23 Novembre

Teor u una soluzione di Leray t.c. $\exists\, T_1 \in \mathbb{R}_+$ t.c.

$*$ $u \in L^r((0,T_1), L^s(\mathbb{R}^3))$ $\quad \frac{2}{r} + \frac{3}{s} = 1$

$r \geq 2 \quad s > 3$

Allora $u \in C^\infty((0,T_1] \times \mathbb{R}^3, \mathbb{R}^3)$

E se v è un'altra soluzione analoga con anche altra coppia (r,s), allora $u = v$ in $[0,T]$.

Dim Iniziamo con $u_0 \in V$. Supponiamo che $\exists\ T^* > 0$ t.c. $u \in L^\infty([0,T], V)$ $\forall\ T \in (0,T^*)$ con $u \in C^\infty((0,T^*) \times \mathbb{R}^3, \mathbb{R}^3)$

Si tratta di dimostrare che $T_1 < T^*$.

Se per assurdo questo è falso e che $T_1 \geq T^*$.

Ricordiamoci $|\nabla u(t)|_{L^2_x} \xrightarrow{t \nearrow T^*} +\infty$.

In $[0,T^*)$ vale

$$\frac{d}{dt} |\nabla u|^2_{L^2} + 2 |\Delta u|^2_{L^2} = 2 \langle \operatorname{div}(u \otimes u), \Delta u \rangle$$

In $[0,T^*)$ vale

$$\frac{d}{dt}|\nabla u|_{L^2}^2+2|\Delta u|_{L^2}^2=2\langle \overset{u\cdot\nabla u}{\mathrm{div}(u\otimes u)},\Delta u\rangle$$

$$\le c\,|u|_{L^s}|\nabla u|_{L^{\frac{2s}{s-2}}}|\Delta u|_{L^2}$$

$$\frac{1}{s}+\frac{s-2}{2s}+\frac{1}{2}=1 \qquad \le c\,|u|_{L^s}|\nabla u|_{L^2}^{\frac{s-3}{s}}|\nabla u|_{L^6}^{\frac{3}{s}}|\Delta u|_{L^2}$$

$$\frac{s-2}{2s}=\frac{\frac{s-3}{s}}{2}+\frac{\frac{3}{s}}{6}\implies |f|_{L^{\frac{2s}{s-2}}}\le |f|_{L^2}^{\frac{s-3}{s}}|f|_{L^6}^{\frac{3}{s}}$$

$$\dot{H}^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)\quad \frac{1}{6}=\frac{1}{2}-\frac{1}{3}$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2+2|\Delta u|_{L^2}^2\le c\,|u|_{L^s}|\nabla u|_{L^2}^{\frac{s-3}{s}}|\nabla u|_{L^6}^{\frac{3}{s}}|\Delta u|_{L^2}$$

$$\le c_1\,|u|_{L^s}|\nabla u|_{L^2}^{\frac{s-3}{s}}|\Delta u|_{L^2}^{\frac{s+3}{s}}$$

$$1=\frac{s-3}{2s}+\frac{s+3}{2s}$$

$$\le c_2\left(|u|_{L^s}|\nabla u|_{L^2}^{\frac{s-3}{s}}\right)^{\frac{2s}{s-3}}+|\Delta u|_{L^2}^2$$

$$\frac{d}{dt}|\nabla u|_{L^2}^2+\cancel{|\Delta u|_{L^2}^2}\le c_2\,|u|_{L^s}^{\frac{2s}{s-3}}|\nabla u|_{L^2}^2$$

$$|\nabla u(t)|_{L^2}^2\le|\nabla u(0)|_{L^2}^2\,e^{c_2\int_0^t|u|^{\overset{=r}{\frac{2s}{s-3}}}dt'}$$

$$\frac{2}{r}+\frac{3}{s}=1\qquad \frac{2}{r}=1-\frac{3}{s}=\frac{s-3}{s}\qquad \frac{1}{r}=\frac{s-3}{2s}$$

