Teorema del confronto

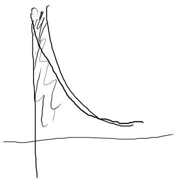
$f, g :]a,b] \to \mathbb{R}$ localmente integrabili $0 \leq f(x) \leq g(x)$

Sia g integrabile in senso generalizzato su $]a,b]$, allora anche f lo è $\int_a^b f\,dx \leq \int_a^b g\,dx$

Sia f non integrabile in senso gen., allora anche g non lo è.

<u>Dim</u> $\forall \varepsilon$ $\int_{a+\varepsilon}^b f(x)\,dx \leq \int_{a+\varepsilon}^b g(x)\,dx \Rightarrow \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f(x)\,dx \leq \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b g(x)\,dx$

Criterio dell'ordine di infinito



Sia $f:]a,b] \to \mathbb{R}$ localmente integrabile, $f(x) \geq 0 \ \forall x \in]a,b]$

$\lim_{x \to a} f(x) = +\infty$; allora

se $\mathrm{Ord}_a f \leq \alpha < 1$ $\alpha \in \mathbb{R}$, allora f è integrabile in senso generalizzato su $]a,b]$;

se $\mathrm{Ord}_a f \geq 1$ f non è integrabile

⚠ Se $\mathrm{Ord}_a f < 1$ ma non esiste alcun $\alpha \in \mathbb{R}$ tale che $\mathrm{Ord}_a f \leq \alpha < 1$, non si può concludere nulla

__Dim__ Sia $\mathrm{Ord}_a f \leq \alpha < \beta < 1$

$$\lim_{x \to a} \frac{f(x)}{\frac{1}{(x-a)^\beta}} = 0$$

Def. limite $\varepsilon = 1$ $\exists \delta$: $\forall x \; 0 < |x-a| < \delta$

$$\left| \frac{f(x)}{\frac{1}{(x-a)^\beta}} - 0 \right| < 1$$

allora se $x < a + \delta$ $f(x) < \frac{1}{(x-a)^\beta}$

la funzione $\frac{1}{(x-a)^\beta}$ è integrabile //

se $\mathrm{Ord}_a f \geq 1$ allora $\lim_{x \to a} \frac{f(x)}{\frac{1}{(x-a)}} = +\infty$

e quindi in un intorno di a si ha $f(x) > \frac{1}{(x-a)}$ $\frac{1}{x-a}$ non è integrabile

Es: $\displaystyle \int_0^{\pi/2} \operatorname{tg}x\,dx = \lim_{\varepsilon\to 0^+} \int_0^{\pi/2-\varepsilon} \operatorname{tg}x\,dx = \lim_{\varepsilon\to 0^+} \Big[-\log|\cos x|\Big]_0^{\frac{\pi}{2}-\varepsilon} =$

I modo: $D\left(-\log|\cos x|\right) = \operatorname{tg}x$

$\displaystyle = \lim_{\varepsilon\to 0^+} -\log\left|\cos\left(\frac{\pi}{2}-\varepsilon\right)\right| = +\infty$

II modo: studiamo $\displaystyle \operatorname{Ord}_{\frac{\pi}{2}^-} \underline{\operatorname{tg}x} = \operatorname{ord}_{\frac{\pi}{2}^-} \underbrace{\left(\frac{\cos x}{\sin x}\right)}_{=\varphi(x)}$

$\displaystyle \left(\lim_{x\to\frac{\pi}{2}^-} \frac{\operatorname{tg}x}{\left(\frac{1}{\frac{\pi}{2}-x}\right)^1} \leftarrow \text{l'H} \;---\right.$

