14 Dicembre

Suitable pair (u, p) in $(a,b)\times U$

1) $u \in L^\infty((a,b), L^2(U))$ $\qquad \nabla u \in L^2((a,b)\times U)$

$p \in L^{\frac{3}{2}}((a,b)\times U)$

2) $-\Delta p = \partial_i \partial_j (u_i u_j)$

3) $\forall \; t \in (a,b)$ e $\forall \; \phi \in C^\infty_c((a,b)\times U, [0,+\infty))$

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(t)\, dx + 2\int_a^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi \; dx\, ds \le$$

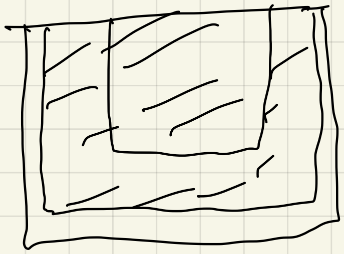
$$\le \int_a^t \int_{\mathbb{R}^3} |u|^2 (\phi_t + \Delta\phi) + \int_a^t \int_{\mathbb{R}^3} (|u|^2 + 2p)\, u\cdot\nabla\phi \; dx\, ds$$

Teor $\exists$ $\varepsilon_0^* > 0$ e $C_M > 0$ t.c. se (u,p) è una suit. pair in $Q_R(t_0, x_0)$ e se

$$R^{-2} \int_{Q_R} \left(|u|^3 + |p|^{\frac{3}{2}}\right) dt\, dx < \varepsilon_0 \quad \text{con}$$

$\varepsilon_0 \in [0, \varepsilon_0^*)$ e con $R > 0$, allora

$$|u|_{L^\infty(Q_{\frac{1}{2}R}(t_0, x_0))} < C_M \; \varepsilon_0^{\frac{1}{3}}$$



Osservazioni

$$\frac{1}{R^2}\int_{Q_R\left(t_0+\frac{R^2}{8},\,x_0\right)} \left(|u|^3+|p|^{\frac{3}{2}}\right) < \varepsilon_0$$

$$\Rightarrow \quad |u|_{L^\infty\left(Q_{R/2}\left(t_0+\frac{R^2}{8},\,x_0\right)\right)} < C'_M\,\varepsilon_0 ,$$

$$Q_{R/2}\left(t_0+\frac{R^2}{8},x_0\right) = \Big(t_0 \underbrace{+\cancel{\frac{R^2}{8}}-\cancel{\frac{R^2}{4}}}_{-\frac{R^2}{8}},\ t_0+\frac{R^2}{8}\Big)\times B\left(x_0,\frac{R}{2}\right)$$

Def Un punto $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^3$ è regolare per una fissata soluzione di Leray Hopf u, se $\exists\ (a,b)\times U$ un intorno di (t_0,x_0) t.c.

$$|u|_{L^\infty((a,b)\times U)}<\infty .$$

S sarà l'insieme di punti singolari (cioè non regolari)

Corollario Sia u una soluzione di Leray-Hopf t.c. (u,p) sia una suitable weak in $\mathbb{R}_+\times\mathbb{R}^3$. Allora S è sottoinsieme limitato di $\mathbb{R}_+\times\mathbb{R}^3$

Dim Per prima cosa $\exists\ T>0$ t.c. in $(T,\infty)\times\mathbb{R}^3$ risulta che u è C^∞.

