# Teorema di integrabilità delle funzioni continue

$f:[a,b] \to \mathbb{R}$ continua. Allora $f$ è integrabile.

Dim

$f$ è uniformemente continua (Heine-Cantor) $\left[ \forall \sigma>0 \ \exists \eta>0 : \forall x_1, x_2 \in I \text{ se } |x_1-x_2|<\eta \text{ allora } |f(x_1)-f(x_2)|<\sigma \right]$

Si vuole dimostrare che $\forall \varepsilon>0$ $\exists$ una decomposizione $\mathcal{S}$ di $I$ tale che

$$S(f,\mathcal{S}) - s(f,\mathcal{S}) < \varepsilon$$

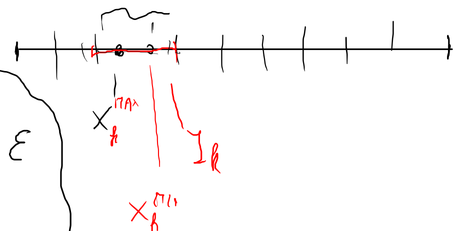
se $\forall k$ $m(I_k)<\eta$

$$= \sum_{k=1}^{n} \Big( \underbrace{\max_{x\in I_k} f(x)}_{= f(x_k^{MAX})} - \underbrace{\min_{x\in I_k} f(x)}_{= f(x_k^{MIN})} \Big) \cdot m(I_k) < \sum_{k=1}^{n} \sigma \cdot m(I_k) = \sigma \cdot m(I) < \varepsilon$$

$$\sigma < \frac{\varepsilon}{m(I)}$$

$$\underbrace{f(x_k^{MAX}) - f(x_k^{MIN})}_{<\sigma}$$

$$|x_k^{MAX} - x_k^{MIN}| \leq m(I_k) < \eta \Rightarrow |f(\ ) - f(\ )| < \sigma$$

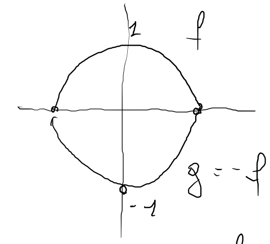


# Proprietà delle funzioni integrabili

## Linearità

Siano $f, g : [a,b] \to \mathbb{R}$ integrabili, $\alpha, \beta \in \mathbb{R}$, allora $\alpha f + \beta g$ è integrabile e si ha

$$\int_{[a,b]} (\alpha f + \beta g)(x)\, dm = \alpha \int_{[a,b]} f\, dm + \beta \int_{[a,b]} g\, dm.$$




**Dim** proviamo prima per $\alpha = \beta = 1$

OSS $f, g : [a,b] \to \mathbb{R} \Rightarrow \boxed{\inf f + \inf g \leq \inf(f+g)}$

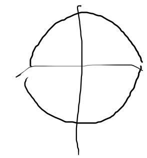
$f + g = 0$

$\inf f + \inf g = -1 < 0 = \inf(f+g)$

( infatti $\forall x$ $\inf f \leq f(x)$, $\inf g \leq g(x)$ $\Rightarrow \underbrace{\inf f + \inf g}_{\text{costante}} \leq [f(x) + g(x)]$

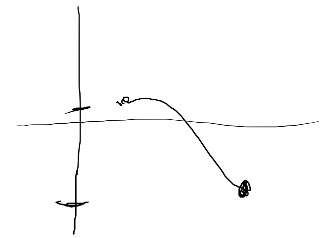
$\Rightarrow \inf f + \inf g \leq \inf(f(x) + g(x))$

$$\sup f + \sup g \geq \sup(f+g)$$

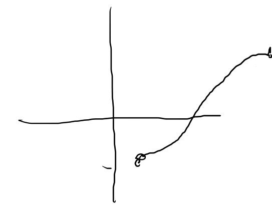


$-f$

$$\boxed{\begin{aligned} \inf(-f) &= -\sup(f) \\ \sup(-f) &= -\inf(f) \end{aligned}}$$



