$$f(x) = \frac{(1+x)^{\alpha} - \log_{\alpha}(1+x) - 1}{x^2}$$

i) Si verifichi che esiste uno ed un solo $\alpha_0 \in \mathbb{R}^+$ tale che $\alpha_0 \cdot \log(\alpha_0) = 1$

Poniamo $h(\alpha) = \alpha \log \alpha$, si ha $h(1) = 0$ $\quad h(2) = 2 \log 2 > 1$

perché $\log 2 > \frac{1}{2}$
$2 > \sqrt{e}$
$4 > e$

Per il teorema di connessione, essendo h continua su $[1,2]$, esiste $\alpha_0 \in [1,2]$ tale che $h(\alpha_0) = 1$.

L'unicità si può verificare osservando che $h'(\alpha) = \log \alpha + 1 \geq 1$ se $\alpha \geq 1$: quindi h è crescente su $[1, +\infty[$; inoltre, se $\alpha \in]0,1]$ si ha $\alpha \log \alpha \leq 0$.

ii) $\lim_{x\to 0} \dfrac{(1+x)^{\alpha} - \log_{\alpha}(1+x) - 1}{x^2}$ $\qquad \alpha \neq \alpha_0 \qquad \alpha \cdot \log \alpha = 1$

$$\lim_{x\to 0} \underbrace{\frac{(1+x)^{\alpha}-1}{x}}_{\alpha} \cdot \frac{1}{x} - \underbrace{\frac{\log_{\alpha}(1+x)}{x}}_{\frac{1}{\ln\alpha}} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{1}{x}\left[\frac{(1+x)^{\alpha}-1}{x} - \frac{\log_{\alpha}(1+x)}{x}\right] \begin{matrix} -\infty \\ +\infty \end{matrix}$$

$$\downarrow$$

$$\alpha - \frac{1}{\ln\alpha} \neq 0$$

$\alpha = \alpha_0 \qquad \alpha = \dfrac{1}{\log\alpha}$

$$\lim_{x\to 0} \frac{(1+x)^{\frac{1}{\log\alpha}} - \log_{\frac{1}{\log\alpha}}(1+x) - 1}{x^2} = \frac{\frac{1}{\log\alpha}(1+x)^{\overset{1}{\frac{1}{\log\alpha}-1}} - \overset{1}{\frac{1}{(1+x)}} \frac{1}{\log\left(\frac{1}{\log\alpha}\right)}}{2x} = \lim_{x\to 0} \frac{\alpha \cdot (1+x)^{\alpha-1} - \frac{1}{1+x} \cdot \alpha}{2x} =$$

$$\lim_{x\to 0} \qquad = \qquad = \lim_{x\to 0} \frac{\alpha \cdot \alpha^{-1} \cdot (1+x)^{\alpha-2} - \log(1+x)}{2}$$

$$\lim_{x \to \infty} \left[\frac{(1+x)^{\alpha} - 1}{x} - \frac{\log_{\alpha}(1+x) - 1}{x} \right] \frac{1}{x}$$

$$\alpha = \frac{1}{\log \alpha}$$

$\alpha = \alpha_0$

$\alpha \in \,]1,2[$

$\alpha = 2$

$$\lim_{x \to +\infty} \frac{x^{\alpha}}{x^2} \left[\left(\left(\frac{1}{x}+1\right)^{\alpha} - \frac{1}{x^{\alpha}} \right) - \left[\frac{\log_{\alpha}(1+x) - 1}{x^{\alpha}} \right] \right)$$

$\left(\frac{1}{x}+1\right)^{\alpha} \to 1$, $\quad \frac{1}{x^{\alpha}} \to 0$, $\quad \frac{\log_{\alpha}(1+x) - 1}{x^{\alpha}} \to 0$

$$x^{\alpha - 2} \left(\underset{\downarrow}{u} - 0 \right) \to 0$$

$u \to 1$

$\alpha - 2 < 0$

Es 1 dato $f(x) = 2^x$

9/Gen./2018

i) $\forall n \in \mathbb{N}^+$ calcolare derivata ordine n di f

$$f(x) = 2^x$$
$$f'(x) = 2^x \cdot \log 2$$
$$f''(x) = 2^x \log 2 \cdot \log 2 = 2^x (\log 2)^2 \quad \rightsquigarrow \quad f^n(x) = 2^x \cdot (\log 2)^n$$

- Passo base

ii) Retta tangente in $(x_0, f(x_0))$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$f'(x) = 2^x \cdot \log 2$$

