15 dicembre

Fujita

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1} u \\ u(0) = \varphi \in C_c^\infty(\mathbb{R}^d, [0, +\infty)) \end{cases}$$

$1 < p < p(d)$

$1 + \frac{2}{d}$

Keel Teo — Disuguaglianze di Strichartz

$$\| e^{it\Delta} \|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \leq \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{|t|^{\frac{d}{2}}}$$

$$\| e^{it\Delta} \|_{L^2(\mathbb{R}^d) \supseteq} = 1$$

$$\| e^{it\Delta} u_0 \|_{L^r(\mathbb{R}, L^q(\mathbb{R}^d))}$$

$d = 3$

$(r, q) = (0, \infty)$!

$(r, q) = (2, 6)$

$u_0 \in \dot{H}^{\frac{1}{2}}$

$C^0([0, T], \dot{H}^{\frac{1}{2}}) \longrightarrow \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$

$\Downarrow$

$C^0([0, T], L^3(\mathbb{R}^3)) \longrightarrow L^3(\mathbb{R}^3)$

$$\boxed{\dot{H}^{\frac{1}{2}} \subset L^3(\mathbb{R}^3)} \subset \ldots\ldots \subset \dot{B}^{-1}_{\infty\,\infty}$$

$\Uparrow$

Koch-Tataru

Lemma $\exists\, C_0 > 0$ t.c. $\forall\, (s,a) \in \mathbb{R} \times \mathbb{R}^3$ e $\forall\, r > 0$

se $\quad u \in L^\infty((s-r^2, s), L^2(B_r(a)))$

$\quad \nabla u \in L^2(Q_r(s,a))$

$$r^{-2} \int_{Q_r(s,a)} |u|^3 \, dx \le C_0 \left[r^{-1} \sup_{s-r^2<t<s} \int_{B_r(a)} |u(t)|^2 \, dx + r^{-1} \int_{Q_r(s,a)} |\nabla u|^2 \, dx\, dt \right]^{3/2}$$

Dim $\quad r = 1 \qquad (s,a) = (0,0).$

$$|u|_{L^3(B_1)} \le |u|^{\frac12}_{L^6(B_1)} \, |u|^{\frac12}_{L^2(B_1)}$$

$$\frac13 = \frac{\frac12}{6} + \frac{\frac12}{2}$$

$$\frac16 = \frac12 - \frac13$$

$$|u|_{L^6(B_1)} \le C_0 \, |u|_{L^2(B_1)} + C_0 \, |\nabla u|_{L^2(B_1)}$$

$$|u|_{L^3(B_1)} \le C_0^{\frac12} \left(|u|^{\frac12}_{L^2(B_1)} + |\nabla u|^{\frac12}_{L^2(B_1)} \right) |u|^{\frac12}_{L^2(B_1)}$$

$$= C_0^{\frac12} \left(|u|_{L^2(B_1)} + |\nabla u|^{\frac12}_{L^2(B_1)} |u|^{\frac12}_{L^2(B_1)} \right)$$

$$\int_{B_1} |u|^3 \, dx \le C_0^{\frac32} \left(|u|_{L^2(B_1)} + |\nabla u|^{\frac12}_{L^2(B_1)} |u|^{\frac12}_{L^2(B_1)} \right)^3$$

$$\boxed{\int_{B_1} |u|^3 \, dx \le 4\, C_0^{\frac32} \left(|u|^3_{L^2(B_1)} + |\nabla u|^{\frac32}_{L^2(B_1)} |u|^{\frac32}_{L^2(B_1)} \right)}$$

$$(\alpha+\beta)^q \le 2^{q-1} (\alpha^q + \beta^q) \qquad q \ge 1$$

$$\left(\frac{\alpha+\beta}{2}\right)^q \le \frac12 \alpha^q + \frac12 \beta^q \qquad \alpha \longrightarrow \alpha^q$$

$$\int_{-1}^{0} dt$$

$$\int_{Q_1} |u|^3 \, dx\, dt \le 4 C_0^{\frac32} \int_{-1}^{0} |\nabla u|^{\frac32}_{L^2(B_1)} |u|^{\frac32}_{L^2(B_1)} \, dt \; +$$