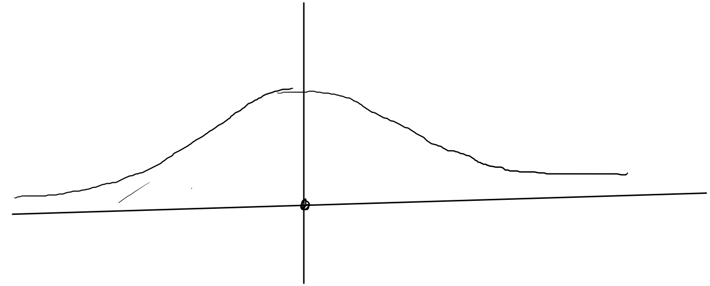
$\varphi\left(\frac{\pi}{2}\right) = 0$

$\displaystyle \varphi'(x) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \qquad \varphi'\left(\frac{\pi}{2}\right) = -1 \qquad \operatorname{ord}_{\frac{\pi}{2}^-} \varphi(x) = 1$

$\displaystyle \Rightarrow \operatorname{Ord}_{\frac{\pi}{2}^-} \operatorname{tg}x = 1 \Rightarrow \operatorname{tg}x$ non è integrabile $\displaystyle \int_0^{\frac{\pi}{2}} \operatorname{tg}x\,dx = +\infty$

Cosi misli

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx$$



$$\|$$

$$\int_{-\infty}^{0} e^{-x^2}\,dx + \int_{0}^{+\infty} e^{-x^2}\,dx$$

$$\int_{-\infty}^{+\infty} f(x)\,dx =$$

$$\underbrace{\int_{-\infty}^{x_0} f(x)\,dx} + \underbrace{\int_{x_0}^{+\infty} f(x)\,dx}$$

$$\lim_{b \to +\infty} \int_{-b}^{b} f(x)\,dx$$

Es:

$$\int_{-\infty}^{+\infty} x\,dx$$

$$\lim_{b\to+\infty} \int_{-b}^{b} x\,dx = 0$$

$$PV\int_{-\infty}^{+\infty} f(x)\,dx$$

$$\int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx$$ NON esiste!

$-\infty$

$+\infty$

Sia f localmente integrabile su $]a,b[$ $\quad a \in \mathbb{R} \cup \{-\infty\}$ $\quad b \in \mathbb{R} \cup \{+\infty\}$ $\quad a < b$

f si dice integrabile in senso generalizzato su $]a,b[$ se, preso $x_0 \in]a,b[$ qualsiasi, f è integrabile in senso gen. su $]a,x_0]$ e su $[x_0,b[$; in questo caso diremo integrale generalizzato di f su $]a,b[$ il numero

$$\int_a^b f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx$$

<u>OSS</u>: la definizione non dipende dalla scelta di x_0: $\quad x_1 < x_2$

$$\int_a^{x_1} f(x)\,dx + \int_{x_1}^b f(x)\,dx = \int_a^{x_2} f(x)\,dx + \int_{x_2}^b f(x)\,dx.$$

Si dice invece valor principale di f su $]a,b[$, se esiste, il numero ottenuto come limite simmetrico

$$\lim_{b\to+\infty}\int_{-b}^{b} f(x)\,dx \qquad \lim_{b\to+\infty}\int_{a+\frac{1}{b}}^{b} f(x)\,dx \qquad \text{o con simili}$$

Scriveremo $PV\displaystyle\int_a^b f(x)\,dx$

Es

$$PV\int_{-\infty}^{+\infty}\frac{1}{x}\,dx = \lim_{b\to+\infty}\left[\int_{]-b,-\frac{1}{b}[\,\cup\,]\frac{1}{b},b[}\frac{1}{x}\,dx\right] = 0$$

mentre

$$\int_{-\infty}^{+\infty}\frac{1}{x}\,dx = \lim_{b\to+\infty}\int_{-b}^{-1}\frac{1}{x}\,dx + \lim_{b\to+\infty}\int_{-1}^{-\frac{1}{b}}\frac{1}{x}\,dx + \lim_{b\to+\infty}\int_{1/b}^{1}\frac{1}{x}\,dx + \lim_{b\to+\infty}\int_{1}^{b}\frac{1}{x}\,dx$$

NON ESISTE!