Questo segue considerando la disug dell'energia

$$|u(t)|^2_{L^2_x} + 2\int_0^t |\nabla u|^2_{L^2_x}\,dt' \le |u_0|^2_{L^2_x}$$

$$|u(t)|_{L^2_x} \leq |u_0|_{L^2_x}$$

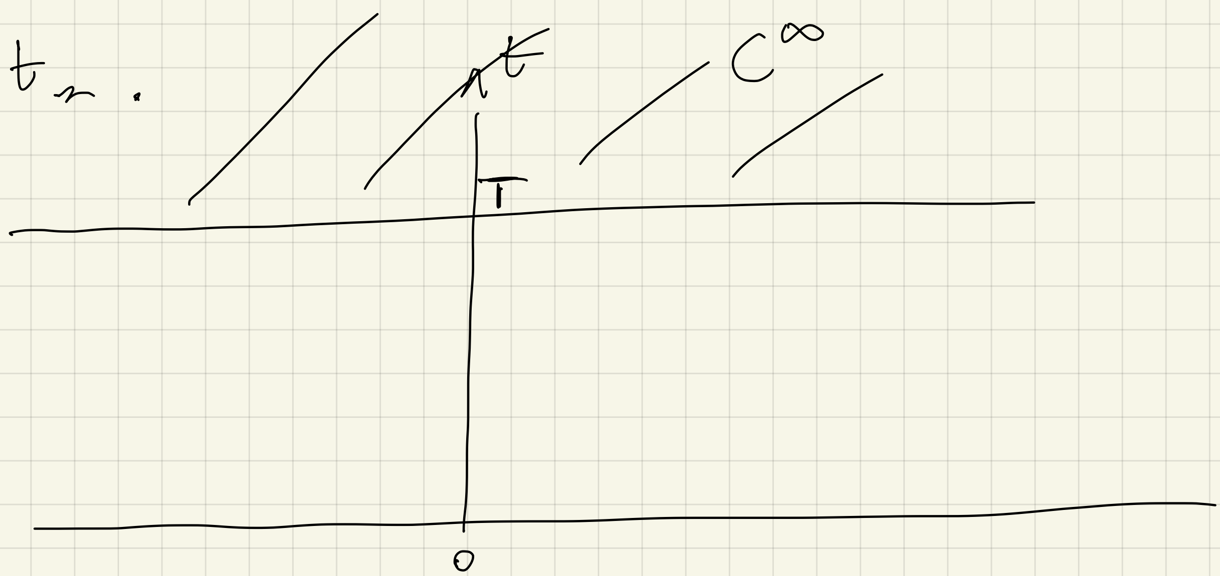
$$2\int_0^{+\infty} |\nabla u|^2_{L^2_x}\, dt' \leq |u_0|^2_{L^2_x} \Rightarrow \exists\ t_n \text{ t.c.}$$

$$|\nabla u(t_n)|_{L^2_x} \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow \quad |u(t_n)|_{L^2_x}\, |\nabla u(t_n)|_{L^2_x} \xrightarrow{n\to\infty} 0$$

$\exists\, t_n$ t.c. $\quad |u(t_n)|_{L^2_x}\, |\nabla u(t_n)|_{L^2_x} \leq \varepsilon_0$

t.c. risulta che $u \in L^\infty([t_n, T], H^1_x)$ $\forall\ T > t_n$ ed inoltre $u \in C^\infty((t_n, T) \times \mathbb{R}^3)$ $\forall\ T > t_n$.



$$S \subset [0,T] \times \mathbb{R}^3$$

vogliamo dimostrare che $\exists\ R_0 > 0$ t.c.

$$S \subset (0,T] \times B(0,R_0)$$

Suppongo che $\quad \forall \ (t,x) \in S$

$$R^{-2} \int_{Q_R\left(t+\frac{R^2}{8},x\right)} \left(|u|^3 + |p|^{\frac{3}{2}}\right) \geq \varepsilon_0^* \qquad \forall \ R>$$

Supponiamo però che $\quad u \in L^{\frac{10}{3}}\left([0,T]\times\mathbb{R}^3\right)$