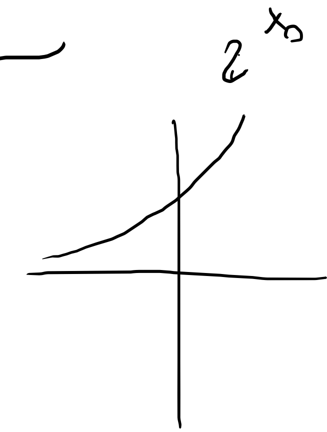
$$y = f'(x_0)(x - x_0) + f(x_0)$$

$$y = 2^{x_0} \cdot \log(2) \cdot x + 2^{x_0}(1 - \log 2 \cdot x_0)$$

$$y = 2^{x_0} \cdot \log(2)(x - x_0) + 2^{x_0}$$

iii) Pass. per origine

$$y = 2^{x_0}\cdot \log(2)\cdot x + \underbrace{2^{x_0}(1-\log(2)x_0)}_{0}$$



Impongo

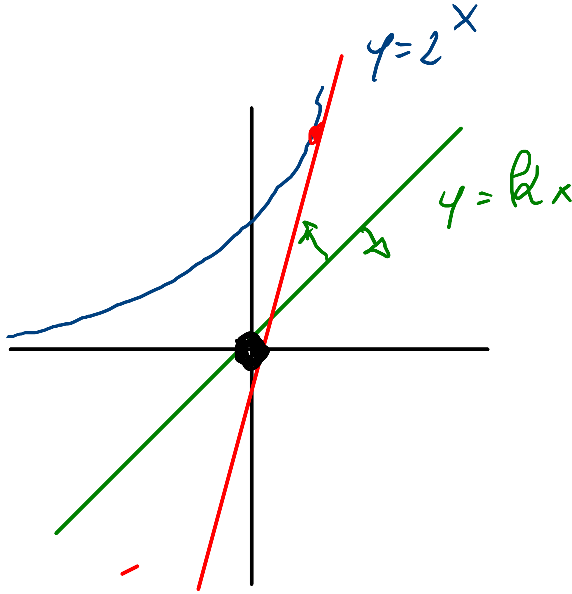
$$f = mx + q$$

$$\underset{\neq 0}{2^{x_0}}(1-\log 2\cdot x_0) = 0$$

$$1-\log 2\cdot x_0 = 0 \rightsquigarrow x_0 = \frac{1}{\log 2} = \log_2 e$$

iv) variare di $k \in \mathbb{R}$ numero delle soluzioni

$$2^x = kx$$



- $x_0 = \frac{1}{\log 2} = \log_2 e$
- $y = 2^{x_0} \cdot \log(2) \cdot x + \cancel{2^{x_0}\left(1 - \log(2) \cdot x_0\right)}$

Sostituendo x_0 otteniamo

$$y = \underbrace{2^{\log_2 e}}_{e} \cdot \log(2) \cdot x$$

$$y = e \cdot \log(2)\, x$$

$y = e \log 2 \cdot x$

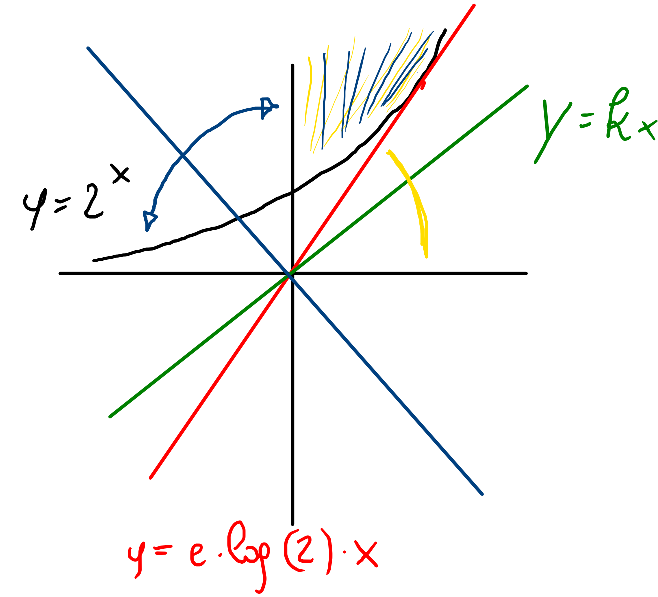
Determiniamo n° delle soluzioni:

conv. $k = e \cdot \log 2 \rightsquigarrow 1$ SOLUZIONE

ord sopranzale. $k \in \left] e \cdot \log 2 , +\infty \right[\rightsquigarrow 2$ SOLUZIONI