$$\frac{d}{dt}|\nabla u|^2_{L^2} + \cancel{|\Delta u|^2_{L^2}} \leq C_2 \, |u|^{\frac{2s}{s-3}}_{L^s} \, |\nabla u|^2_{L^2}$$

$$|\nabla u(t)|^2_{L^2} \leq |\nabla u(0)|^2_{L^2} \, e^{C_2\int_0^t |u|^{\frac{2s}{s-3}}_{L^s} dt'} \qquad \frac{2s}{s-3} = r$$

$$\frac{2}{r} + \frac{3}{s} = 1 \qquad \frac{2}{r} = 1 - \frac{3}{s} = \frac{s-3}{s} \qquad \frac{1}{r} = \frac{s-3}{2s}$$

$$|\nabla u(t)|^2_{L^2} \leq |\nabla u(0)|^2_{L^2} \, e^{C_2 |u|^r_{L^r((0,T_1), L^s_x)}}$$

$$0 \leq t < T^* \leq T_1$$

Non c'è blow up per $t \nearrow T^*$. Assurdo

Passiamo all'unicità

u v $(0, T_1)$.

Si utilizza l'una come funzione test dell'altra

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle - \langle v, \partial_t u\rangle + \langle \mathrm{div}(v\otimes v), u\rangle\right) dt' = |u_0|^2_{L^2} - \langle u(t), v(t)\rangle$$

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle - \langle \partial_t v, u\rangle + \langle \mathrm{div}(u\otimes u), v\rangle\right) dt' = |u_0|^2_{L^2} - \langle u(t), v(t)\rangle$$

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle + \langle v, \partial_t u\rangle + \langle \mathrm{div}(u\otimes u), v\rangle\right) dt' = 0$$

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle - \langle v, \partial_t u\rangle + \langle \mathrm{div}(v\otimes v), u\rangle \right) dt' = |u_0|_{L^2}^2 - \langle u(t), v(t)\rangle$$

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle - \langle \partial_t v, u\rangle + \langle \mathrm{div}(u\otimes u), v\rangle \right) dt' = |u_0|_{L^2}^2 - \langle u(t), v(t)\rangle$$

$$\int_0^t \left(\langle \nabla u, \nabla v\rangle + \langle v, \partial_t u\rangle + \langle \mathrm{div}(u\otimes u), v\rangle \right) dt' = 0$$

$$-\int_0^t \langle \partial_t v, u\rangle \, dt' = -\lim_{\varepsilon\to 0^+} \int_\varepsilon^t \langle \partial_t v, u\rangle \, dt =$$

$$= \lim_{\varepsilon\to 0^+} \left[\int_\varepsilon^t \langle \partial_t u, v\rangle - \langle v(t), u(t)\rangle + \langle v(\varepsilon), u(\varepsilon)\rangle \right]$$

$$= \int_0^t \langle \partial_t u, v\rangle \, dt - \langle v(t), u(t)\rangle + |u_0|_{L^2_x}^2$$

$$u(\varepsilon) \xrightarrow{\varepsilon\to 0^+} u_0 \quad \text{in } L^2_x$$

$$v(\varepsilon) \xrightarrow[\varepsilon\to 0^+]{} v_0$$

$$\int_0^t \left(\langle \nabla u, \nabla v \rangle - \langle v, \partial_t u \rangle + \langle \mathrm{div}(v \otimes v), u \rangle \right) dt' = |u_0|_{L^2}^2 - \langle u(t), v(t) \rangle$$

$$\int_0^t \left(\langle \nabla u, \nabla v \rangle + \langle v, \partial_t u \rangle + \langle \mathrm{div}(u \otimes u), v \rangle \right) dt' = 0$$

$$w = v - u$$

$$\frac{1}{2} |w(t)|_{L^2}^2 + \int_0^t \left(|\nabla w|_{L^2}^2 - \langle \mathrm{div}(w \otimes w), u \rangle \right) dt'$$