$\qquad\qquad\qquad\qquad\qquad\quad p \in L^{\frac{5}{3}}\left([0,T]\times\mathbb{R}^3\right)$

$$|u|_{L^3\left(Q_R\left(t+\frac{R^2}{8},x\right)\right)} \leq |u|_{L^{\frac{10}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \left|Q_R\left(t+\frac{R^2}{8},x\right)\right|^{\frac{1}{30}}$$

$$= |u|_{L^{\frac{10}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \ c \ R^{\frac{1}{6}}$$

$$|p|_{L^{\frac{3}{2}}\left(Q_R\left(t+\frac{R^2}{8},x\right)\right)} \leq |p|_{L^{\frac{5}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \left|Q_R\left(t+\frac{R^2}{8},x\right)\right|^{\frac{1}{15}}$$

$$= |p|_{L^{\frac{5}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \ c \ R^{\frac{1}{3}}$$

$$\varepsilon_*^{\frac{10}{9}} \le |u|^{\frac{10}{3}}_{L^3\left(Q_R\left(t+\frac{R^2}{8},x\right)\right)} \le |u|^{\frac{10}{3}}_{L^{\frac{10}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \; c \; R^{\frac{5}{9}}$$

$$\frac{1}{6}\cdot\frac{10}{3} = \frac{5}{9}$$

$$\underbrace{|p|^{\frac{5}{3}}_{L^{\frac{3}{2}}\left(Q_R\left(t+\frac{R^2}{8},x\right)\right)}}_{\varepsilon_*^{\frac{10}{9}} \le} \le |p|^{\frac{5}{3}}_{L^{\frac{5}{3}}\left(Q_R\left(t+\frac{R^2}{2},x\right)\right)} \; c \; R^{\frac{5}{9}}$$

$$|u|^3_{L^3} \ge \varepsilon_* \qquad |u|^{\frac{10}{3}}_{L^3} \ge \varepsilon_*^{\frac{10}{9}}$$

$$|p|^{\frac{3}{2}}_{L^{\frac{3}{2}}} \ge \varepsilon_* \qquad |p|^{\cancel{\frac{3}{2}}\cdot\cancel{\frac{2}{3}}\cdot\frac{5}{3}}_{L^{\frac{3}{2}}} \ge \varepsilon_*^{\frac{10}{9}}$$

$$\int_{Q_R\left(t+\frac{R^2}{8},x\right)} \left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) \ge C \; \varepsilon_*^{\frac{10}{9}} \; R^{-\frac{5}{3}}$$

Fisso R. Se S è illimitato, esiste una successione $(t_n, x_n) \in S$ con $Q_R\left(t_n+\frac{R^2}{8}, x_n\right)$ a due a due disgiunti e contenuti nella regione $(0,2T)\times\mathbb{R}^3$

$$\infty > \int_{(0,2T)\times\mathbb{R}^3} \left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) dt\,dx \ge$$

$$\ge \sum_n \int_{Q_R\left(t_n+\frac{R^2}{8},x_n\right)} \left(|u|^{\frac{10}{3}}+|p|^{\frac{5}{3}}\right) dt\,x$$

$$\geq \sum_n C \, \varepsilon_p^{\frac{10}{3}} R^{-\frac{5}{3}} = +\infty$$

$(a,b) \times U$

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(s) \, dx + 2 \int_a^s \int_{\mathbb{R}^3} |\nabla u|^2 \phi \leq$$

$$\leq \int_a^s \int_{\mathbb{R}^3} |u|^2 (\partial_t + \Delta)\phi + \int_a^s \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \phi$$

$$\forall \, \phi \in C_c^\infty((a,b) \times U)$$

Prop $\exists \, \varepsilon_0^* > 0$ $\exists \, C_M$ t.c. se u è come sopra in $Q_R(t_0,x_0)$ con

$$R^{-2} \int_{Q_R(t_0,x_0)} |u|^3 < \varepsilon_0$$

con $\varepsilon_0 \in [0, \varepsilon_0^*)$, allora $|u|_{L^\infty(Q_{\frac{R}{2}}(t_0,x_0))} < C_M \varepsilon_0^{\frac{2}{9}}$.

