3 Dicembre

$$A_{j\ell} = \frac{1}{(m_j-\ell)!}\left(\frac{d}{dz}\right)^{m_j-\ell}\left(R(z)(z-z_j)^{m_j}\right)\Big|_{z=z_j}$$

$$R(x) = \frac{1}{x(x-1)^3(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} + \frac{E}{x-2}$$

$$D = R(x)(x-1)^3\Big|_{x=1} = \frac{1}{x(x-2)}\Big|_{x=1} = -1$$

$$C = \frac{1}{(3-2)!}\left(\frac{d}{dx}\right)^{3-2}\left(R(x)(x-1)^3\right)\Big|_{x=1} = \left(\frac{1}{x^2-2x}\right)'\Big|_{x=1} = -\frac{2x-2}{(x^2-2x)^2}\Big|_{x=1} = 0$$

$$B = \frac{1}{2}\left(\frac{d}{dx}\right)^2\left(R(x)(x-1)^3\right)\Big|_{x=1} = \frac{1}{2}\left(-\frac{2x-2}{(x^2-2x)^2}\right)'\Big|_{x=1} =$$

$$= -\left(\frac{x+1}{(x^2-2x)^2}\right)'\Big|_{x=1} = -\frac{(x^2-2x)^2 \cancel{-(x+1)\,2(x^2-2x)(2x-2)}}{(x^2-2x)^4}\Big|_{x=1}$$

$$= -\frac{1}{(x^2-2x)^2}\Big|_{x=1} = -1$$

$$A = xR(x)\Big|_{x=0} = \frac{1}{(x-1)^3(x-2)}\Big|_{x=0} = \frac{1}{2}$$

$$E = (x-2)R(x)\Big|_{x=2} = \frac{1}{x(x-1)^3}\Big|_{x=2} = \frac{1}{2}$$

$z^2+1 = (z-i)(z+i)$

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{1}{\left((z-i)(z+i)\right)^2} = \frac{1}{(z-i)^2(z+i)^2}$$

$$R(z) = \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{(z-i)^2} + \frac{D}{(z+i)^2}$$

$$C = R(z)(z-i)^2\Big|_{z=i} = \frac{1}{(z+i)^2}\Big|_{z=i} = \frac{1}{(2i)^2} = -\frac{1}{4}$$

$x \in \mathbb{R}$

$$R(x) = \overline{R(x)} = \frac{\overline{A}}{x+i} + \frac{\overline{B}}{x-i} + \frac{\overline{C}}{(x+i)^2} + \frac{\overline{D}}{(x-i)^2}$$

$$= \frac{A}{x-i} + \frac{B}{x+i} + \frac{C}{(x-i)^2} + \frac{D}{(x+i)^2}$$

$D = \overline{C}$

$B = \overline{A}$

$$D = R(z)(z+i)^2\Big|_{z=-i} = \frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{1}{(-2i)^2} = -\frac{1}{4}$$

$$A = \frac{d}{dz}\frac{1}{(z+i)^2}\Big|_{z=i} = -\frac{2}{(z+i)^3}\Big|_{z=i} = \frac{-2}{(2i)^3} = \frac{-2}{2^3(-i)} =$$

$$= \frac{1}{4i} = \boxed{-\frac{i}{4} = A}$$

$$B = \frac{i}{4}$$

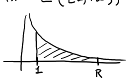
$$B = \frac{d}{dz}R(z)(z+i)^2\Big|_{z=-i} = \frac{d}{dz}\frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{-2}{(z-i)^3}\Big|_{z=-i}$$

$$= \frac{-2}{(-2i)^3} = \frac{1}{4(-i)} = \frac{i}{4}$$

Integrali impropri

Ieri abbiamo visto che $x^{-p} \in L((0,1])$ se e solo se $p < 1$

<u>Teorema</u> x^{-p} con $p > 0$ è in $L([1,+\infty))$ se e solo se $p > 1$.



Dim. Sia $p \neq 1$

$$\lim_{R\to+\infty}\int_1^R x^{-p}\,dx = \lim_{R\to+\infty}\left.\frac{x^{-p+1}}{-p+1}\right]_1^R =$$

$$= \boxed{\lim_{R\to+\infty}\left(\frac{R^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)}$$

Se $-p+1>0 \iff p<1$, allora

$$= \lim_{R\to+\infty}\frac{R^{-p+1}}{-p+1} = +\infty$$

Cioè $x^{-p} \notin L([1,+\infty))$ se $p<1$.

