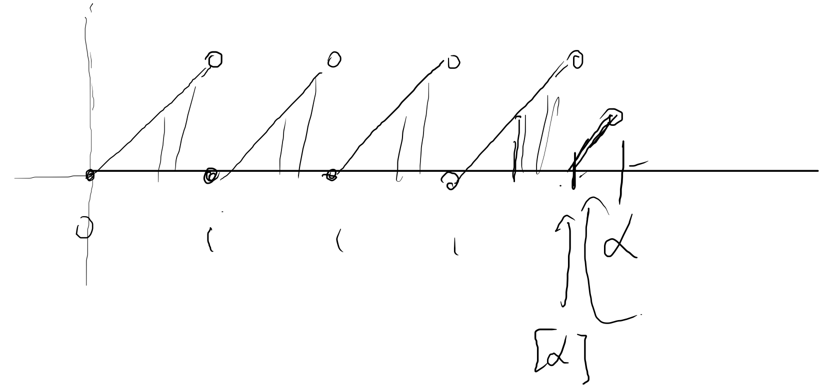
$\alpha > 0$

$f : [0,\alpha] \to \mathbb{R}$ $\qquad f(x) = x - [x]$



$$\int_{[0,\alpha]} f\,dm = \frac{1}{2}[\alpha] + \int_{[\alpha]}^{\alpha} f(x)\,dx$$

$$\| $$

$$\int_{0}^{\alpha-[\alpha]} x\,dx = \frac{1}{2}\left(\alpha - [\alpha]\right)^2$$

$$\int_0^1 \sqrt{\frac{1+\operatorname{senh}^2(x)}{\operatorname{senh}(x)+2}}\, dx$$

$$\cosh^2 x - \operatorname{senh}^2 x = 1$$

$$\sqrt{1+\operatorname{senh}^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

$$\operatorname{senh} x = z \qquad dz = \cosh x \, dx$$

$$x=0 \to z=0$$

$$x=1 \to z=\operatorname{senh}(1)$$

$$= \int_0^{\operatorname{senh}(1)} \frac{1}{\sqrt{z+2}}\, dz = \int_0^{\operatorname{senh}(1)} (z+2)^{-\frac{1}{2}}\, dz = \Big[\, 2\,(z+2)^{\frac{1}{2}} \Big]_0^{\operatorname{senh}(1)} = 2\left(\sqrt{\operatorname{senh}(1)+2} - \sqrt{2}\right)$$

$$\frac{p(x)}{q(x)}$$

grado di $p(x)$ < grado di $q(x)$

Se grado di $p(x) \geq$ grado di $p(x)$? Si divido $p(x) : q(x) = s(x)$

$r(x)$

$$\int_a^b \frac{x^2+1}{x-1}\,dx = \int_a^b \left((x+1) + \frac{2}{x-1} \right) dx$$

$$\frac{p(x)}{q(x)} = \frac{q(x) \cdot s(x) + r(x)}{q(x)}$$

grado di r < grado di q

$$= \boxed{s(x) + \frac{r(x)}{q(x)}}$$

$x^2 + 1 : x - 1 = x + 1$

$x^2 - x$

$/ \; x + 1$

$x \; -1$

2

$$\frac{a}{x-\alpha} \qquad \frac{ax+b}{x^2+\beta x+\gamma} \qquad \frac{d}{dx}\frac{N(x)}{D(x)}$$

$$\log|x-\alpha| \qquad ? \qquad \frac{N(x)}{D(x)}$$

$$\frac{ax+b}{x^2+x+1}$$

$$D\log(x^2+x+1) = \frac{2x+1}{x^2+x+1}$$

$$\frac{a}{2}\,\frac{2x+2b/a}{x^2+x+1} = \frac{a}{2}\left[\frac{2x+1}{x^2+x+1} + \frac{2b/a-1}{x^2+x+1}\right]$$

$$\frac{1}{x^2+x+1} = \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{\frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right]} = \frac{4}{3}\,\frac{1}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}$$

$$\underbrace{\left(x^2+2\,\frac{1}{2}\cdot x+\frac{1}{4}\right)+\frac{3}{4}}_{x^2+x+1}$$

$$y = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$$

$$\int \frac{1}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}\,dx = \frac{\sqrt{3}}{2}\int \frac{1}{1+y^2}\,dy = \frac{\sqrt{3}}{2}\,\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

