14 dicembre

$$\lim_{x\to 0^+} \frac{\log(1+x^a+x^{2a})-\operatorname{th}x}{\int_0^{x^2}\sin\left(\frac{1}{t}\right)dt+1-\cos x}$$

$F(x)=\int_0^x \sin\left(\frac{1}{t}\right)dt$. $\quad F'(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & \text{per } x\neq 0\\ 0 & x=0\end{cases}$

$F(0)=0$

$F(x)=\underset{0}{\underset{\|}{F(0)}}+\underset{0}{\underset{\|}{F'(0)}}x+o(x)=o(x)$

$F(x^2)=o(x^2)$

denominatore $\quad F(x^2)+1-\cos x=o(x^2)+\frac{x^2}{2}+o(x^2)=\frac{x^2}{2}+o(x^2)$

$=\frac{x^2}{2}\left(1+o(1)\right)$

In altre parole il nostro limite coincide con

$$\lim_{x\to 0^+}\frac{\log(1+x^a+x^{2a})-\operatorname{th}x}{\frac{x^2}{2}}=:L_a \qquad a>0$$

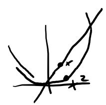
$\operatorname{th}(x)=x+o(x^2)$

$\log(1+y)=y-\frac{y^2}{2}+o(y^2) \qquad \log(1+y)=y+o(y)=x^a+x^{2a}+o(x^a)$

$\log(1+x^a+x^{2a})=x^a+x^{2a}-\frac{(x^a+x^{2a})^2}{2}+\underbrace{o\left((x^a+x^{2a})^2\right)}_{o(x^{2a})}$

$x^a+x^{2a}=x^a\,(1+o(1))$

$o\left((x^a+x^{2a})^2\right)=o\left(x^{2a}\cancel{(1+o(1))}\right)=o(x^{2a})$



$=x^a+x^{2a}-\frac{x^{2a}}{2}+o(x^{2a})=x^a+\frac{1}{2}x^{2a}+o(x^{2a})$

numeratore $=\log(1+x^a+x^{2a})-\operatorname{th}(x)=x^a+\frac{x^{2a}}{2}-x+o(x^{2a})+o(x^2)$

numeratore $=\begin{cases}x^a+o(x^a) & \text{se } 0<a<1\\ -x+o(x) & \text{se } a>1\end{cases}$

Se $a=1$

numeratore $=\frac{x^2}{2}+o(x^2)$

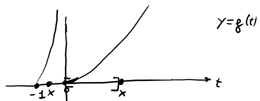
$$L_a=\lim_{x\to 0^+}\frac{\text{num}}{\frac{x^2}{2}}=\begin{cases}\lim_{x\to 0^+}\frac{2x^a}{x^2}=+\infty & 0<a<1\\ \lim_{x\to 0^+}\frac{-2x}{x^2}=-\infty & a>1\\ \lim_{x\to 0^+}\frac{x^2/2}{x^2/2}=1 & \end{cases}$$

Studiare $f(x)=\int_0^x \underbrace{e^{-\frac{1}{t}}(t+1)}_{g(t)}\,dt$

Prima cosa, trovare il dominio di f.

Il dominio è formato dagli x t.c. nell'intervallo di estremi 0 e x la funzione $g(t)$ è integrabile.

$g \in C^\infty(\mathbb{R}\setminus\{0\})$ $\qquad g(t)=e^{-\frac{1}{t}}(t+1)$



$\lim_{t\to 0^+} g(t)=0$

$\lim_{t\to 0^-} g(t)=+\infty$

$g\in L[0,x] \quad \forall x>0 \Rightarrow (x>0 \Rightarrow x\in \mathrm{Dom} f)$

