Esercizio $f:(a,b)\to\mathbb{R}$ primitivabile, G_1 una primitiva ($G_1'(x)=f(x)\ \forall\, x\in(a,b)$). Valgono le seguenti prop.

1) Se $G_2(x)=G_1(x)+c$ dove $c\in\mathbb{R}$, allora anche G_2 è una primitiva di f in (a,b). [Ovvio perché

$$G_2'(x)=G_1'(x)+(c)'=f(x)\qquad \forall\, x\in(a,b)]$$

2) Se $G_3(x)$ è una primitiva di $f(x)$ in (a,b) allora $\exists\, c\in\mathbb{R}$ t.c. $G_3(x)=G_1(x)+c$. [$\left(G_3(x)-G_1(x)\right)'=G_3'(x)-G_1'(x)=$ $=f(x)-f(x)=0$ quindi G_3-G_1 è derivabile in (a,b) con derivata nulla. Per Lagrange G_3-G_1 deve essere $G_3-G_1\equiv c$ con $c\in\mathbb{R}$]

Teor (di valutazione) Se $f \in C^0([a,b])$ e se $G(x)$ e' una primitiva, allora $\int_a^b f(x)\,dx = G(b) - G(a)$

Dim Se pongo $F(x) = \int_a^x f(t)\,dt$, ho che

$F(a) = 0$ e $F(b) = \int_a^b f(t)\,dt$. $(= F(b) - F(a))$

Esiste una $c \in \mathbb{R}$ t.c. $G(x) = F(x) + c$

$$G(b) - G(a) = (F(b) + \cancel{c}) - (F(a) + \cancel{c}) = F(b) - F(a) =$$

$$= \int_a^b f(x)\,dx$$

$f(x)$	$\int f(x)\,dx$
$x^a \quad a \neq -1$	$\frac{x^{a+1}}{a+1} + C$
x^{-1}	$\lg \lvert x \rvert + C$
e^x	$e^x + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$

Proposizione Dato un intervallo I e date $f, g \in C^1(I)$, vale

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx \qquad (1)$$

Se $f, g \in C^1([a,b])$ allora vale

$$\int_a^b f'(x)\, g(x)\, dx = f(x)\, g(x) \Big]_a^b - \int_a^b f(x)\, g'(x)\, dx . \qquad (2)$$

Dim Assumiamo (1) e verifichiamo (2). Applicando il teor. di valutazione

$$\int_a^b f'(x)\, g(x)\, dx = \left(\int f'(x)\, g(x)\, dx \right) \Big]_a^b \overset{(1)}{=} \left(f(x)\, g(x) - \int f(x)\, g'(x)\, dx \right) \Big]_a^b =$$

$$= f(x)\, g(x) \Big]_a^b - \left(\int f(x)\, g'(x)\, dx \right) \Big]_a^b = f(x)\, g(x) - \int_a^b f(x)\, g'(x)\, dx$$

$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx \qquad (1)$$

$$(fg)' = f'g + fg' \iff f'(x)\,g(x) = \big(f(x)\,g(x)\big)' - f(x)\,g'(x)$$

$$\int f'(x)\,g(x)\,dx = \int \Big[\big(f(x)\,g(x)\big)' - f(x)\,g'(x)\Big]\,dx$$

$$\int f'(x)\,g(x)\,dx = \int \big(f(x)\,g(x)\big)'\,dx - \int f(x)\,g'(x)\,dx$$

$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx$$

$$\int x e^x \, dx = \int x (e^x)' \, dx = x e^x - \int \underline{\quad} e^x \, dx = x e^x - e^x + C$$

$$\int x^2 e^x \, dx = \int \cancel{x}^2 (e^x)' \, dx = x^2 e^x - 2 \int x e^x \, dx$$

$$\int x^n e^x \, dx = \int x^n (e^x)' \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$\int x \, 2^x \, dx = \int x \, e^{x \lg 2} \, dx = \int x \left(e^{x \lg 2}\right)' \frac{1}{\lg 2} \, dx =$$

$$= \frac{1}{\lg 2} \int x \left(e^{x \lg 2}\right)' dx = \frac{x \, 2^x}{\lg 2} - \frac{1}{\lg 2} \int e^{x \lg 2} \, dx$$

$$2 = e^{\lg 2}$$

$$= \frac{x \, 2^x}{\lg 2} - \frac{1}{\lg^2 2} \, 2^x + C$$

$$\int f' g = f g - \int f g'$$

$$\int x^n \sin x \, dx = \int x^n (-\cos x)' \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$\int e^x \sin x \, dx = \int (e^x)' \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \int (e^x)' \cos x \, dx = e^x \sin x - e^x \cos x + \int e^x (\cos x)' \, dx$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$$

$$\int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + C$$

$$\int e^{ax} \sin(bx) \, dx$$

$$\int \lg x \, dx = \int 1 \lg x \, dx = \int (x)' \lg x \, dx =$$

$$= x \lg x - \int x (\lg x)' dx = x \lg x - \int x \frac{1}{x} dx = x \lg x - \int 1 \, dx$$

$$= x \lg x - x + c \qquad\qquad (\lg x)' = \frac{1}{x}$$

$$\int (x^3 + x^2 + x) \lg x \, dx = \int \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \right)' \lg x \, dx =$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \right) \lg x - \int \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \right) \frac{1}{x} dx = \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \right) \lg x - \int \left(\frac{x^3}{4} + \frac{x^2}{3} + \frac{x}{2} \right) dx$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} \right) \lg x - \left(\frac{x^4}{16} + \frac{x^3}{9} + \frac{x^2}{4} \right) + c$$

$$\int \arctan x \, dx =$$

$$(\operatorname{artg} x)' = \frac{1}{1+x^2}$$

$$= \int 1 \arctan x \, dx = \int (x)' \arctan x \, dx =$$

$$= x \arctan x - \int x \, (\arctan x)' \, dx =$$

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \int \frac{(1+x^2)'}{1+x^2} \, dx$$

$$= x \arctan x - \frac{1}{2} \lg(1+x^2) + C$$

Calcolare i polinomi di McLaurin di $\arctan x$

$$\int t^{2j} = \frac{t^{2j+1}}{2j+1} + c$$

$$(\arctan(x))' = \frac{1}{1+x^2}$$

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \sum_{j=0}^{n} (-1)^j \int_0^x t^{2j}\,dt + \int_0^x o(t^{2n})\,dt$$

$$\frac{1}{1+t^2} = \sum_{j=0}^{n} (-1)^j t^{2j} + o(t^{2n})$$

$$= \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{2j+1} + \int_0^x o(t^{2n})\,dt$$

$$\int_0^x o(t^{2n})\,dt = o(x^{2n+1})$$

$$\frac{1}{1+y} = \sum_{j=0}^{n} (-1)^j y^j + o(y^n)$$

$$\lim_{x\to 0} \frac{\int_0^x o(t^{2n})\,dt}{x^{2n+1}}$$

$$\lim_{x\to 0} \int_0^x o(t^{2n})\,dt = \int_0^0 o(t^{2n})\,dt = 0$$

$$\lim_{x\to 0} \frac{\int_0^x o(t^{2n})\,dt}{x^{2n+1}} \overset{\text{Hop.}}{=} \lim_{x\to 0} \frac{o(x^{2n})}{(2n+1)x^{2n}} = 0$$

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x)$$