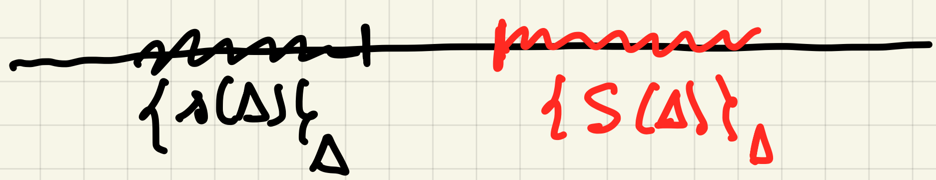
19 Novembre

<u>Corollario</u> Data $f:[a,b]\to\mathbb{R}$, limitata (cioè $\exists\ m\le M$ in $\mathbb{R}$ t.c. $m\le f(x)\le M$ $\forall\, x\in[a,b]$) e date due decomposizioni qualsiasi Δ e Δ', si ha $s(\Delta)\le S(\Delta')$.

Il Corollario può essere equivalentemente espresso dicendo che $\{s(\Delta)\}_\Delta$ e $\{S(\Delta)\}_\Delta$ sono una coppia di classi separate



$$\sup\{s(\Delta)\}_\Delta \le S(\Delta') \quad \forall\ \Delta'$$

$$s(\Delta)\le \inf\{S(\Delta')\}_{\Delta'}$$

In particolare

$$\underline{\int_a^b} f(x)\,dx := \sup\{s(\Delta)\}_\Delta \le \inf\{S(\Delta')\}_{\Delta'} =: \overline{\int_a^b} f(x)\,dx$$

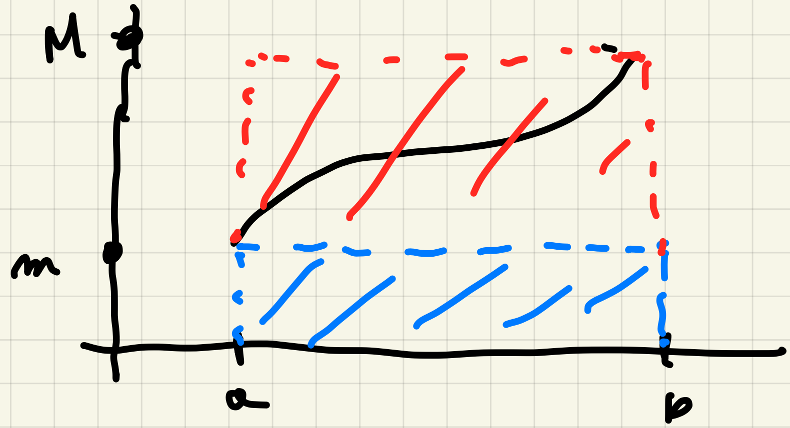
Poniamo

$$\begin{array}{l} \underline{\int_a^b} f(x)\,dx := \sup\{s(\Delta)\}_\Delta \\ \qquad\quad \rotatebox{-90}{$\le$} \\ \overline{\int_a^b} f(x)\,dx := \inf\{S(\Delta')\}_{\Delta'} \end{array}$$

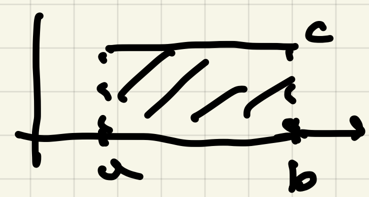
$$\underline{\int_a^b} f(x)\,dx := \sup\{s(\Delta)\}_{\Delta} \le \inf\{S(\Delta')\}_{\Delta'} =: \overline{\int_a^b} f(x)\,dx$$

Osservazioni

1) Se m, M in $\mathbb{R}$ sono tali che $m \le f(x) \le M$ $\forall x$ in $[a,b]$, abbiamo visto che
$$m(b-a) \le s(\Delta) \le S(\Delta) \le M(b-a) \quad \forall \Delta.$$