Se invece $x<0$ $\quad g\notin L[x,0)$ per confronto asintotico

$g(t)=e^{-\frac{1}{t}}(t+1)$ con $\frac{1}{|t|}\notin L[x,0)$

$$\lim_{t\to 0^-}\frac{g(t)}{\frac{1}{|t|}}=\lim_{t\to 0^-}\frac{e^{-\frac{1}{t}}\cancel{(t+1)}}{-\frac{1}{t}} \qquad y=-\frac{1}{t}$$

$$=\lim_{y\to+\infty}\frac{e^y}{y}=+\infty \Rightarrow g\notin L[x,0) \Rightarrow (x<0 \Rightarrow x\notin \mathrm{Dom} f)$$

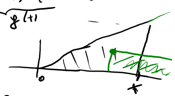
$\mathrm{Dom}\, f=[0,+\infty)$

$f(x)=\int_0^x \underbrace{e^{-\frac{1}{t}}(t+1)}_{g(t)}\,dt \qquad g(t)\geq 0$

$\lim_{t\to+\infty} g(t)=+\infty$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\int_0^x \underbrace{(1+o(1))\,(t+1)}_{g(t)}\,dt$$

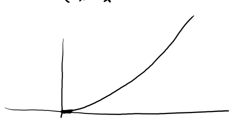
$$=+\infty$$



$f'(x)=e^{-\frac{1}{x}}(x+1)>0 \qquad \forall x>0$

$$f''(x)=\left(e^{-\frac{1}{x}}\right)'(x+1)+e^{-\frac{1}{x}}=(x+1)\,e^{-\frac{1}{x}}\frac{1}{x^2}+e^{-\frac{1}{x}}=$$

$$=e^{-\frac{1}{x}}\left(\frac{1}{x}+\frac{1}{x^2}+1\right)>0 \qquad x>0$$



$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{\int_0^x e^{-\frac{1}{t}}(t+1)\,dt}{x} \qquad \frac{+\infty}{+\infty}$$

$$=\lim_{x\to+\infty} e^{-\frac{1}{x}}(x+1)=+\infty$$ non c'è retta asintotica

$$f(x)=\begin{cases}\dfrac{\int_0^x e^{[t]}\,dt-2x}{\int_0^x t^t\,dt} & x>0\\ -1 & x=0\\ e^{\frac{1}{x}}-1 & x<0\end{cases}$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\left(e^{\frac{1}{x}}-1\right)=0$$

$$\lim_{x\to0^-}\left(e^{\frac{1}{x}}-1\right)=-1=f(0)$$

$$\lim_{x\to0^+}\frac{\int_0^x e^{[t]}\,dt-2x}{\int_0^x t^t\,dt}=\lim_{x\to0^+}\frac{\overbrace{x-2x}^{-x}}{\int_0^x t^t\,dt}=\lim_{x\to0^+}\frac{-1}{x^x}$$

$$=-\lim_{x\to0^+} e^{-x\lg x}=-e^0=-1=f(0)$$

$$\lim_{x\to0^+} x\lg x=\lim_{x\to0^+}\frac{\lg x}{\frac{1}{x}}=\lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=$$

$$=-\lim_{x\to0^+} x=0$$

Quindi f continua in $\mathbb{R}$

$$\lim_{x\to+\infty}\frac{\int_0^x e^{[t]}\,dt-2x}{\int_0^x t^t\,dt}=0 \qquad (3)$$

1) $$\lim_{x\to+\infty}\frac{\int_0^x e^{[t]}\,dt}{\int_0^x t^t\,dt}=0$$

2) $$\lim_{x\to+\infty}\frac{2x}{\int_0^x t^t\,dt}=0 \qquad (1)+(2)\Rightarrow(3)$$

$$=\lim_{x\to+\infty}\frac{2}{x^x}=0$$

1) $$\lim_{x\to+\infty}\frac{\int_0^x e^{[t]}\,dt}{\int_0^x t^t\,dt} \qquad \frac{\infty}{\infty}$$