Es: $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx$

Si stabilisca se la funzione $f(x) = \dfrac{1}{\sqrt{|x|}\,(x^2+1)}$ è integrabile su $\mathbb{R}$.

$$\int_{-\infty}^{-1} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx + \int_{-1}^{0} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx + \int_{0}^{1} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx + \int_{1}^{+\infty} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx$$

f pari $\Rightarrow 2\displaystyle\int_{0}^{+\infty} \frac{1}{\sqrt{|x|}\,(x^2+1)}\,dx = 2\left[\int_0^1 \frac{1}{\dots}\,dx + \int_1^{+\infty} \frac{1}{\dots}\,dx\right]$

$f(x) = \dfrac{1}{\sqrt{x}\,(x^2+1)}$ su $[0,1]$ $\quad$ $\mathrm{Ord}_0 f = \frac{1}{2}$ $\quad$ f è integrabile

$$f(x) = \frac{1}{\sqrt{x}\,(x^7+1)} \qquad a\ +\infty \qquad \lim_{x \to +\infty} \frac{1}{\sqrt{x}\,(x^7+1)} = 0$$

$$\int_1^{+\infty}$$

$$\operatorname{ord}_{+\infty} f = \frac{5}{2} > 1 \qquad \frac{1}{x^{2+\frac{1}{7}}\left(1+\frac{1}{x^7}\right)}$$

f è integrabile

$\Rightarrow$ f è integrabile su $\mathbb{R}$

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{|x|}\,(1+x^7)}\,dx = 2\int_0^{+\infty} \frac{1}{\sqrt{x}\,(1+x^7)}\,dx = 2\left[\lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{\sqrt{x}\,(1+x^7)}\,dx + \lim_{b \to +\infty} \int_1^b \frac{1}{\sqrt{x}\,(1+x^7)}\,dx\right]$$

$$\int \frac{1}{\sqrt{x}\,(1+x^2)}\,dx = 2\int \frac{1}{1+y^4}\,dy$$

$y=\sqrt{x}$ $\quad \uparrow$

$\sqrt{x} = y \qquad dy = \frac{1}{2\sqrt{x}}\,dx$

$$(y^4+2y^2+1) - 2y^2 =$$

$$(y^2+1)^2$$

$$\frac{1}{(\quad)\circ(\quad)} = \frac{ay+b}{y^2+1-\sqrt{2}y} + \frac{cy+d}{y^2+1+\sqrt{2}y}$$

$$= (y^2+1-\sqrt{2}y)(y^2+1+\sqrt{2}y)$$

$$f(x) = \int_0^{x^2} e^{t^2}\, dt \qquad f(0)=0 \qquad P_{10}(x)$$

$$e^z = 1 + z + \frac{1}{2} z^2 + \frac{1}{3!} z^3 + \dots$$

$$f'(x) = e^{x^4} \cdot 2x$$

$$= \left(1 + x^4 + \frac{1}{2} x^8 + \frac{1}{6} x^{12} + o(1)\right) \cdot 2x = 2x + 2x^5 + x^9 + o(x^{12})$$

$$f(x) = \int_0^x \left[2t + 2t^5 + t^9 + o(t^{12})\right] dt = \underbrace{x^2 + \frac{1}{3} x^6 + \frac{1}{10} x^{10}} + o(x^{13})$$

$$\frac{x^2+2}{x(2x^2+1)^2} = \frac{1}{4}\,\frac{x^2+2}{x\left(x^2+\frac{1}{2}\right)^2}$$

$a = 8 \qquad b = -8$

$4 - 2e = 1 \qquad e = \frac{3}{2}$

$c = d = 0$

$$\frac{x^2+2}{x\left(x^2+\frac{1}{2}\right)^2} = \frac{a}{x} + \frac{bx+c}{x^2+\frac{1}{2}} + \frac{d}{dx}\,\frac{dx+e}{x^2+\frac{1}{2}}$$

$dx^2 + \frac{1}{2}d - 2dx^2 - 2ex$

$$+\ \frac{d\left(x^2+\frac{1}{2}\right) - (dx+e)\cdot 2x}{\left(x^2+\frac{1}{2}\right)^2}$$