$$+ \, 4 C_0^{\frac32} \int_{-1}^{0} |u|^3_{L^2(B_1)} \, dt$$

$$\le 4 C_0^{\frac32} \left\| \, |\nabla u|^{\frac32}_{L^2(B_1)} \right\|_{L^{\frac43}(-1,0)} \left\| \, |u|^{\frac32}_{L^2(B_1)} \right\|_{L^4(-1,0)}$$

$$+ \, 4 C_0^{\frac32} \left(\sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac32}$$

$$\le 2 C_0^{\frac34}\sqrt{2} \, |\nabla u|^{\frac32}_{L^2(Q_1)} \sqrt{2}\, C_0^{\frac34} \left(\sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac34} + 4 C_0^{\frac32} \left(\sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac32}$$

$$\le C_0^{\frac32} \, 2 \, |\nabla u|^3_{L^2(Q_1)} + 6 C_0^{\frac32} \left(\sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac32} +$$

$$\le 6 C_0^{\frac32} \left(|\nabla u|^{2\,\frac32}_{L^2(Q_1)} + \left(\sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac32} \right)$$

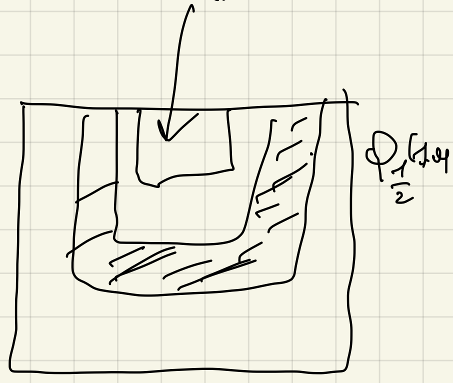
$$\le 6 C_0^{\frac32} \left(|\nabla u|^2_{L^2(Q_1)} + \sup_{-1<t<0} |u|^2_{L^2(B_1)} \right)^{\frac32}$$

$$\alpha^q + \beta^q \le (\alpha+\beta)^q \qquad q \ge 1$$

$$\left(\frac{\alpha}{\alpha+\beta}\right)^q + \left(\frac{\beta}{\alpha+\beta}\right)^q \le \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1$$

$$\phi_n \sim 2^n \qquad Q_{2^{-n}}(1,a)$$

$$|\nabla \phi_n| \lesssim \begin{cases} 2^{2n} & Q_{2^{-n}}(1,a) \\ 2^{4k} \ge 2^{2n} & Q_{2^{-k}}(1,a) \setminus Q_{2^{-(k+1)}}(1,a) \end{cases}$$



$$(\partial_t + \Delta)\phi_n = 0$$

Lemma $\exists C_1 > 0$ t.c. $\forall (s,a) \in \mathbb{R}^4$ $\exists$

$\phi_n \in C_c^\infty\left(\left(s - \frac{1}{9}, s\right] \times B_{\frac{1}{3}}(a), [0, +\infty)\right)$ t.c.

(i) $\frac{1}{C_1} 2^n < \phi_n < C_1 2^n$ e $|\nabla \phi_n| < C_1 2^{2n}$ in $Q_{2^{-n}}(s,a)$

(ii) $\phi_n \leq C_1 2^{-2n} 2^{3k}$ in $Q_{2^{-(k-1)}}(s,a) \setminus Q_{2^{-k}}(s,a)$

$|\nabla \phi_n| \leq C_1 2^{-2n} 2^{4k}$

(iii) $\operatorname{supp} \phi_n \cap \left((-\infty, s] \times \mathbb{R}^3 \right) \subset \overline{Q_{\frac{1}{3}}(s,a)}$

(iv) $|(\partial_t + \Delta) \phi_n| \leq C_1 2^{-2n}$ in $(-\infty, s] \times \mathbb{R}^3$

$(s,a) = (0,0)$

$$\phi_n(t,x) = 2^{-2n} \vartheta_n(t,x) = 2^{-2n} \chi_n(t,x) \, \Psi_n(t,x)$$

$(\partial_t + \Delta) \Psi_n (t,x) = 0$ $\quad t < 2^{-2n}$

$\Psi_n |_{t = 2^{-2n}} = \delta(x)$

Noi sappiamo già che

$$\begin{cases} (\partial_t - \Delta) K_t(x) = 0 & t > 0 \\ K_t(x) |_{t=0} = \delta(x) \end{cases} \qquad K_t(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$$