$$= \frac{1}{2} |u(t)|_{L^2}^2 + \int_0^t |\nabla u|_{L^2}^2 - \frac{1}{2} |u_0|_{L^2}^2 \quad (\le 0)$$

$$+ \frac{1}{2} |v(t)|_{L^2}^2 + \int_0^t |\nabla u|_{L^2}^2 - \frac{1}{2} |v_0|_{L^2}^2 \quad (\le 0)$$

$$\le 0$$

$$|w(t)|_{L^2}^2 + 2 \int_0^t |\nabla w|_{L^2}^2 \, dt' \le 2 \int_0^t \langle \overbrace{\mathrm{div}(w \otimes w)}^{w \cdot \nabla w}, u \rangle \, dt'$$

$$\le 2 \, |u|_{L^s} |w|_{L^{\frac{2s}{s-2}}} |\nabla w|_{L^2}$$

$$1 = \frac{1}{s} + \frac{s-2}{2s} + \frac{1}{2}$$

$$\frac{s-2}{2s} = \frac{1}{2} - \frac{k}{3} \qquad \frac{s-2}{2s} - \frac{1}{2} = -\frac{k}{3}$$

$$\frac{\cancel{s} - \cancel{2} - \cancel{s}}{\cancel{2} s} = -\frac{k}{3}$$

$$k = \frac{3}{s}$$

$$|w|_{L^{\frac{2s}{s-2}}} \le C |w|_{\dot{H}^{\frac{3}{s}}} \le C \, |\nabla w|_{L^2}^{\frac{3}{s}} \, |w|_{L^2}^{1 - \frac{3}{s}}$$

$$|w(t)|^2_{L^2} + 2\int_0^t |\nabla w|^2_{L^2}\,dt' \leq 2\int_0^t \langle \operatorname{div}(\overset{w\cdot\nabla w}{w\otimes w}), u\rangle\,dt'$$

$$\leq 2\int_0^t |u|_{L^s}\, |w|_{L^{\frac{2s}{s-2}}}\, |\nabla w|_{L^2}\,dt'$$

$$\leq C\int_0^t |u|_{L^s}\, |w|_{L^2}^{\frac{s-3}{s}}\, |\nabla w|_{L^2}^{\frac{s+3}{s}}\,dt'$$

$$1 = \underbrace{\frac{s-3}{2s}}_{\frac{1}{r}} + \frac{s+3}{2s}$$

$$\leq C_1\int_0^t |u|^r_{L^s}\, |w|^2_{L^2}\,dt' + \cancel{\int_0^t |\nabla w|^2_{L^2}\,dt}$$

$$|w(t)|^2_{L^2} \leq C\int_0^t |u|^r_{L^s_x}\, |w|^2_{L^2_x}\,dt'$$

Gronwell $\Longrightarrow$ $|w(t)|^2_{L^2_x} \equiv 0$

$\forall \quad 0 \leq t < T_1$

Vorticità

Lemma Data $f \in \mathcal{S}(\mathbb{R}^3)$ e sia $u \in \mathcal{S}'(\mathbb{R}^3)$ soluzione di $-\Delta u = f$. Allora

$$u = K * f + h \qquad K(x) = \frac{1}{4\pi |x|}$$

$h(x)$ è un polinomio armonico

$$u \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$$

$$\omega = \nabla \times u$$

Lemma (Biot-Savart) $u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3)$ $1<p<3$ con $\operatorname{div} u = 0$ allora

$$u = T\omega = -\frac{1}{4\pi} \int_{\mathbb{R}} \frac{x-y}{|x-y|^3} \times \omega(y)\, dy$$

<u>Dim</u> Cominciamo con $\omega \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$$\frac{1}{|x|} * \partial_j \omega_k = \partial_j \frac{1}{|x|} * \omega_k$$

$$\boxed{-\Delta u = \nabla \times \omega} = \epsilon_{ijk} \partial_j \omega_k \vec{e}_i$$