$-f$



$$f(x) \leq \sup(f) \quad \rightsquigarrow \quad -f(x) \geq -\sup(f)$$

$$\inf(-f(x)) \geq -\sup(f) \quad ..$$

$f, g$ integrabili $\Rightarrow$ $\underbrace{f+g \text{ integrabile}}$ e $\int_{[a,b]} (f+g)\, dm = \int_{[a,b]} f\, dm + \int_{[a,b]} g\, dm$

Dim

Osserviamo che

① $\inf_{I_k} f + \inf_{I_k} g \leq \inf_{I_k} (f+g)$ ② $\sup_{I_k} f + \sup_{I_k} g \geq \sup_{I_k} (f+g)$

Si vuole provare che $\forall \varepsilon > 0$ esiste una decomposizione $\mathcal{S}$ tale che $S(f+g, \mathcal{S}) - s(f+g, \mathcal{S}) < \varepsilon$

① $-s(f, \mathcal{S}) - s(g, \mathcal{S}) \geq -s(f+g, \mathcal{S})$ ② $S(f+g, \mathcal{S}) \leq S(f, \mathcal{S}) + S(g, \mathcal{S})$

$$S(f+g, \mathcal{S}) - s(f+g, \mathcal{S}) \leq \underbrace{S(f, \mathcal{S}) - s(f, \mathcal{S})}_{< \frac{\varepsilon}{2} \text{ per } \mathcal{S} \text{ opportuno}} + \underbrace{S(g, \mathcal{S}) - s(g, \mathcal{S})}_{< \frac{\varepsilon}{2} \text{ per } \mathcal{S} \text{ opportuno}} < \varepsilon$$

esiste $\delta_\varepsilon$ dec. tale che

$$\int_{[a,b]} f\,dm > S(f,\delta_\varepsilon) - \frac{1}{2}\varepsilon$$

$\inf + \frac{\varepsilon}{2}$

$$\int_{[a,b]} g\,dm > S(g,\delta_\varepsilon) - \frac{1}{2}\varepsilon$$

$$\Rightarrow \int_{[a,b]} (f+g)\,dm \le S(f+g,\delta) \le S(f,\delta) + S(g,\delta) <$$

$$< \int_{[a,b]} f\,dm + \frac{1}{2}\varepsilon + \int_{[a,b]} g\,dm + \frac{1}{2}\varepsilon = \int_{[a,b]} f\,dm + \int_{[a,b]} g\,dm + \varepsilon$$

per arbitrarietà di $\varepsilon > 0$ si conclude che $\int_{[a,b]} (f+g)\,dm \le \int_{[a,b]} f\,dm + \int_{[a,b]} g\,dm$

In modo simile $\int_{[a,b]} f\,dm = \sup\{ s(f,\delta),\ \delta \text{ decomp.} \}$

$\forall \frac{\varepsilon}{2} > 0 \quad \exists\, \delta_\varepsilon : \quad \int_{[a,b]} f\,dm - \frac{\varepsilon}{2} < s(f,\delta_\varepsilon)$

$-\frac{\varepsilon}{2}$

$$\int_{[a,b]} g\,dm - \frac{\varepsilon}{2} < s(g,\delta_\varepsilon)$$

$$\Rightarrow \int_{[a,b]} f\,dm + \int_{[a,b]} g\,dm < s(f,\delta_\varepsilon) + s(g,\delta_\varepsilon) + \varepsilon \le s(f+g,\delta_\varepsilon) + \varepsilon \le$$

$$\le \int_{[a,b]} (f+g)\,dm + \varepsilon$$

$$\Rightarrow \int_{[a,b]} f\,dm + \int_{[a,b]} g\,dm \le \int_{[a,b]} (f+g)\,dm$$