$k \in \left[0 , e \cdot \log 2 \right[\rightsquigarrow 0$ SOLUZIONI

$k < 0 \rightsquigarrow 1$ SOLUZIONE



② $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \frac{1}{3}\sqrt[3]{(3x-2)(3x+4)^2}$

I) $\lim_{x\to-\infty} \frac{1}{3}\sqrt[3]{(3x-2)(3x+4)^2} = \lim_{x\to-\infty} \frac{x}{3}\sqrt[3]{\left(3-\frac{2}{x}\right)\left(3+\frac{4}{x}\right)^2} = -\infty$

$\lim_{x\to+\infty}$ " $= \lim_{x\to+\infty}$ " $= +\infty$

II) asintoti a $+\infty$ e a $-\infty$ per f

$$m = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{\frac{1}{3}\sqrt[3]{(3x-2)(3x+4)^2}}{x} \quad \lim_{x\to\pm\infty} \frac{\cancel{x}}{3}\frac{\sqrt[3]{\left(3-\frac{2}{x}\right)\left(3+\frac{4}{x}\right)^2}}{\cancel{x}} = \frac{1}{3}\cdot 3 = 1$$

$$q = \lim_{x\to\pm\infty}(f(x)-x) = \lim_{x\to\pm\infty}\left[\frac{x}{3}\sqrt[3]{\left(3-\frac{2}{x}\right)\left(3+\frac{4}{x}\right)^2} - x\right] = \qquad x = \frac{1}{y} \qquad y \to 0$$

$$\lim_{y\to 0}\left[\frac{1}{3y}\sqrt[3]{(3-2y)(3+4y)^2} - \frac{1}{y}\right] = \lim_{y\to 0} \frac{\sqrt[3]{1+\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)}-1}{\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)} \cdot \frac{\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)}{y} =$$

$$\lim_{y\to 0} \underbrace{\frac{\left[1+\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)\right]^{\frac{1}{3}}-1}{\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)}}_{\to \frac{1}{3}} \cdot \underbrace{\frac{\left(\left(1-\frac{2}{3}y\right)\left(1+\frac{4}{3}y\right)^2-1\right)}{y}}_{\to 2} = \frac{2}{3}$$

$$\lim_{y\to 0} \frac{\cancel{1} + \frac{8}{3}y + \cancel{\frac{16}{9}y^2} - \frac{2}{3}y - \cancel{\frac{16}{9}y^2} - \frac{32}{27}y^3 - \cancel{1}}{y} =$$

$$= \lim_{y\to 0} \frac{\cancel{y}\left(2 - \frac{32}{27}y^2\right)}{\cancel{y}} = 2$$

$\Rightarrow y = x + \frac{2}{3}$

III) Zeri e segno di f

$\frac{1}{3}\sqrt[3]{(3x-2)(3x+4)^2} > 0$

$(3x-2)(3x+4)^2 > 0$

$x > \frac{2}{3}$

$(3x+4)^2 > 0 \quad \forall x \neq -\frac{4}{3}$

$f(x) = 0$ se $x = \frac{2}{3}$, $x = -\frac{4}{3}$

	$-\frac{4}{3}$		$\frac{2}{3}$	
$-$		$-$		$+$
$+$		$+$		$+$
$-$		$-$		$+$

$f(x) > 0$ se $x \in]\frac{2}{3}, +\infty[$

$f(x) < 0$ se $x \in]-\infty, -\frac{4}{3}[\cup]-\frac{4}{3}, \frac{2}{3}[$

IV) f'?

$f'(x) = \frac{1}{3} \frac{1}{3\sqrt[3]{((3x-2)(3x+4)^2)^2}} [3(3x+4)^2 + (3x-2)[2(3x+4)]3] =$

$= \frac{1}{9} \frac{3(3x+4)[3x+4+6x-4]}{(3x+4)\sqrt[3]{(3x-2)^2(3x+4)}} = \frac{1}{3}\frac{9x}{\sqrt[3]{(3x-2)^2(3x+4)}} = \frac{3x}{\sqrt[3]{(3x-2)^2(3x+4)}}$

$f'\left(\frac{2}{3}\right) = +\infty \qquad f'\left(-\frac{4}{3}\right) = \frac{3\left(-\frac{4}{3}\right)}{\sqrt[3]{\left(3\left(-\frac{4}{3}\right)-2\right)^2\left(3\left(-\frac{4}{3}\right)+4\right)}} \longrightarrow$ non esiste

$f'_-\left(-\frac{4}{3}\right) = +\infty \qquad f'_+\left(-\frac{4}{3}\right) = -\infty$

V) punti critici di f e segno di f'