Se $-p+1<0 \iff p>1$, allora

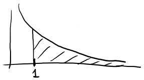
$(-p+1<0)$

allora

$$\lim_{R\to+\infty}\left(\frac{R^{-p+1}}{-p+1}-\frac{1}{-p+1}\right)=-\frac{1}{-p+1}=\frac{1}{p-1}\ (>0)$$

$\frac{1}{p-1} \in \mathbb{R}$

Cioè per $p>1$ si ha $x^{-p}\in L([1,+\infty))$



Per $p=1$

$$\lim_{R\to+\infty}\int_1^R x^{-1}\,dx=\lim_{R\to+\infty}\log R=+\infty$$

$$\Rightarrow\quad x^{-1}\notin L([1,+\infty))$$

<u>Teor</u> Date $f,g\in L([a,b))$ e date due costanti α,β, allora $\alpha f+\beta g\in L([a,b))$ con

$$\int_a^b(\alpha f(x)+\beta g(x))\,dx=\alpha\int_a^b f(x)\,dx+\beta\int_a^b g(x)\,dx.$$

Inoltre, se $f(x)\le g(x)\ \forall x\in[a,b)$, risulta che

$$\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx$$

<u>Teor</u> (Aut-aut) Sia $f\in L_{loc}([a,b))$ con $f(x)\ge0\ \forall x\in[a,b)$.
Allora $\lim_{R\to b^-}\int_a^R f(x)\,dx$ esiste ed appartiene a $[0,+\infty]$.

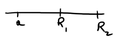
$F:[a,b)\to\mathbb{R}$

<u>Dim</u> Poniamo $F(R)=\int_a^R f(x)\,dx$. Risulta che $F(R)$ è crescente visto che, se $a\le R_1<R_2$, si ha

$$F(R_2)=\int_a^{R_2}f(x)\,dx=$$

$$=\int_a^{R_1}f(x)\,dx+\underbrace{\int_{R_1}^{R_2}f(x)\,dx}_{\ge0}$$

$$\ge\int_a^{R_1}f(x)\,dx=F(R_1)$$



Allora $\lim_{R\to b^-}F(R)=\sup\{F(R);\ R\in[a,b)\}$

<u>Teor</u> (confronto) Siano $f,g\in L_{loc}([a,b))$ con $0\le f(x)\le g(x)\ \forall x\in[a,b)$. Allora

$$g\in L([a,b))\Longrightarrow f\in L([a,b))$$

$$\left(f\notin L[a,b)\Longrightarrow g\notin L[a,b)\right)$$

<u>Dim</u> $g\in L[a,b)\Longleftrightarrow\lim_{R\to b^-}\int_a^R g(x)\,dx\in[0,+\infty)$

Per la monotonia dell'integrale di Riemann, risulta

$$\int_a^R f(x)\,dx\le\int_a^R g(x)\,dx\qquad\forall\ R\in[a,b).$$

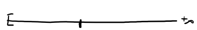
E pertanto, se esiste il $\lim_{R\to b^-}\int_a^R f(x)\,dx$ questo deve essere $\le\lim_{R\to b^-}\int_a^R g(x)\,dx=\int_a^b g(x)\,dx\in[0,+\infty)$

Per Aut Aut sappiamo che $\lim_{R\to b^-}\int_a^R f(x)\,dx$ esiste e quindi è un numero reale, e quindi $f\in L[a,b)$

Esempio $\int_0^{+\infty}\frac{x^3}{1+x^4}\,dx$ verificare se è sommabile.

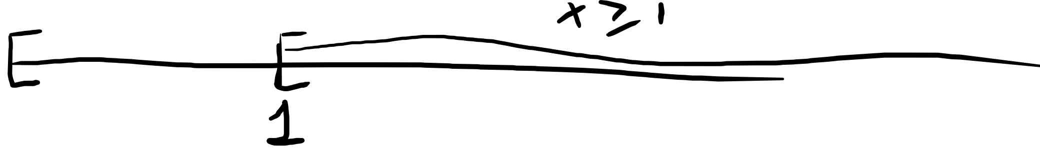
$f = \frac{x^3}{1+x^4}$

$f\in C^0([0,+\infty))$



Risulta $f\in L[0,+\infty)\Longleftrightarrow f\in L[a,+\infty)$

$$\frac{x^3}{1+x^4} =$$



$$= \frac{x^3}{x^4} \frac{1}{1+x^{-4}} \geq \frac{1}{x} \frac{1}{2}$$

$$\frac{1}{2} \frac{1}{x} \notin L[1, +\infty) \Longrightarrow \frac{x^3}{1+x^4} \notin L[1, +\infty)$$

$$\Longrightarrow \frac{x^3}{1+x^4} \notin L[0, +\infty)$$