2) $f(x) \equiv c$, $m = M = c$



$$c(b-a) \underset{=}{\le} s(\Delta) \underset{=}{\le} S(\Delta) \underset{=}{\le} c(b-a) \quad \forall \Delta$$

$$\underline{\int_a^b} c\,dx = \sup\{s(\Delta)\} = \sup\{c(b-a)\}$$
$$= c(b-a)$$

$$\overline{\int_a^b} c\,dx = \inf\{S(\Delta)\}_{\Delta} = \inf\{c(b-a)\}$$
$$= c(b-a)$$

$$\overline{\int_a^b} c\,dx = \underline{\int_a^b} c\,dx = c(b-a)$$

3) $$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \qquad [a,b]$$

$$s(\Delta) = 0 \qquad \forall \Delta$$

$$S(\Delta) = b-a \qquad \forall \Delta$$

$$\underline{\int_a^b} D(x)\,dx = \sup\{s(\Delta)\}_\Delta = \sup\{0\} = 0$$

$$\overline{\int_a^b} D(x)\,dx = \inf\{S(\Delta)\}_\Delta = \inf\{b-a\} = b-a$$

$$\underline{\int_a^b} D(x)\,dx = 0 < \overline{\int_a^b} D(x)\,dx = b-a$$

<u>Def</u> Data $f: [a,b] \to \mathbb{R}$, diciamo che è integrabile per Darboux, se

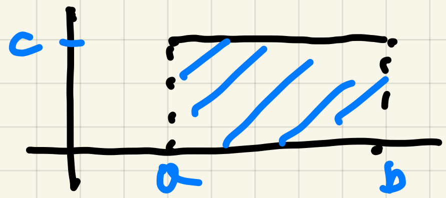
$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

Allora chiamiamo integrale di Darboux di $f(x)$, che denotiamo con $\int_a^b f(x)\,dx$, il comune valore dell'integrale superiore e dell'integrale inferiore

$$\boxed{\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx}$$

Esempi

1) $\int_a^b c\,dx = \underline{\int_a^b} c\,dx = \overline{\int_a^b} c\,dx = c(b-a)$



2) $\int_a^b D(x)\,dx$ non esiste perché

$$\underline{\int_a^b} D(x)\,dx = 0 < \overline{\int_a^b} D(x)\,dx = b-a$$

Teor Sia $f: [a,b] \to \mathbb{R}$, f limitata. Sono equivalenti:

1) f è integrabile per Darboux.

2) $\forall \varepsilon > 0 \ \exists \Delta_\varepsilon$ t.c. $\quad 0 \leq S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon$.

Dim $1 \Rightarrow 2$. f sia integrabile e sia $\varepsilon > 0$

Allora, so che $\exists \Delta$ t.c.

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} \leq s(\Delta) \leq \underline{\int_a^b} f(x)\,dx$$

ed $\exists \Delta'$ t.c.

$$\overline{\int_a^b} f(x)\,dx \leq S(\Delta') < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$$

1) f è integrabile per Darboux.

2) $\forall \varepsilon > 0 \ \exists \ \Delta_\varepsilon$ t.c. $\quad 0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon.$

Dim $1 \Rightarrow 2$. f sia integrabile e sia $\varepsilon > 0$

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$$\overline{\int_a^b} f(x)\,dx \le S(\Delta') < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$$

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < s(\Delta) \le S(\Delta') < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

Sia ora Δ_ε un raffinamento di Δ e Δ'. Allora

$$s(\Delta) \le s(\Delta_\varepsilon) \le S(\Delta_\varepsilon) \le S(\Delta')$$

e quindi otteniamo

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < s(\Delta_\varepsilon) \le S(\Delta_\varepsilon) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$$

siccome per ipotesi f è integrabile, abbiamo

$$\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < s(\Delta_\varepsilon) \le S(\Delta_\varepsilon) < \int_a^b f(x)\,dx + \frac{\varepsilon}{2}$$

$0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon$. Conclusione:

f integrabile $\Rightarrow \forall \varepsilon > 0 \ \exists \ \Delta_\varepsilon$ t.c.

$$0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon.$$

1) f è integrabile per Darboux.