Si dimostrerà che $\forall \; Q_\rho(s,a) \subset Q_R(t_0,x_0)$ si ha

$$\rho^{-5} \int_{Q_\rho(s,a)} |u|^3 < \varepsilon_0^{\frac{2}{3}}$$

e da cui si conclude che q.o. $|u(t,x)|^3 \leq \varepsilon_0^{\frac{2}{3}}$ usando la differenziazione di Lebesgue poiché per q.o. (t,x)

$$|u(t,x)|^3 = \lim_{\rho \to 0^+} \rho^{-5} \int_{Q_\rho(t,x)} |u|^3$$

$$\lambda\, u(\lambda^2 t, \lambda x)$$

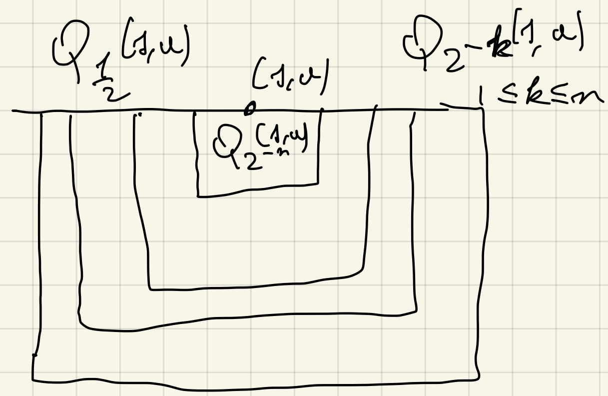
$$\frac{1}{R^2}\int_{Q_R} \qquad\qquad Q_1 = (-1,0)\times B(0,1)$$

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi(s)\, dx + 2\int_{-\infty}^{s}\int_{\mathbb{R}^3} |\nabla u|^2\phi \le$$

$$\le \int_{-\infty}^{s}\int_{\mathbb{R}^3} |u|^2(\partial_t+\Delta)\phi + \int_{-\infty}^{s}\int_{\mathbb{R}^3} |u|^2\, u\cdot\nabla\phi$$

Dimostreremo $\forall (s,a)\in Q_{\frac12}$

$$2^{5n}\int_{Q_{2^{-n}}(s,a)} |u|^3\, dt\, dx < \varepsilon_0^{\frac23} \qquad (14.7_n)$$



$$\phi_n \qquad (\partial_t+\Delta)\phi_n \simeq 0$$

$$\phi_n \sim 2^n \quad \text{in } Q_{2^{-n}}(s,a)$$

$$|\nabla\phi_n| \lesssim \begin{cases} C\, 2^{2n} & \text{in } Q_{2^{-n}}(s,a) \\ C\, 2^{-2n}\, 2^{4k} & \text{in } Q_{2^{k}}(s,a)\setminus Q_{2^{k+1}}(s,a) \end{cases}$$

Supponendo per induzione che

$$2^{5k}\int_{Q_{2^{-k}}(s,a)} |u|^3 < \varepsilon_0^{\frac23} \qquad \forall\ 1\le k\le n$$

$$\int_{\mathbb{R}^3} |u(t)|^2 \phi_n(t)\,dx + 2\int_{-\infty}^{t}\int_{\mathbb{R}^3} |\nabla u|^2 \phi_n \le$$

$$\le \int_{-\infty}^{t}\int_{\mathbb{R}^3} \cancel{|u|^2(\partial_t+\Delta)\phi} + \int_{-\infty}^{t}\int_{\mathbb{R}^3} |u|^2\, u\cdot\nabla\phi_n$$

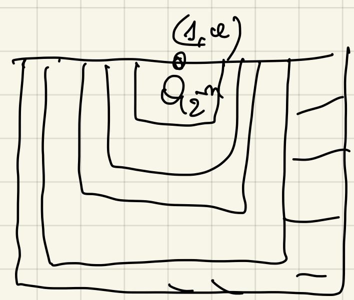
$\phi_n \sim 2^n$ in $Q_{2^{-n}}(1,a)$

$$|\nabla\phi_n| \lesssim \begin{cases} C\,2^{2n} & \text{in } Q_{2^{-n}}(1,a) \\ C\,2^{-2n}\,2^{4k} & \text{in } Q_{2^{k}}(1,a)\setminus Q_{2^{k+1}}(1,a) \end{cases}$$