$$y = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$$

$$dy = \frac{2}{\sqrt{3}}\,dx$$

$$\int \frac{ax+b}{x^2+x+1}\,dx = \frac{a}{2}\left(\log\left(x^2+x+1\right) + \left(\frac{2b}{a}-1\right)\cdot\frac{2}{\sqrt{3}}\,\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right)$$

$$\frac{x+1}{x^2-2x+5} = \boxed{\frac{x+1}{4+(x-1)^2}}$$

$$\uparrow$$

$$(x^2-2x+1)+4$$

$$\frac{x+1}{(x-1)^2+4} = \boxed{\frac{1}{4}} \frac{x+1}{\left(\frac{x-1}{2}\right)^2+1} =$$

$$\frac{1}{4}(x+1) = \frac{1}{2}\frac{x+1}{2} = \frac{1}{2}\frac{x-1+2}{2}$$

$$\left(\frac{x-1}{2}\right)^2 \cdot \frac{(x-1)^2}{4}$$

$$\log\left(\left(\frac{x-1}{2}\right)^2+1\right) = \frac{\frac{x-1}{2}}{\left(\frac{x-1}{2}\right)^2+1}$$

$$\frac{1}{1+y^2}$$

$$= \frac{1}{2} \frac{\frac{x-1}{2}+1}{\left(\frac{x-1}{2}\right)^2+1} = \frac{1}{2}\frac{\frac{x-1}{2}}{\left(\frac{x-1}{2}\right)^2+1} + \frac{1}{2}\frac{1}{1+\left(\frac{x-1}{2}\right)^2}$$

$$\operatorname{arctg} y$$

$$y = \frac{x-1}{2} \qquad dy = \frac{1}{2}dx$$

$$\int \frac{1}{2}\frac{1}{1+\left(\frac{x-1}{2}\right)^2}dx = \int \frac{1}{1+y^2}dy$$

$$\frac{1}{2}\log\left(\left(\frac{x-1}{2}\right)^2+1\right) + \operatorname{arctg}\left(\frac{x-1}{2}\right)$$

$$\frac{1}{2}\cdot\frac{1}{\left(\frac{x-1}{2}\right)^2+1}\cdot\cancel{2}\left(\frac{x-1}{2}\right)\cdot\frac{1}{\cancel{2}} + \frac{1}{1+\left(\frac{x-1}{2}\right)^2}\cdot\frac{1}{2} = \frac{1}{2}\frac{\frac{x-1}{2}+1}{1+\left(\frac{x-1}{2}\right)^2} = \frac{x+1}{4+(x-1)^2}$$

$$\int \frac{x+2}{x^2(x+1)^2}\,dx$$

$$2cx^2 - cx + 2dx - d$$

$$= \frac{c(x^2-x) - (cx+d)\cdot(2x-1)}{x^2(x-1)^2}$$

$$\frac{x+2}{x^2(x-1)^2} = \frac{a}{x} + \frac{b}{x-1} + \frac{d}{dx}\frac{cx+d}{x(x-1)}$$

$5 \qquad -5 \qquad -5 \qquad 2$

$$x^2 - x$$

$$1$$

$$cx^2 - \cancel{cx} - 2cx^2 + \cancel{cx} - 2dx + d$$

$$-cx^2 - 2dx + d$$

$$= \frac{ax(x-1)^2 + bx^2(x-1) - cx^2 - 2dx + d}{x^2(x-1)^2}$$

$$ax(x^2-2x+1) + \cancel{bx^3} - bx^2 - cx^2 - 2dx + d$$

$$\cancel{ax^3} - 2ax^2 + ax$$

$$x^3(a+b) + x^2(-2a-b-c) + x(a-2d) + d$$

$$\begin{cases} a+b=0 & \rightsquigarrow \quad b=-a \\ +2a+b+c=0 & \quad -a=+c \quad \rightsquigarrow b=c \\ a-2d=1 & \quad a = 1+4 = 5 \\ d=2 & \end{cases}$$

$a=5 \qquad b=c=-5$

$d=2$

$$\int \left(\frac{5}{x} - \frac{5}{x-1} + \frac{\phi}{dx} \frac{-5x+2}{x^2-x} \right) dx$$

$$= 5 \log|x| - 5 \log|x-1| + \frac{-5x+2}{x^2-x}$$

$$\int_2^3 \frac{x+2}{(x-1)(x^2+1)}\,dx$$

$(bx+c)(x-1) = bx^2 - bx + cx - c$

$(a+b)x^2 + (-b+c)x + a - c$

$$\frac{x+2}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1} = \frac{ax^2 + a + bx^2 - bx + cx - c}{(x-1)(x^2+1)} = *$$

$$\begin{cases} a+b = 0 \\ c-b = 1 \\ a-c = 2 \end{cases} \quad \begin{matrix} b = -a \\ c = b+1 \\ a = c+2 \end{matrix}$$