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)$$

$$\boxed{-\Delta u_i = \epsilon_{ijk} \partial_j \omega_k}$$

$$\boxed{\frac{1}{|x|} * \partial_j w_k = \partial_j \frac{1}{|x|} * w_k}$$

$$\boxed{-\Delta u = \nabla \times w} = \epsilon_{ijk} \partial_j w_k \vec{e}_i$$

$$\Delta u = \nabla(\nabla \cdot u) - \nabla \times (\nabla \times u)$$

$$\boxed{-\Delta u_i = \epsilon_{ijk} \partial_j w_k}$$

$$u_i = \epsilon_{ijk} \frac{1}{4\pi} \quad \frac{1}{|x|} * \partial_j w_k + h_i =$$

$$u = -\vec{e}_i \, \epsilon_{ijk} \frac{1}{4\pi} \quad \frac{x_j}{|x|^3} * w_k + h$$

$$= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times w(y)\, dy + h$$

$$u = Tw + h$$

$$u \in W^{1,p}(\mathbb{R}^3, \mathbb{R}^3), \quad w \in L^p(\mathbb{R}^3, \mathbb{R}^3)$$

$T: L^p(\mathbb{R}^3, \mathbb{R}^3) \to L^q(\mathbb{R}^3, \mathbb{R}^3)$ per Hardy-Littlewood-Sobolev

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{3}$$

$$Tw \in L^q(\mathbb{R}^3, \mathbb{R}^3), \quad u \in L^q(\mathbb{R}^3, \mathbb{R}^3$$

$$\Longrightarrow h \in L^q(\mathbb{R}^3, \mathbb{R}^3) \Rightarrow h \equiv 0$$

Caso $w \in L^p(\mathbb{R}^3, \mathbb{R}^3)$

$\tilde{w}_n \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^3)$

$\tilde{w}_n \longrightarrow w$ in $L^p(\mathbb{R}^3, \mathbb{R}^3)$

$u_n = T\tilde{w}_n \longrightarrow \tilde{u} = Tw$ in $L^q(\mathbb{R}^3, \mathbb{R}^3)$

E' facile verificare $\nabla \cdot u_n = 0$

Scopo è dimostrare che $\tilde{u} = u$

$\mathbb{P}$ operatore di Leray è di C - Z

$$\tilde{w}_n = \mathbb{P}\tilde{w}_n + (1-\mathbb{P})\tilde{w}_n = \tilde{w}_n^{(1)} + \tilde{w}_n^{(2)}$$

$\tilde{w}_n \to w$ in L^p dove $w = \nabla \times u$ è t.c. $\nabla \cdot w = 0$

$\Rightarrow$ $\tilde{w}_n^{(1)} \to w$ in L^p

$\tilde{w}_n^{(2)} \to 0$ in L^p

$$\nabla \times u_n = \tilde{w}_n^{(1)}$$

$$\nabla \times \tilde{u} = w$$

$u_n = Tw_n \longrightarrow \tilde{u}$ in $L^q(\mathbb{R}^3, \mathbb{R}^3)$

$\nabla \times u_n \longrightarrow \nabla \times \tilde{u}$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$

$\shortparallel$

$\tilde{w}_n^{(1)} \longrightarrow w$ in $L^p(\mathbb{R}^3, \mathbb{R}^3)$

$$\nabla \cdot u = 0 \qquad \nabla \cdot \tilde{u} = 0$$

$$\nabla \times u = \nabla \times \tilde{u} \implies u = \tilde{u}$$

$$\mathbb{P} v = - \Delta^{-1} \nabla \times (\nabla \times v)$$

$$\begin{cases} u = \mathbb{P} u = - \Delta^{-1} \nabla \times \omega \\ \tilde{u} = \mathbb{P} \tilde{u} = - \Delta^{-1} \nabla \times \omega \end{cases}$$

$$\nabla \times u_n = \omega^{(1)}_n \, .$$