$t \in (1-2^{-2n}, 1)$

$$\int_{B_{2^{-n}}(a)} |u(t)|^2\, 2^n + 2\int_{Q_{2^{-n}}(1,a)\cap\{t'<t\}} |\nabla u|^2\, 2^n$$

$$\le \sum_{k=1}^{n-1}\int_{Q_{2^{-k}}(1,a)\setminus Q_{2^{-(k+1)}}(1,a)} |u|^3\, \overset{2^{-2n}2^{4k}}{|\nabla\phi|} + \int_{Q_{2^{-n}}(1,a)} |u|^3\, \overset{2^{2n}}{|\nabla\phi|}$$



$$\int_{B_{2^{-n}}(a)} |u(t)|^2\, 2^n + 2\int_{Q_{2^{-n}}(1,a)\cap\{t'<t\}} |\nabla u|^2\, 2^n \lesssim 2^{-2n}\,\varepsilon_0^{\frac{2}{3}}$$

$$\le \sum_{k=1}^{n-1} 2^{-2n}\,2^{4k} \underbrace{\int_{Q_{2^{-k}}(1,a)} |u|^3}_{<\,2^{-5k}\varepsilon_0^{\frac{2}{3}}} + 2^{2n}\underbrace{\int_{Q_{2^{-n}}(1,a)} |u|^3}_{<\,2^{-5n}\varepsilon_0^{\frac{2}{3}}}$$

$$\leq \varepsilon_0^{\frac{2}{3}} \, 2^{-2n} \underbrace{\sum_{k=1}^{n} 2^{-k}}_{<1} < \varepsilon_0^{\frac{2}{3}} \, 2^{-2n}$$

$$2^n \int_{B_{2^{-n}}(a)} |u(t)|^2 + 2 \, 2^n \int_{Q_{2^{-n}}(s,a) \cap \{t' < t\}} |\nabla u|^2 < \varepsilon_0^{\frac{2}{3}} \, 2^{-2n}$$

$$t \in (s - 2^{-2(n+1)}, s) \qquad B_{2^{-(n+1)}}(a)$$

$$2^n \int_{B_{2^{-(n+1)}}(a)} |u(t)|^2 + 2 \, 2^n \int_{Q_{2^{-(n+1)}}(s,a) \cap \{t' < t\}} |\nabla u|^2 < \varepsilon_0^{\frac{2}{3}} \, 2^{-2n}$$

$$2^{n+1} \int_{B_{2^{-(n+1)}}(a)} |u(t)|^2 + 2 \, 2^{n+1} \int_{Q_{2^{-(n+1)}}(s,a) \cap \{t' < t\}} |\nabla u|^2 < 2^3 \varepsilon_0^{\frac{2}{3}} \, 2^{-2(n+1)}$$

Vale una stima di Sobolev, che garantisce

$$2^{2(n+2)} \int_{Q_{2^{-n-1}}(s,a)} |u|^3 \leq C_0 \Big[2^{+n+1} \sup_{\substack{t \\ \uparrow \\ (s-2^{-2n-2}, s)}} \int_{B_{2^{-n-1}}(s,a)} |u(t)|^2 + 2^{+n+1} \int_{Q_{2^{-n-1}}(s,a)} |\nabla u|^2 \Big]^{\frac{3}{2}}$$

$$\leq C_0 \Big[2^3 \varepsilon_0^{\frac{2}{3}} \, 2^{-2(n+1)} \Big]^{\frac{3}{2}} =$$

$$= C_0 \, 2^{\frac{9}{2}} \, \varepsilon_0 \, 2^{-3(n+1)} < \varepsilon_0^{\frac{2}{3}} \, 2^{-3(n+1)}$$

$$C_0 \, 2^{\frac{9}{2}} \, \varepsilon_0^{\frac{1}{3}} < 1$$