2) $\forall \varepsilon > 0 \ \exists \ \Delta_\varepsilon$ t.c. $\quad 0 \leq S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon.$

Ora 2) $\Rightarrow$ 1

Per ogni Δ valgono

$$s(\Delta) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq S(\Delta)$$

Siccome 2) è vera, preso $\varepsilon > 0$ arbitrario considero il Δ_ε della 2).

$$s(\Delta_\varepsilon) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq S(\Delta_\varepsilon)$$

$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx \leq S(\Delta_\varepsilon) - s(\Delta) < \varepsilon$$

Quindi concludiamo che

$$0 \leq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \varepsilon \qquad \forall \varepsilon > 0.$$

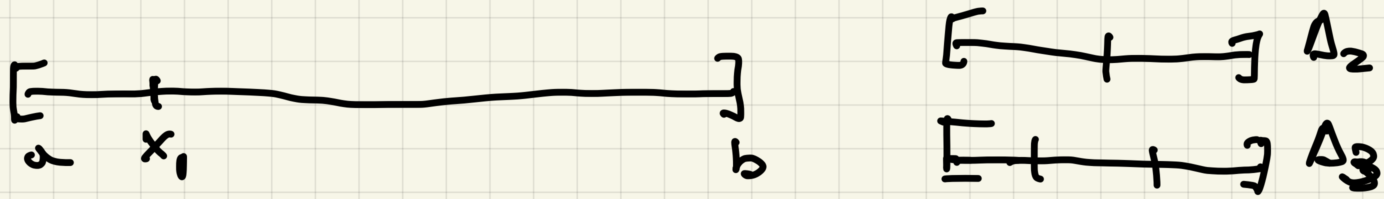
$$\Rightarrow \quad \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx = 0$$

f è integrabile per Darboux in $[a,b]$.

<u>Corollario</u> Se $f: [a,b] \to \mathbb{R}$ è monotona, allora è integrabile in $[a,b]$.

<u>Dim</u> sia f crescente. Applicheremo il criterio di

$$\forall \varepsilon > 0 \quad \exists\, \Delta_\varepsilon \text{ t.c. } \quad 0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon.$$



Per ogni $n \in \mathbb{N}$ suddividiamo $[a,b]$ in n intervalli di lunghezza $\frac{b-a}{n}$, Δ_n denoteranno le corrispondenti decomposizioni

$$\Delta_n \quad x_0 = a < x_1 = a + \frac{b-a}{n} < x_2 = a + 2\,\frac{b-a}{n} < \cdots <$$

$$< x_n = b = a + n\,\frac{b-a}{n}$$

$$x_j = a + j\,\frac{b-a}{n} \quad . \qquad f \text{ crescente}$$

$$S(\Delta_n) = \sum_{j=1}^{n} (x_j - x_{j-1})\, f(x_j)$$

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$$S(\Delta_n) - s(\Delta_n) = \frac{b-a}{n} \left(\sum_{j=1}^{n} f(x_j) - \sum_{j=1}^{n} f(x_{j-1}) \right)$$

$$= \frac{b-a}{n} \left(f(b) - f(a) \right)$$

$\forall n \in \mathbb{N}$

$$S(\Delta_n) - s(\Delta_n) = \frac{b-a}{n} \left(f(b) - f(a) \right) \xrightarrow{n \to +\infty} 0$$

$\forall \varepsilon > 0 \quad \exists\, n \quad$ t.c.

$$0 \le S(\Delta_n) - s(\Delta_n) = \frac{b-a}{n} \left(f(b) - f(a) \right) < \varepsilon$$

Abbiamo verificato che

$\forall \varepsilon > 0 \ \exists\, \Delta_\varepsilon \quad$ t.c.

$$0 \le S(\Delta_\varepsilon) - s(\Delta_\varepsilon) < \varepsilon$$

$\Delta_\varepsilon = \Delta_n \quad$ per $\quad n >> 1$.

Teor $f \in C^0([a,b]) \Rightarrow f$ è integrabile in $[a,b]$

$L[a,b] = \{ f : [a,b] \to \mathbb{R} ; f$ è integrabile per Darboux in $[a,b] \}$

Teor (Linearità dell'integrale)

1) Se $f, g \in L[a,b] \Rightarrow f+g \in L[a,b]$ e si ha

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

2) Se $f \in L[a,b]$ e $c \in \mathbb{R}$ allora $cf \in L[a,b]$

$$\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$$

(1) e 2) dicono che $L[a,b]$ è uno spazio vettoriale e che $\int_a^b : L[a,b] \longrightarrow \mathbb{R}$

$$f \longrightarrow \int_a^b f(x)\,dx$$

è un operatore lineare.)

3) Se $f, g \in L[a,b]$ allora anche $f \cdot g \in L[a,b]$.

(Nessun nesso, in generale, tra $\int_a^b f(x) g(x)\,dx$ e gli integrali $\int_a^b f(x)\,dx$ e $\int_a^b g(x)\,dx$).