \neq j,\ W^{i,i} = E^{i,i}$. ($E^{i,j}$ HA TUTTI ZERI TRANNE 1 IN POSIZIONE i,j)

- DIMENSIONE?

- $\left| \{V^{i,j}\}_{i\leq j} \right| = \sum_{i=1}^{n}\sum_{j=i}^{n} 1 = \sum_{i=1}^{n} i = \frac{n(n-1)}{2}$.

- $A = \left(\begin{array}{c|c} \begin{matrix} \text{MATRICE} \\ \text{SIMMETRICA} \\ (n-1)\times(n-1) \end{matrix} & \begin{matrix} a_{1,n} \\ \vdots \\ a_{n-1,n} \end{matrix} \\ \hline a_{1n} \ \text{—} \ a_{n-1,n} & a_{nn} \end{array}\right) \longrightarrow \dim(\mathrm{Sym}(n\times n, \mathbb{R})) = n + \dim(\mathrm{Sym}(n-1\times n-1, \mathbb{R})) = \sum_{j=1}^{n} j.$

$(\mathrm{Sym}(1\times 1, \mathbb{R}) = \mathbb{R}) \longrightarrow \dim 1.$

- $\boxed{\sum_{j=1}^{n} j = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.}$

- $n=1 \longrightarrow \sum_{j=1}^{1} j = 1 = \frac{1\cdot\cancel{(1+1)}}{\cancel{2}}$ ✓.

- $n-1 \longrightarrow n \longrightarrow \sum_{j=1}^{n} j = \sum_{j=1}^{n-1} j + n = \frac{(n-1)\cdot n}{2} + n = \frac{n(n+1)}{2}$ ✓.

$\Longrightarrow \dim(\mathrm{Sym}(n\times n, \mathbb{R})) = \frac{n^2}{2} + \frac{n}{2}.$

(ii) STESSA COSA PER $\mathrm{Alt}(n\times n, \mathbb{R})$.

- SOTTOSPAZIO?

$A, B \in \mathrm{Alt}(n\times n, \mathbb{R}),\ \lambda, \mu \in \mathbb{R} \Rightarrow \lambda A + \mu B \in \mathrm{Alt}(n\times n, \mathbb{R}).$

$$(\lambda A+\mu B)_{i,j} = \lambda a_{ij} + \mu b_{ij} = -\lambda a_{ji} - \mu b_{ji} = -(\lambda A + \mu B)_{j,i} \ V.$$

• BASE?

$a_{ii} = -a_{ii} \longrightarrow a_{ii} = 0.$

3×3

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 0 & a_{23} \\ a_{13} & a_{23} & 0 \end{pmatrix} = a_{12}\boxed{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} + a_{13}\boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}} +$$

$$+ a_{23}\boxed{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}} \longrightarrow$$ QUESTE TRE MATRICI SARANNO LA BASE DI $Alt(3\times 3, \mathbb{R})$

• $n\times n$

$\{\hat{V}_{ij}\}_{\substack{i<j \\ i,j=1,\dots,n}}$ con 1 al posto i,j, con -1 al posto j,i e 0 in tutti gli altri posti.

$(\{E_{ij}\}$ base canonica $\Longrightarrow \hat{V}_{ij} = E_{ij} - E_{ji})$

DIMENSIONE?

$$A = \left(\begin{array}{c|c} \boxed{\begin{matrix}\text{MATRICE} \\ \text{ANTISIMMETRICA} \\ (n-1)\times(n-1)\end{matrix}} & \begin{matrix} a_{1n} \\ \vdots \\ a_{n-1,n} \end{matrix} \\ \hline -a_{1n} \quad \cdots \quad -a_{n-1,n} & 0 \end{array}\right) \longrightarrow$$

$\dim(Alt(n\times n, \mathbb{R})) =$

$= \dim(Alt(n-1\times n-1, \mathbb{R})) + (n-1) =$

$= \dim(Alt((n-2)\times(n-2), \mathbb{R})) + (n-2) + (n-1) = \underline{\text{(RIPETIAMO)}} =$

$= \underline{\dim(Alt(1\times 1, \mathbb{R}))} + 1 + \_\_\_ + n-1.$ ⊛

$M(1\times 1, \mathbb{R}) = \mathbb{R}$ $\quad (a_{11}) \longrightarrow \quad a_{11} = -a_{11}$

$\Longrightarrow a_{11} = 0$

$\Longrightarrow Alt(1\times 1, \mathbb{R}) = \{0\} \Longrightarrow \dim = 0.$

$$\sum^{n-1} \ldots (n-1)(n-1+1)$$

$$\Rightarrow \otimes \Rightarrow \dim(\mathrm{Alt}(n\times n,\mathbb{R})) = \sum_{j=1}^{n-1} j = \frac{(n-\cancel{1})(n-\cancel{1}+\cancel{1})}{2} =$$

$$= \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}\,. \checkmark\checkmark$$

(iii) $\underline{\mathrm{Sym} \oplus \mathrm{Alt}}$?

$W := \mathrm{Sym} \cap \mathrm{Alt} = \{\underline{0}\}$?

$A \in W \quad A = (a_{ij}) \xrightarrow{\forall i,j} a_{ij} = -a_{ji} = -a_{ij} \Rightarrow a_{ij} = 0$

$\Rightarrow A = 0 \checkmark$.

- $\underbrace{\mathrm{Sym} \oplus \mathrm{Alt}}\ -\ \underbrace{M(n\times n,\mathbb{R})}_{n^2}$?

$$\boxed{\left(\frac{n^2}{2}+\frac{n}{2}\right) + \left(\frac{n^2}{2}-\frac{n}{2}\right)} \longrightarrow$$

$$\| \qquad \frac{n^2}{2} + \frac{n^2}{2} = n^2$$

stessa dimensione (finita!),

$\mathrm{Sym} \oplus \mathrm{Alt} \subseteq M(n\times n,\mathbb{R})$

$\Rightarrow \mathrm{Sym} \oplus \mathrm{Alt} = M(n\times n,\mathbb{R}). \checkmark$