2° passo $\quad f$ integrabile $\Rightarrow$ $\boxed{-f \text{ integrabile}}$ $\quad \int_{[a,b]} -f \, dm = -\int_{[a,b]} f \, dm$

oss: $\inf(-f) = -\sup(f) \qquad \sup(-f) = -\inf(f)$

$$s(-f, S) = -S(f, S) \qquad S(-f, S) = -s(f, S)$$

Fisso $\varepsilon > 0$

$$S(-f, S) - s(-f, S) = \underline{-s(f, S) + S(f, S)} < \varepsilon$$

$$f + (-f) = 0 \qquad 0 = \int_{[a,b]} [f + (-f)] \, dm = \int_{[a,b]} f \, dm + \int_{[a,b]} -f \, dm$$

$$\Rightarrow \int_{[a,b]} -f \, dm = -\int_{[a,b]} f \, dm$$

3° passo $\boxed{\alpha > 0}$ $f$ integrabile $\Rightarrow$ $\alpha \cdot f$ è integrabile $\int_{[a,b]} \alpha f \, dm = \alpha \int_{[a,b]} f \, dm$

$\inf(\alpha f) = \alpha \inf f$ $\qquad s(\alpha f, \mathcal{S}) = \alpha \, s(f, \mathcal{S})$

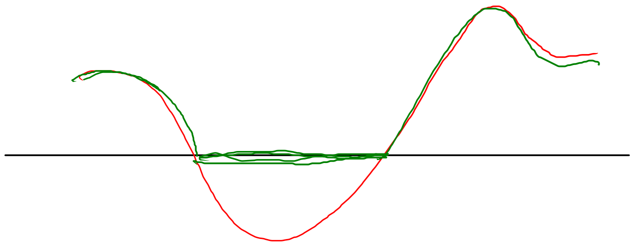
$\sup(\alpha f) = \alpha \sup f$ $\qquad S(\alpha f, \mathcal{S}) = \alpha \, S(f, \mathcal{S})$

$$S(\alpha f, \mathcal{S}) - s(\alpha f, \mathcal{S}) = \alpha \, [\underbrace{S(f, \mathcal{S}) - s(f, \mathcal{S})}_{< \varepsilon / \alpha}] < \varepsilon$$

Parte positiva / negativa di una funzione

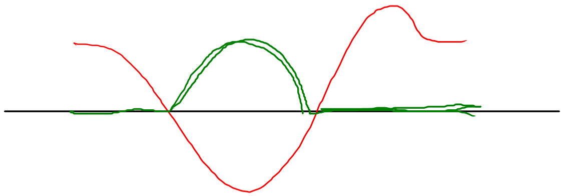
$f: E \subseteq \mathbb{R} \to \mathbb{R}$ si dice parte positiva di $f$ la funzione

$$f^+(x) = \begin{cases} f(x) & \text{se } f(x) \geq 0 \\ 0 & \text{se } f(x) < 0 \end{cases}$$



si dice parte negativa di $f$ la funzione

$$f^-(x) = \begin{cases} -f(x) & \text{se } f(x) \leq 0 \\ 0 & \text{se } f(x) > 0 \end{cases}$$



OSS: $f^+(x) \geq 0 \quad f^-(x) \geq 0 \quad \forall x \in E$

**OSS**

$$f(x) = f^+(x) - f^-(x)$$

$$|f(x)| = f^+(x) + f^-(x)$$

OSS

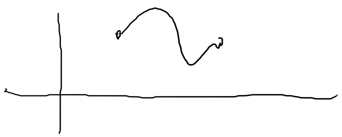
$$\max\{f(x), g(x)\} = \frac{1}{2}\left( |f(x) - g(x)| + f(x) + g(x) \right)$$

$$\min\{f(x), g(x)\} = \frac{1}{2}\left( f(x) + g(x) - |f(x) - g(x)| \right)$$

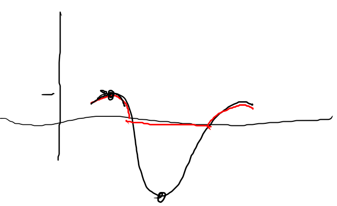
**Teorema** $f$ integrabile $\Rightarrow$ $f^+$ integrabile

$$S(f,\delta) - s(f,\delta) < \varepsilon$$

$$\underbrace{S(f^+,\delta) - s(f^+,\delta)}_{} = \sum_{k=1}^{n} \underbrace{\left[\sup_{I_k} f^+(x) - \inf_{I_k} f^+(x)\right]}_{(*)} m(I_k) \le \underbrace{\sum_{k=1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f\right) m(I_k)}_{}$$