$ax^4 + ax^2 + \frac{1}{4}a \qquad bx^4 + \frac{1}{2}bx^2 + cx^3 + \cancel{\frac{1}{2}cx}$

$$= \frac{a\left(x^2+\frac{1}{2}\right)^2 + (bx+c)\,x\left(x^2+\frac{1}{2}\right) + \cancel{dx^3} + \cancel{\frac{1}{2}dx} - \cancel{2dx^3} - 2ex^2}{x\left(x^2+\frac{1}{2}\right)^2}$$

$\frac{1}{4}a = 2 \qquad a = 8$

$a + b = 0$

$c - d = 0$

$a + \frac{1}{2}b - 2e = 1$

$\frac{1}{2}c + \frac{1}{2}d = 0$

$$\frac{1}{4}\left[\frac{8}{x}-\frac{8x}{x^2+1/2}+\frac{d}{dx}\frac{3/2}{x^2+1/2}\right]$$

$$\frac{2}{x}-\frac{2x}{x^2+1/2}+\frac{3}{8}\frac{d}{dx}\frac{1}{x^2+1/2}$$

$$2\log|x|-\log\left(x^2+1/2\right)+\frac{\frac{3}{8}\frac{1}{x^2+1/2}}{\frac{3}{4}\frac{1}{2x^2+1}}$$

$$\log\left(\frac{1}{2}(2x^2+1)\right)$$

$$\log 1/2+\log(2x^2+1)$$

$$\int_0^1 \frac{x^5}{\sqrt{1-x^4}}\,dx$$

$$y = \sqrt{1-x^4}$$

$$dy = \frac{1}{\cancel{2}\sqrt{1-x^4}} \cdot \left(-\cancel{4}^{\,2} x^3\right) dx$$

$$\int_0^1 \frac{-2x^3}{\sqrt{1-x^4}} \cdot \left(-\frac{1}{2}x^2\right) dx = \int_1^0 -\frac{1}{2}x^2\,dy = \int_1^0 -\frac{1}{2}\sqrt{1-y^2}\,dy \stackrel{*}{=}$$

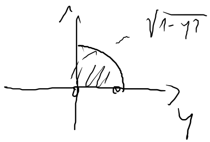
$$y^2 = 1-x^4$$

$$x^4 = 1-y^2$$

$$x^2 = \sqrt{1-y^2}$$

$x=0 \rightsquigarrow y=1$

$x=1 \rightsquigarrow y=0$



$$\stackrel{*}{=} \frac{1}{2}\int_0^1 \sqrt{1-y^2}\,dy = \frac{\pi}{8}$$

$y = \sin t$ ...

$\alpha \in \mathbb{R} \quad \alpha > 0$

$$\lim_{x \to 0^+} \frac{\int_x^{x^2} \log(1+t^\alpha)\, dt}{1-\cos(x^2)}$$

$$\boxed{\frac{\int_x^{x^2} \log(1+t^\alpha)\, dt}{x^4}} \cdot \frac{x^4}{1-\cos(x^2)} \to 2$$

$1-\cos^2 x = \sin^2 x$

$1-\cos(x^2)$

x^4

$\Downarrow$ l'H

$$\frac{\log(1+x^{2\alpha}) \cdot 2x - \log(1+x^\alpha)}{2x^3} = \frac{x^\alpha \left[\frac{\log(1+x^{2\alpha})}{x^{2\alpha}} \cdot 2 x^{1-\alpha} - \frac{\log(1+x^\alpha)}{x^\alpha} \right]}{2x^3}$$

$x^{\alpha-3}$ $\quad \alpha = 3$ $\quad x^{\alpha+1}$ $\quad \to 1$

$\boxed{\alpha = 3}$

$$\left[\frac{\log(1+x^{2\alpha})}{x^{2\alpha}} \cdot 2 x^{\alpha+1} - \frac{\log(1+x^\alpha)}{x^\alpha} \right] \cdot \frac{1}{2} \cancel{x^{\alpha-3}} \to -\frac{1}{2} \quad \text{se } \alpha = 3$$

