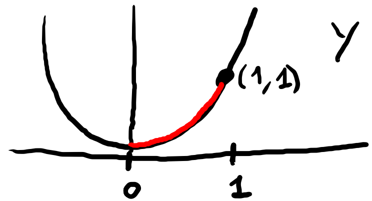
30 Nov

$u : [a,b] \to J \qquad u \in C^1([a,b]), \quad f \in C^0(J).$

Allora
$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

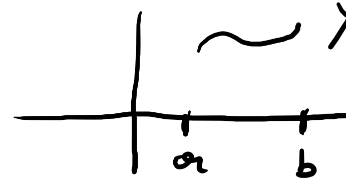
$$(x^2)' = 2x$$

$$L = \int_0^1 \sqrt{1+4x^2}\, dx$$



$y = x^2$

L è la lunghezza dell'arco di parabola tra $(0,0)$ e $(1,1)$.



$y = f(x)$

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$\int_0^1 \sqrt{1+4x^2}\,dx$$

$$= \frac{1}{2}\int_0^2 \sqrt{1+y^2}\,dy$$

$y = 2x$

$dy = 2\,dx$

$y(x) = 2x$

$$\int_{y(a)}^{y(b)} f(y)\,dy = \int_a^b f(y(x))\,y'(x)\,dx$$

$$\int_0^1 \sqrt{1+4x^2}\,dx = \frac{1}{2}\int_0^2 \sqrt{1+y^2}\,dy =$$

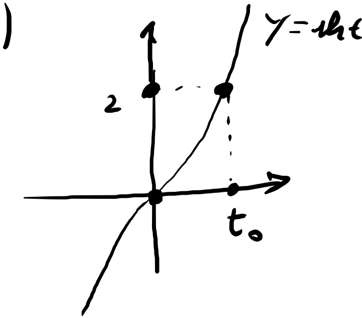
$$R\left(y, \sqrt{1+y^2}\right)$$

$$R(x_1, x_2) = x_2$$

$$y = sh(t)$$

$$dy = ch(t)\,dt$$

$$ch^2(t) = 1 + sh^2(t)$$



$$= \frac{1}{2}\int_0^{t_0} \sqrt{1+sh^2(t)}\; ch(t)\,dt =$$

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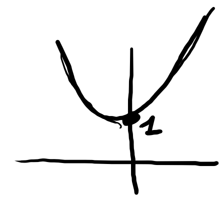


$$= \frac{1}{2}\int_0^{t_0} ch^2(t)\,dt$$

$$= \frac{1}{2} \int_0^{t_0} ch^2(t)\, dt$$

$$ch^2(t) = \frac{1 + ch(2t)}{2} \qquad ch(t) = \frac{e^t + e^{-t}}{2}$$

$$ch^2(t) = \frac{(e^t + e^{-t})^2}{4} = \frac{e^{2t} + e^{-2t} + 2}{4} = \frac{\frac{e^{2t} + e^{-2t}}{2} + 1}{2} =$$

$$= \frac{ch(2t) + 1}{2}$$

$$= \frac{1}{2} \int_0^{t_0} \frac{1 + ch(2t)}{2}\, dt = \frac{1}{4} \int_0^{t_0} dt + \frac{1}{4} \int_0^{t_0} ch(2t)\, dt$$

$$= t_0/4 + \frac{1}{4} \left. \frac{sh(2t)}{2} \right]_0^{t_0} =$$

$$= \frac{t_0}{4} + \frac{1}{8}\, sh(2t_0) =$$

$$sh\, t = \frac{e^{t} - e^{-t}}{2}$$

$$ch\, t = \frac{e^{t} + e^{-t}}{2}$$

$$\sin(2t_0) = 2 \sin(t_0) \cos(t_0)$$

$$\boxed{sh(2t) = 2\, sh(t)\, ch(t)}$$

$$2\, sh(t)\, ch(t) = 2\, \frac{e^{t} - e^{-t}}{2}\, \frac{e^{t} + e^{-t}}{2}$$

$$= \frac{1}{2}\left(e^{2t} - e^{-2t}\right) =$$

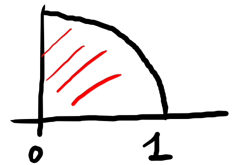
$$= sh(2t)$$

$$= \frac{t_0}{4} + \frac{1}{\cancel{8}\,4}\, \cancel{2}\, \underbrace{sh(t_0)}_{2}\, ch(t_0) = \frac{t_0}{4} + \frac{1}{2}\sqrt{1 + sh^2(t_0)} = \frac{t_0}{4} + \frac{\sqrt{5}}{2}$$

