29 Nov.

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

Cambio di variabile

<u>Lemma</u> (int. indefinito) Siano I e J due intervalli $u : I \to J$ $u \in C^1(I)$ e sia $f : J \to \mathbb{R}$, $f \in C^0(J)$. Denotiamo con $\int f(u)\, du$ le primitive di $f(u)$. Vale allora la formula

$$\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du \right)(u(x))$$

Dobbiamo dimostrare $\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du\right)[u(x)]$

$$\frac{d}{dx} \int f(u(x))\, u'(x)\, dx = f(u(x))\, u'(x)$$

$$\frac{d}{dx} \left(\int f(u)\, du\right)[u(x)] = \left(\int f(u)\, du\right)'(u(x)) \quad u'(x) =$$

qui ho applicato la regola della catena

$$\frac{d}{dx} F(u(x)) = F'(u(x))\, u'(x)$$

$$= f(u(x)) \quad u'(x)$$

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

$$\int \frac{x}{1+x^2}\,dx$$

$$u = 1+x^2 \qquad \frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$= \int \frac{1}{u}\,\frac{du}{2} = \frac{1}{2}\int \frac{1}{u}\,du = = \frac{1}{2}\lg|u| + C$$

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\lg|u| + C = \frac{1}{2}\lg(1+x^2) + C$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx =$$

$u = \cos x$

$du = -\sin x \, dx$

$$= -\int \frac{1}{u} \, du = -\lg|u| + c$$

$$= -\lg|\cos x| + c$$

$$\int \cos^{2n+1}(x)\,\sin^m(x)\,dx =$$

$m, n \in \{0, 1, 2, \dots$

$\cos^2(x) = 1 - \sin^2(x)$

$$= \int \cos^{2n}(x)\,\sin^m(x)\,\cos x\,dx$$

$$= \int \left(1-\sin^2(x)\right)^n \sin^m(x)\,\underbrace{\cos x\,dx}_{du}$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int \left(1-u^2\right)^n u^m\,du$$

$$\int \cos^3 x\,\sin^2 x\,dx = \int \cos^2 x\,\sin^2 x\,\cos x\,dx =$$

$$= \int (1-\sin^2 x)\sin^2 x\,\cos x\,dx = \int (1-u^2)\,u^2\,du = \int (u^2-u^4)\,du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C \qquad u = \sin x$$

$$\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx =$$

$$= \int \cos^2 x \, (1-\cos^2 x) \sin x \, dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int u^2 (1-u^2) \, (-1) du =$$

$$= \int (u^4 - u^2) \, du = \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

con $u = \cos x$

$$\int \sin^{2n}(x) \cos^{2m}(x)\, dx = \qquad n, m \in \mathbb{Z}_{\geq 0}$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^n \left(\frac{1+\cos(2x)}{2}\right)^m dx$$

$$\int \sin^4(x)\cos^2(x)\,dx =$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \int \frac{(1-\cos(2x))^2}{4}\,\frac{1+\cos(2x)}{2}\,dx$$

$$= \frac{1}{8}\int (1-\cos(2x))\,(1-\cos^2(2x))\,dx =$$

$$= \frac{1}{8}\int \left(1-\cos(2x)-\cos^2(2x)+\cos^3(2x)\right)dx$$

$$= \underbrace{\int \frac{1}{8}dx}_{\frac{x}{8}} - \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{16}\int (1+\cos(4x))\,dx + \frac{1}{8}\int \cos^3(2x)\,dx$$

$$-\frac{1}{8}\int \cos(2x)\,dx = \qquad y=2x \qquad dy = 2\,dx$$

$$= -\frac{1}{16}\int \cos(y)\,dy = -\frac{1}{16}\sin(y) = -\frac{1}{16}\sin(2x)$$

$$-\frac{1}{16}\int \left(1+\cos(4x)\right)dx = -\frac{1}{16}\int dx - \frac{1}{16}\int \cos(4x)\,dx =$$

$$= -\frac{x}{16} - \frac{1}{16\cdot 4}\int \cos(y)\,dy \qquad y = 4x \quad dy = 4\,dx$$

$$= -\frac{x}{16} - \frac{1}{16\cdot 4}\sin(y) = -\frac{x}{16} - \frac{1}{16\cdot 4}\sin(4x)$$

$$\frac{1}{8}\int \cos^3(2x)\,dx \overset{y=2x}{=} \frac{1}{16}\int \cos^3(y)\,dy = \frac{1}{16}\int \left(1-\sin^2(y)\right)\cos y\,dy$$