$\downarrow \qquad \downarrow \qquad \downarrow$

$1 \qquad 0 \qquad 1$

$x^{\alpha-3} \to 0$ ($\alpha > 3$), $\to +\infty$

se $\alpha > 3$ $\quad \lim = 0$

se $0 < \alpha < 3$ $\quad \lim = -\infty$

$$f(x) = \frac{1+(x+1)^2}{1+x^2} = \frac{1+x^2+2x+1}{1+x^2} = 1 + \frac{2x}{1+x^2} + \frac{1}{1+x^2}$$

i) Si calcoli una primitiva di f:

$$F(x) = x + \log(1+x^2) + \operatorname{arctg}(x)$$

$$\int_0^x f(t)\,dt = \Big[t + \log(1+t^2) + \operatorname{arctg}(t)\Big]_0^x \quad *$$

ii) $$\int_0^1 \left(\int_0^x f(t)\,dt \right) dx = \int_0^1 \Big[x + \log(1+x^2) + \operatorname{arctg} x\Big]\,dx$$

$\frac{1}{2}$

$$\int_0^1 \underset{1\cdot}{\operatorname{arctg} x}\,dx = \Big[x \operatorname{arctg} x\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{1}{2}\log(2)$$

$$\int_0^1 \underset{1\cdot}{\log(1+x^2)}\,dx = \Big[x \log(1+x^2)\Big]_0^1 - \int_0^1 x \cdot \frac{2x}{1+x^2}\,dx = \log 2 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$

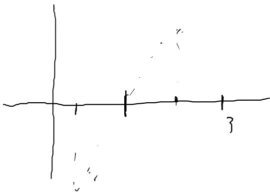
$$= \log 2 - 2\left[\int_0^1 \left(1 - \frac{1}{x^2+1}\right) dx\right] = \log 2 - 2 + 2\,\frac{\pi}{4} = \log 2 - 2 + \frac{\pi}{2}$$

$$\frac{3}{4}\pi - \frac{3}{2} + \frac{1}{2}\log 2$$

$$f(x) = \int_0^{\log x} e^{-t^2}\,dt$$

$\operatorname{dom} f =]0,+\infty[$ segni $\log x > 0$ se $x > 1$ $\log x < 0$ se $x \in]0,1[$

$f(x) < 0$ se $x \in]0,1[$ $f(1) = 0$ $f(x) > 0$ se $x > 1$



$\forall x \in \operatorname{dom} f$ $f(1/x) = -f(x)$

$$f(1/x) = \int_0^{\log 1/x} e^{-t^2}\,dt = \int_0^{-\log x} e^{-t^2}\,dt = -\int_0^{\log x} e^{-s^2}\,ds = -f(x)$$

$s = -t \quad ds = -dt$

$$f(x) = \int_0^{\log x} e^{-t^2}\, dt$$

$$f'(x) = \underbrace{e^{-(\log x)^2} \cdot \frac{1}{x}}_{>0\ \forall x}$$

$f'(x) > 0 \quad \forall x$

$$f''(x) = e^{-(\log x)^2} \cdot (-2\log x) \cdot \frac{1}{x^2} + e^{-(\log x)^2} \cdot \left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\, e^{-(\log x)^2} \left(2\log x + 1\right)$$

$\log x = -\frac{1}{2} \qquad x = e^{-\frac{1}{2}}$

$f''(e^{-\frac{1}{2}}) = 0 \qquad f''(x) < 0$ se $x > e^{-\frac{1}{2}}$

$f''(x) > 0$ se $x \in \,]0, e^{-\frac{1}{2}}[$

f è convessa su $]0, e^{-\frac{1}{2}}]$

f è concava su $[e^{-\frac{1}{2}}, +\infty[$

$e^{-\frac{1}{2}}$ è punto di flesso discendente

f è limitata su $]0,+\infty[$? Basta ragionare su $[1,+\infty[$ $\quad f$ è strettamente crescente ;

f è limitata se e solo se $\lim_{x\to+\infty} f(x) \in \mathbb{R}$ $\quad [= \sup f] \quad \left[\underbrace{\inf f}_{=\lim_{x\to 0} f(x)} = -\sup f\right]$