$-a = c - 1 \quad -a = 1 - c$

$a = c + 2$

$\Rightarrow 1 - c - c - 2 = 0$

$2c = -1$

$c = -\frac{1}{2}$

$a = \frac{3}{2} \quad b = -\frac{3}{2}$

$$* = \int \frac{3}{2}\,\frac{1}{x-1} + \int \frac{-\frac{3}{2}x - \frac{1}{2}}{x^2+1} \Rightarrow -\frac{1}{2}\int \frac{3x+1}{x^2+1}$$

$$\frac{3x+1}{x^2+1}$$

$$D(x^2+1) = 2x$$

$$-\frac{3}{4}\log(10) + \frac{3}{4}\log 5$$

$$-\frac{3}{4}\log 2$$

$$\frac{3}{2}\,\frac{2x+2/3}{x^2+1} = \frac{3}{2}\,\frac{2x}{x^2+1} + \frac{1}{x^2+1}$$

$$\frac{3}{2}\,\frac{1}{x-1} - \frac{1}{2}\cdot\frac{3}{2}\,\frac{2x}{x^2+1} - \frac{1}{2}\,\frac{1}{1+x^2}$$

$$\frac{3}{2}\log|x-1| - \frac{3}{4}\log(x^2+1) - \frac{1}{2}\operatorname{arctg} x$$

$$\int_2^3 \frac{x+2}{(x-1)(x^2+1)}\,dx = \left[\frac{3}{2}\log(x-1) - \frac{3}{4}\log(x^2+1) - \frac{1}{2}\operatorname{arctg}\right]_2^3$$

$$= \frac{3}{2}\log 2 - \frac{3}{4}\log\frac{\cancel{10}^{\,2}}{\cancel{5}} - \frac{1}{2}\operatorname{arctg} 3 + \frac{1}{2}\operatorname{arctg} 2$$

$$\frac{3}{4}\log 2 + \frac{1}{2}\left(\operatorname{arctg} 2 - \operatorname{arctg} 3\right)$$

Sostituzioni "razionalizzanti"

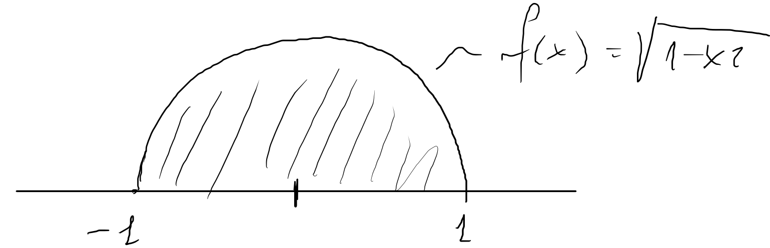
$$\int \frac{e^{2x}}{e^x + e^{-x}} \frac{e^x}{e^x} dx = \int \frac{y^2}{y + \frac{1}{y}} \frac{1}{y} dy = \int \frac{y^2+1-1}{y^2+1} dy = \int \left(1 - \frac{1}{y^2+1}\right) dy$$

$$= y - \operatorname{arctg} y$$

$y = e^x \qquad dy = e^x dx$

$$e^x - \operatorname{arctg}(e^x)$$

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$$



$$\int \sqrt{1-x^2}\, dx$$

$\text{sen}^2 t + \cos^2 t = 1$

$1 - \text{sen}^2 t = \cos^2 t \qquad \sqrt{1-\text{sen}^2 t} = |\cos t|$

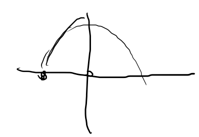
y tale che $x = \text{sen}\, y$

$dx = \cos y\, dy$

$$\int \sqrt{1-x^2}\, dx = \int |\cos y| \cos y\, dy = \int \cos^2 y\, dy = \int \frac{1}{2}\left(1+\cos(2y)\right) dy$$