$t_0 = \lg(2+\sqrt{5})$

$f(x) = \lg\left(x+\sqrt{1+x^2}\right)$ la funzione inversa del sh(x)

$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$



$y=\sqrt{1-x^2}$

$x=\sin t$ $\quad dx = \cos t\,dt$ $\quad \sqrt{1-\sin^2 t} = \cos t$ in $[0,\frac{\pi}{2}]$

$\sin t : [0,\frac{\pi}{2}] \to [0,1]$

$\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt$

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \cos^2 t\,dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2}\,dt =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} dt + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt =$$

$$= \frac{\pi}{4} + \frac{1}{2}\left.\frac{\sin(2t)}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Esame 20/6/2016 $\quad P_6(x)$ pol. di McLaurin di

$$f(x) = \arctan(1+x^2)$$

$$\boxed{f(x) = P_6(x) + o(x^6)}$$

$$P_6(x) = \sum_{j=0}^{6} \frac{f^{(j)}(0)}{j!} x^j$$

$$\operatorname{artg} y = \sum_{j=0}^{n} (-1)^j \frac{y^{2j+1}}{2j+1} + o(y^{2n+1})$$

$$f(x) = \operatorname{artg}(1+x^2) = \sum_{j=0}^{n} (-1)^j \frac{(1+x^2)^{2j+1}}{2j+1} + \underline{o\left((1+x^2)^{2n+1}\right)}$$

grado $4n+2$

non è della forma $o(x^{4n+2})$

$$\left(\operatorname{artg}(1+x^2)\right)' = \operatorname{artg}'(1+x^2)\,2x = \frac{1}{1+(1+x^2)^2}\,2x =$$

$x - x^3 + \frac{x^5}{2} + o(x^5)$

$1+2x^2+x^4$

$$= \frac{2x}{2+2x^2+x^4} =$$

$$\frac{x}{1+y}$$

$$= \frac{x}{1+x^2+\frac{x^4}{2}} = x\,\frac{1}{1+x^2+\frac{x^4}{2}}$$

$$\frac{1}{1+y} = \sum_{j=0}^{2}(-1)^j y^j + o(y^2)$$

$x^2+2x^2\frac{x^4}{2}+\left(\frac{x^4}{2}\right)^2$

$$= x\left[1-\left(x^2+\frac{x^4}{2}\right)+\left(x^2+\frac{x^4}{2}\right)^2+o\left(\left(x^2+\frac{x^4}{2}\right)^2\right)\right] = x\left[1-x^2-\frac{x^4}{2}+x^4+o(x^4)\right]$$

$$\lim_{x\to0}\frac{o\left(\left(x^2+\frac{x^4}{2}\right)^2\right)}{x^4} = \lim_{x\to0}\frac{o\left(\left(x^2+\frac{x^4}{2}\right)^2\right)}{\left(x^2+\frac{x^4}{2}\right)^2}\,\frac{\left(x^2+\frac{x^4}{2}\right)^2}{x^4} = 0$$

$o(x^4)$

$\to 1$

$$= x\left[1-x^2+\frac{x^4}{2}+o(x^4)\right]$$

$$f'(x) = x - x^3 + \frac{x^5}{2} + o(x^5)$$

$$f(x) = \operatorname{artg}(1+x^2)$$

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

$$\operatorname{artg}(1+x^2) = \operatorname{artg} 1 + \int_0^x \left(t - t^3 + \frac{t^5}{2} + o(t^5)\right) dt$$

$$= \frac{\pi}{4} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^6}{12} + \underbrace{\int_0^x o(t^5)\,dt}_{o(x^6)}$$

$$\lim_{x \to 0} \frac{\int_0^x o(t^5)\,dt}{x^6} \overset{H}{=} \lim_{x \to 0} \frac{o(x^5)}{6\,x^5} = 0$$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$0 < \frac{y^n}{e^y} e^{\overset{<0}{y-y^2}} < \frac{y^n}{e^y} \xrightarrow{y \to +\infty} 0$$

$f \in C^\infty(\mathbb{R})$ con in particolare $f^{(n)}(0) = 0$

Osservare che rispetto a $x = 0$ si ha

che $f(x) = o(x^n) \quad \forall n \in \mathbb{N}$

$$\lim_{x \to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^n} = \lim_{y \to +\infty} y^n e^{-y^2} = \lim_{y \to +\infty} \frac{y^n}{e^{y^2}} = \lim_{y \to +\infty} \frac{y^n}{e^y} e^{y-y^2} = 0$$

$y = \frac{1}{x}$

Esercizio Verificare che $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{per } x > 0 \\ 0 & \text{per } x \leq 0 \end{cases}$

è $f \in C^\infty(\mathbb{R})$ e calcolare $f^{(n)}(0)$ $\forall n$.