$[t]>t-1$

$e^{[t]}> e^{t-1}$

$\int_0^x e^{[t]}\,dt\geq\int_0^x e^{t-1}\,dt$

$= e^{t-1}\Big]_0^x=$

$e^{[t]}\leq e^t$

$$0<\frac{\int_0^x e^{[t]}\,dt}{\int_0^x t^t\,dt}\leq\frac{\int_0^x e^t\,dt}{\int_0^x t^t\,dt} \qquad \frac{\infty}{\infty} \qquad = e^{x-1}-e^{-1}\xrightarrow[x\to+\infty]{}+\infty$$

0

$$\lim_{x\to+\infty}\frac{\int_0^x e^t\,dt}{\int_0^x t^t\,dt}=\lim_{x\to+\infty}\frac{e^x}{x^x}=0$$

0

$$0<\frac{e^x}{x^x}=\left(\frac{e}{x}\right)^x< \qquad \text{per} \quad x>e^2$$

$$<\left(\frac{e}{e^2}\right)^x=\left(\frac{1}{e}\right)^x=\frac{1}{e^x}\xrightarrow{x\to+\infty}0$$

per $x<0$ $\qquad f(x)=e^{\frac{1}{x}}-1 \qquad f'(x)=-\frac{1}{x^2}e^{\frac{1}{x}}$

$\lim_{x\to-\infty} f(x)=0$

$f_1'(0)=0$

$$f''(x)=\frac{1}{x^4}e^{\frac{1}{x}}+\frac{2}{x^3}e^{\frac{1}{x}}=\frac{e^{\frac{1}{x}}}{x^4}(1+2x)$$

$1+2x=0 \Leftrightarrow x=-\frac{1}{2}$ è un flesso

$f''(x)<0$ per $x<-\frac{1}{2}$, $f''(x)>0$ ~~per~~ $-\frac{1}{2}<x<0$

$f(0)=-1$

$$f_1'(0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x}=\lim_{x\to0^-}\frac{e^{\frac{1}{x}}-\cancel{1}+\cancel{1}}{x}=\lim_{x\to0^-}\frac{e^{\frac{1}{x}}}{x}=0$$

$y=-\frac{1}{x}$

$$=\lim_{y\to+\infty}\frac{e^{-y}}{-\frac{1}{y}}=\lim_{y\to+\infty}-\frac{y}{e^y}=0$$

per $x \in (0,1)$ $\qquad f(x) = \dfrac{-x}{\int_0^x t^t \, dt}$

$$f'(x) = \frac{-\int_0^x t^t \, dt + x \, x^x}{\left(\int_0^x t^t \, dt\right)^2}$$

$$f'_d(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} \overset{?}{=} \lim_{x \to 0^+} f'(x)$$

$$t^t = e^{t \lg t} = 1 + t \lg t + o(t \lg t)$$

$$\text{num} = x \, x^x - \int_0^x t^t \, dt = x \left(1 + x \lg x + o(x \lg x)\right) -$$

$$- \int_0^x \left(1 + t \lg t + o(t \lg t)\right) dt =$$

$$= \cancel{x} + x^2 \lg x + x \, o(x \lg x) - \cancel{x} - \int_0^x t \lg t + \int_0^x o(t \lg t) \, dt$$

$$\int_0^x t \lg t = \int_0^x \left(\frac{t^2}{2}\right)' \lg t \, dt = \frac{x^2}{2} \lg x - \frac{1}{2} \int_0^x t^{\cancel{2}} \, \frac{1}{\cancel{t}} \, dt$$

$$= \frac{x^2}{2} \lg x - \frac{x^2}{2}$$

Numeratore $= \frac{x^2}{2} \lg x + o(x^2 \lg x)$

$$\text{den} = \left(\int_0^x t^t \, dt\right)^2 = \left(\int_0^x dt \, (1 + o(1))\right)^2 = x^2 + o(x^2)$$

$$f'(x) = \frac{\frac{x^2}{2} \lg x + o(x^2 \lg x)}{x^2 + o(x^2)} = \frac{\frac{1}{2} \lg x + o(\lg x)}{1 + o(1)} \xrightarrow[x \to 0^+]{} -\infty$$

$f'_d(0)$ non esiste