$K_{-t}(x)$

$$\begin{cases} (\partial_t + \Delta)\, K_{-t}(x) = 0 & t < 0 \\ K_{-t}(x)\big|_{t=0} = \delta(x) \end{cases}$$

$$\psi_n(t,x) = K_{2^{-2n}-t}(x) = (4\pi)^{-\frac{3}{2}} \left(2^{-2n} - t\right)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}$$

$$\psi_n(t,x) \geq (8\pi)^{-\frac{3}{2}} e^{-\frac{1}{4}}\ 2^{3n}\ \varphi_{2^{-n}}$$

$$\boxed{\psi_n(t,x) \leq (4\pi)^{-\frac{3}{2}}\ 2^{3n}\ \varphi_{2^{-n}}}$$

$$\psi_n(t,x) = (4\pi)^{-\frac{3}{2}} \left(2^{-2n} + |t|\right)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}}$$

$$\leq (4\pi)^{-\frac{3}{2}}\ 2^{3n}$$

$$\psi_n(t,x) \geq (4\pi)^{-\frac{3}{2}}\ 2^{-\frac{3}{2}}\ 2^{3n}\ e^{-\frac{\cancel{2^{-2n}}}{4\,\cancel{2^{-2n}}}}$$

$$e^{-\frac{1}{4}}$$

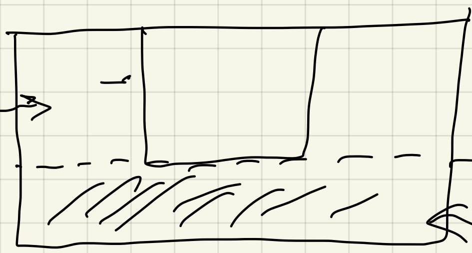
$$\psi_n(t,x) = K_{2^{-2n}-t}(x) = (4\pi)^{-\frac{3}{2}} \left(2^{-2n} - t\right)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}-t)}}$$

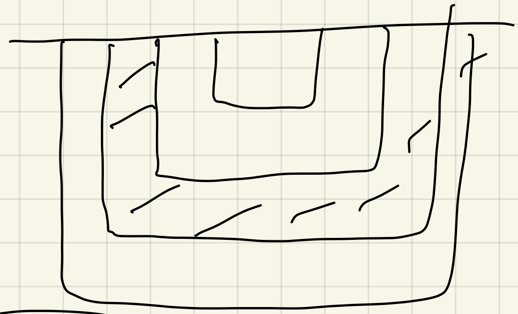
$$\nabla \psi_n(t,x) = -\cancel{2}\,(4\pi)^{-\frac{3}{2}} \left(2^{-2n} - t\right)^{-\frac{5}{2}}\ \cancel{x}\ e^{-\frac{|x|^2}{4(2^{-2n}-t)}}$$

$$|\nabla \psi_n(t,x)| \leq C\ 2^{5n}\ 2^{-n} = C\ 2^{4n}$$

$Q_{2^{-(k-1)}} \setminus Q_{2^{-k}} =$

$=$





$(-2^{-2(k-1)}, -2^{-2k}) \times B_{2^{-(k-1)}}$

$[-2^{-2k}, 0) \times (B_{2^{-(k-1)}} \setminus B_{2^{-k}})$

$$|\Psi_n(t,x)| = (4\pi)^{-\frac{3}{2}} (2^{-2n} + |t|)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4(2^{-2n}+|t|)}}$$

$$\le (2^{-2n} + 2^{-2k})^{-\frac{3}{2}} \quad \cancel{e}$$

$$\le 2^{3k}$$

$$\phi_n \le 2^{-2n} \, 2^{3k}$$

$$|\Psi_n(t,x)| \le (2^{-2n} + |t|)^{-\frac{3}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}}$$

$$= 2^{3k} \left(\frac{2^{-2k}}{2^{-2n} + |t|} \right)^{\frac{3}{2}} e^{-\frac{2^{-2k}}{4(2^{-2n}+|t|)}}$$

$$\le 2^{3k} \underbrace{\sup_{\alpha \ge 0} \alpha^{\frac{3}{2}} e^{-\frac{\alpha}{4}}}_{C}$$