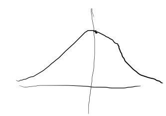
$$\lim_{x \to +\infty} \int_0^{\log x} e^{-t^2}\,dt = \int_0^{+\infty} e^{-t^2}\,dt \in \mathbb{R}$$ perché e^{-t^2} è infinitesimo di ordine superiore a $+\infty$

Si dimostri che $|f(x)| < 2 \quad \forall x$

$e^{-t^2} \le e^{-t}$ se $t \ge 1$

$$\lim_{b \to +\infty} \left[-e^{-t}\right]_1^b = e^{-1}$$

$$\int_0^{+\infty} e^{-t^2}\,dt \le \underbrace{\int_0^1 e^{-t^2}\,dt}_{\le 1} + \underbrace{\int_1^{+\infty} e^{-t}\,dt}_{\le \frac{1}{e}} \le 1 + \frac{1}{e} < 2$$



$0 \le t \le 1$

$e^{-t^2} \le 1$

$$f_n(x) = \int_0^x \frac{2}{(1-t)(t-2)^n}\,dt$$

$n \in \mathbb{N}^+$

determinare $\mathrm{dom}\, f_n$: se $t=1$ la funzione $\frac{2}{(1-t)(t-2)^n}$ non è definita; è infinito di ordine 1, quindi non è integrabile in senso generalizzato su intervalli di estremo 1

$\mathrm{dom}\, f_n =]-\infty, 1[$

$$f_n'(x) = \frac{2}{\underset{>0}{(1-x)}\,\underset{?}{(x-2)^n}} \begin{cases} >0 & \text{se } n \text{ è pari} \\ <0 & \text{se } n \text{ è dispari} \end{cases}$$

$$\lim_{x\to-\infty} f_n(x) = -\int_{-\infty}^{0} \frac{2}{(1-t)(t-2)^n}\,dt \in \mathbb{R}$$

$\mathrm{ord}_{-\infty} f_n = n+1 \geq 2$

$n = 2 \qquad f_2(-1) = \int_0^{-1} \frac{2}{(1-t)\,(t-2)^2}\,dt = \int_0^{-1} -2[\quad]\,dt = -2\left[\log\frac{|t-1|}{|t-2|} - \frac{1}{t-2}\right]_0^{-1}$

$$\frac{2}{(1-t)(t-2)^2} = -2\,\frac{1}{(t-1)(t-2)^2} = -2\left[\frac{1}{t-1} - \frac{1}{t-2} - \frac{d}{dt}\,\frac{1}{t-2}\right]$$

$$= -2\left[\log 2/3 + \frac{1}{3} - \frac{1}{2}\right] \qquad -\log 2$$

$t^2 - 3t + 2$

$$\frac{a}{t-1} + \frac{b}{t-2} + \frac{d}{dt}\,\frac{c}{t-2} = \frac{a(t-2)^2 + b(t-1)(t-2) - c(t-1)}{(t-1)(t-2)^2}$$

$$-\frac{c}{(t-2)^2}$$

$$2\log\frac{3}{2} + 2\log 2 - -$$

$$at^2 - 4at + 4a + bt^2 - 3bt + 2b - ct + c = 1$$

$$\begin{cases} a + b = 0 \\ -4a - 3b - c = 0 \\ 4a + 2b + c = 1 \end{cases} \qquad \begin{aligned} &b = -a \\ &-a - c = 0 \qquad c = -a \\ &2a + c = 1 \end{aligned} \qquad \begin{aligned} &a = 1 \quad c = -1 \\ &b = -1 \end{aligned}$$