$x = \text{sen}\, y$

se $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\cos y \geq 0$



$$\int \frac{1}{2}\left(1+\cos 2y\right) dy = \frac{1}{2}y + \frac{1}{4}\operatorname{sen}(2y) = \frac{1}{2}y + \frac{1}{\cancel{4}_2}\left(\cancel{2}\operatorname{sen}y \cos y\right) =$$

$\cos y = \sqrt{1-\operatorname{sen}^2 y}$

$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\boxed{x = \operatorname{sen} y}$

$\frac{1}{2}\operatorname{sen}(2y)$

$$= \frac{1}{2}\left(\operatorname{arcsen}x + x\sqrt{1-x^2}\right)$$

$$\int \sqrt{1-x^2}\, dx = \frac{1}{2}\left(\operatorname{arcsen}x + x\sqrt{1-x^2}\right)$$

$$\int_{-2}^{-1} \sqrt{x^2-1}\,dx = *$$

$$\cosh^2 t - \operatorname{senh}^2 t = 1$$

$$\sqrt{\cosh^2 t - 1} = \sqrt{\operatorname{senh}^2 t} = |\operatorname{senh} t|$$

$x = -\cosh y$

$dx = -\operatorname{senh} y\, dy$

$e^y = 2+\sqrt{3} \qquad y = \log(2+\sqrt{3})$

$x=-2 \rightsquigarrow y = \operatorname{settcosh}(2) = \log(2+\sqrt{3})$

$x=-1 \rightsquigarrow y = \operatorname{settcosh}(1) = 0$

$\cosh y = 2 \Rightarrow \frac{1}{2}(e^y + e^{-y}) = 2$

$e^{-y}(e^{2y}+1) = 4$

$$* = \int_{\log(2+\sqrt{3})}^{0} \underbrace{|\operatorname{senh} y|}_{=\operatorname{senh} y} \cdot (-\operatorname{senh} y)\, dy = -\int_{\log 2+\sqrt{3}}^{0} \operatorname{senh}^2 y\, dy = \int_{0}^{\log(2+\sqrt{3})} \operatorname{senh}^2 y\, dy$$

$\log(2+\sqrt{3}) > 0$

$e^{2y}+1-4e^y = 0$

$z^2 - 4z + 1 = 0 \quad \begin{cases} 2-\sqrt{3} \\ 2+\sqrt{3} \end{cases}$

$$\int_{-2}^{-1} \sqrt{x^2-1}\,dx = \int_0^{\log(2+\sqrt{3})} \sinh^2(y)\,dy = \frac{1}{4}\int_0^{\log(2+\sqrt{3})} \left(e^{2y}-2+e^{-2y}\right) dy =$$

$$\sinh^2(y) = \left[\frac{1}{2}\left(e^{y}-e^{-y}\right)\right]^2$$

$$= \frac{1}{4}\left(\frac{1}{2}e^{2y} - 2y - \frac{1}{2}e^{-2y}\right]_0^{\log(2+\sqrt{3})} = \frac{1}{4}\left(\frac{1}{2}(2+\sqrt{3})^2 - 2\log(2+\sqrt{3}) - \frac{1}{2}(2+\sqrt{3})^{-2} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}}\right)$$

$$\int \frac{1}{\operatorname{sen} x + 2}\, dx$$

$$\frac{1}{\cos^2\alpha} = 1 + \operatorname{tg}^2\alpha \qquad \cos^2\alpha = \frac{1}{1+\operatorname{tg}^2\alpha}$$

$$\operatorname{sen}(2\alpha) = 2\sin(\alpha)\cdot\cos(\alpha) = 2\operatorname{tg}(\alpha)\cdot\cos^2(\alpha) = \frac{2\operatorname{tg}\alpha}{1+\operatorname{tg}^2\alpha}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \frac{1}{1+\operatorname{tg}^2\alpha} - \frac{\operatorname{sen}^2\alpha}{\cos^2\alpha}\cos^2\alpha = \frac{1-\operatorname{tg}^2\alpha}{1+\operatorname{tg}^2\alpha}$$

$$\operatorname{tg}^2\alpha\cdot\frac{1}{1+\operatorname{tg}^2\alpha}$$

$$\operatorname{tg}(2\alpha) = \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$$

$$\boxed{y = \operatorname{tg}\left(\frac{x}{2}\right)} \longrightarrow \quad dy = \left[1 + \operatorname{tg}^2\left(\frac{x}{2}\right)\right]\cdot\frac{1}{2}\,dx = \frac{1}{2}\left(1+y^2\right)dx$$