(*) 

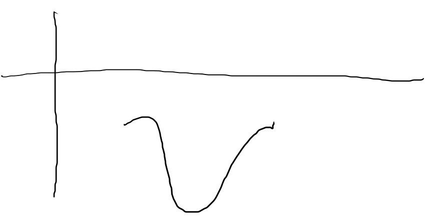
$$f^+(x) = f(x) \Rightarrow (*) = \sup_{I_k} f(x) - \inf_{I_k} f(x)$$



$$\sup f^+(x) = \sup f(x)$$

$$\inf f^+(x) = 0 > \inf f(x)$$

$$(*) \ \sup_{I_k} f(x) - 0 < \sup_{I_k} f - \inf_{I_k} f$$



$$f^+(x) = 0$$

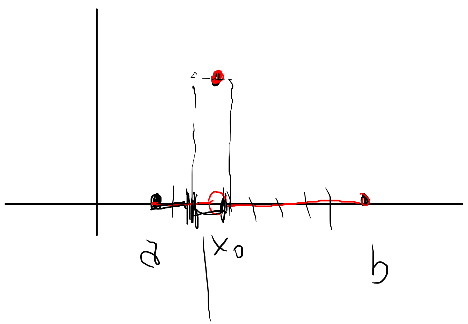
$$(*) \ 0 \le \sup_{I_k} f - \inf_{I_k} f$$

OSS $\quad f^-(x) = (-f)^+(x)$

quindi $f$ integrabile $\Rightarrow$ $f^-$ integrabile

$\Rightarrow$ $|f|$ integrabile, $\quad \begin{matrix}\max\\ \min\end{matrix}\{f, g\}$ integrabili, "Funzioni quasi ovunque nulle"

$f : [a,b] \to \mathbb{R}$, $\quad x_0 \in [a,b] \qquad f(x) = \begin{cases} 0 & \forall x \in [a,b] \setminus \{x_0\} \\ 1 & x = x_0 \end{cases}$




$f$ è integrabile e $\displaystyle\int_{[a,b]} f \, dm = 0$

$$\underline{\int}_{[a,b]} f \, dm = 0$$

$s(f, \mathcal{S}) = 0 \quad \forall \mathcal{S}$ $\qquad S(f, \mathcal{S}) = m(I_k) < \varepsilon$ se $\mathcal{S}$ è opportuna

dove $x_0 \in I_k$

oss

$$f(x) = \begin{cases} \alpha_1 & x = x_1 \\ \alpha_2 & x = x_2 \\ \vdots & \\ \alpha_p & x = x_p \\ 0 & \text{se } x \in [a,b] \setminus \{x_1, x_2, \dots, x_p\} \end{cases} \qquad \alpha_i \in \mathbb{R}$$

$$f = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_p f_p \qquad f_i(x) = \begin{cases} 1 & \text{se } x = x_i \\ 0 & \text{se } x \neq x_i \end{cases}$$

$$\int_{[a,b]} f \, dm = 0$$

**OSS** Sia $f : [a,b] \to \mathbb{R}$ integrabile

$g : [a,b] \to \mathbb{R}$

$$g(x) = \begin{cases} f(x) & \text{se } x \in [a,b] \setminus \{x_1, x_2, \dots, x_p\} \\ \text{qualsiasi} & \text{se } x \in \{x_1, x_2, \dots, x_p\} \end{cases}$$

Allora $g$ è integrabile e $\int_{[a,b]} g \, dm = \int_{[a,b]} f \, dm$

**Dim** considero $g - f$.

Questa osservazione mi permette di estendere la nozione di integrale di Riemann a insiemi più generali.

$]a,b[$.