$u = \sin y$

$$\frac{1}{8}\int \cos^3(2x)\,dy = \frac{1}{16}\int (1-\sin^2(y))\cos(y)\,dy = \qquad =$$

$y = 2x$

$u = \sin y$

$du = \cos y\, dy$

$$= \frac{1}{16}\int (1-u^2)\,du = \frac{u}{16} - \frac{u^3}{16\cdot 3}$$

$u = \sin y = \sin(2x)$

$$= \frac{\sin(2x)}{16} - \frac{\sin^3(2x)}{16\cdot 3}$$

$$R(x,y) = \frac{P(x,y)}{Q(x,y)}$$

con $P(x,y)$ e $Q(x,y)$ polinomi

$$P(x,y) = x^3y^2 + 2x^3 + 5x^6y^7 + 1$$

$$\int R(\cos t, \sin t)\, dt \qquad R(x,y) = x^n y^m$$

$$\int \cos^n(t)\, \sin^m(t)\, dt$$

$$\int \frac{\sin(t)}{\cos(t)}\, dt$$

$$R(x,y) = \frac{y}{x}$$

$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + c$$

$$\int \frac{1}{ax^2+bx+c}\, dx \qquad \Delta = b^2 - 4ac < 0$$

$$\int \frac{1}{ax^2+bx+c}\,dx \qquad \Delta = b^2-4ac<0 \qquad a>0$$

$$= \int \frac{1}{(\sqrt{a}\,x)^2 + 2\sqrt{a}\,\frac{b}{2\sqrt{a}}\,x + c}\,dx = \int \frac{1}{(\sqrt{a}\,x)^2 + 2\sqrt{a}\,x\,\frac{b}{2\sqrt{a}} + \frac{b^2}{4a} - \frac{b^2}{4a} + c}$$

$$= \int \frac{dx}{\left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2 + \underbrace{\left(c - \frac{b^2}{4a}\right)}_{>0}}$$

$$\boxed{y = \sqrt{a}\,x + \frac{b}{2\sqrt{a}}}$$

$$dy = \sqrt{a}\,dx$$

$$= \frac{1}{\sqrt{a}} \int \frac{dy}{y^2 + \left(c - \frac{b^2}{4a}\right)}$$

$$y = \sqrt{c - \frac{b^2}{4a}}\;u$$

$$dy = \sqrt{c - \frac{b^2}{4a}}\;du$$

$$\frac{1}{\sqrt{a}}\int\frac{dy}{y^2+\left(c-\frac{b^2}{4a}\right)}$$

$$y=\sqrt{c-\frac{b^2}{4a}}\;u \qquad \frac{dy}{du}=\sqrt{c-\frac{b^2}{4a}}$$

$$dy=\sqrt{c-\frac{b^2}{4a}}\;du$$

$$=\frac{1}{\sqrt{a}}\sqrt{c-\frac{b^2}{4a}}\int\frac{du}{\left(c-\frac{b^2}{4a}\right)u^2+\left(c-\frac{b^2}{4a}\right)}=$$

$$=\frac{1}{\sqrt{a}}\frac{1}{\sqrt{c-\frac{b^2}{4a}}}\int\frac{du}{u^2+1}=\frac{1}{\sqrt{a}}\frac{1}{\sqrt{c-\frac{b^2}{4a}}}\arctan u+C_1$$

$$=\frac{1}{\sqrt{a}}\frac{1}{\sqrt{c-\frac{b^2}{4a}}}\operatorname{arctg}\left(\frac{y}{\sqrt{c-b^2/4a}}\right)+C_1=\frac{1}{\sqrt{a}}\frac{1}{\sqrt{c-b^2/4a}}\operatorname{arctg}\left(\frac{\sqrt{a}\,x+b/2\sqrt{a}}{\sqrt{c-b^2/4a}}\right)+C_1$$

Lemma (Cambio variabile in integrali definiti) Siano $[a,b]$ e J due intervalli e $u:[a,b]\to J$ ed $f:J\to\mathbb{R}$, con $u\in C^1([a,b])$ ed $f\in C^0(J)$. Allora si ha

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Dim
$$\int_a^b f(u(x))\,u'(x)\,dx = \int f(u(x))\,u'(x)\,dx\Big]_a^b = \left(\int f(u)\,du\right)(u(x))\Big]_a^b$$

$$= \int f(u)\,du\Big]_{u(a)}^{u(b)} = \int_{u(a)}^{u(b)} f(u)\,du\,.$$