$f'(x) = 0 \Rightarrow x = 0$ punto critico

$f'(x) > 0$

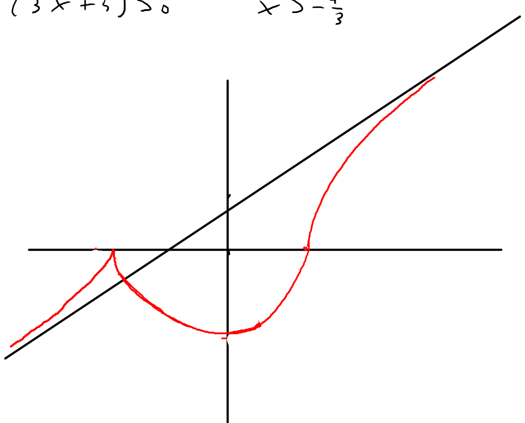
N: $x > 0$

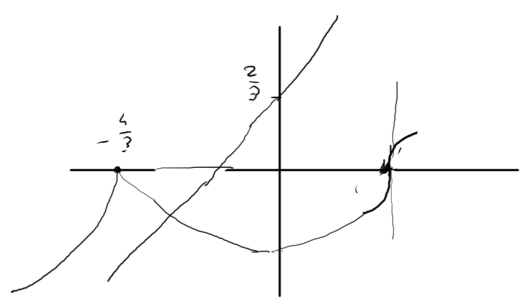
D: $(3x-2)^2(3x+4) > 0$

$(3x-2)^2 > 0 \quad \forall x \neq \frac{2}{3}$

$(3x+4) > 0 \quad x > -\frac{4}{3}$

	$-\frac{4}{3}$		0		$\frac{2}{3}$	
$-$		$-$		$+$		$+$
$+$		$+$		$+$		$+$
$+$		$-$		$+$		$+$
$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$





$$\left(1+x\right)^{\frac{1}{\log a}}$$

$$\frac{1}{\log a}\left(1+x\right)^{\frac{1}{\log a}-1}$$

$$\frac{1}{\log a}-1 \in \left]0,1\right[$$

ESERCIZIO 3

$$f(x) = \frac{(1-|x-1|)^2}{\sqrt{|x|}}$$

$$f(x) = \begin{cases} \frac{(\cancel{1}-x+\cancel{1})^2}{\sqrt{|x|}} & x \geq 1 \\ \frac{x^2}{\sqrt{|x|}} & x < 1 \end{cases} = \begin{cases} \frac{(2-x)^2}{\sqrt{x}} & x \geq 1 \\ |x|^{\frac{3}{2}} & x < 1, \; x \neq 0 \end{cases}$$

i) $D = \mathbb{R} \setminus \{0\}$

Zeri: $f(x) = 0 \quad (2-x)^2 = 0 \Rightarrow f(2) = 0$

$x = 2$

Segni $f(x) > 0$: $\frac{(1-|x-1|)^2}{\sqrt{|x|}} > 0$

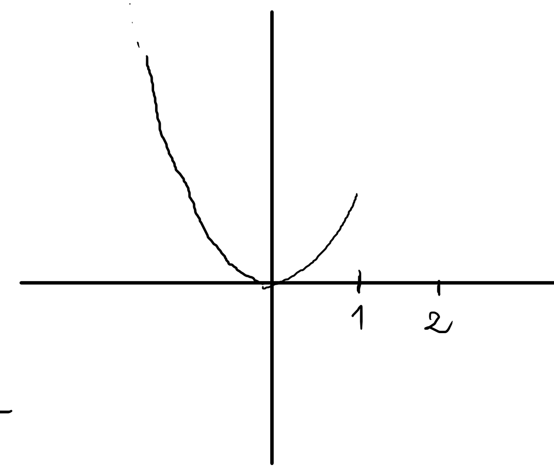
$\forall x \in D \setminus \{2\}$

$$\lim_{x \to 0} x^{3/2} = 0$$

ii)

$$f(x) = \begin{cases} \frac{(2-x)^2}{\sqrt{x}} & x \geq 1 \\ |x|^{\frac{3}{2}} & x < 1 \quad x \neq 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{3x^2-4x-4}{2x^{3/2}} & x > 1 \\ \frac{3}{2}\sqrt{x} & 0 < x < 1 \\ -\frac{3}{2}\sqrt{-x} & x < 0 \end{cases}$$



$$\lim_{x \to 1^+} \frac{3x^2-4x-4}{2x^{3/2}} = -\frac{5}{2} \neq \lim_{x \to 1^-} \frac{3}{2}\sqrt{x} = \frac{3}{2}$$

$f(x)$ non è derivabile in $x = 1$

$f'(x) > 0 \quad \frac{3x^2-4x-4}{2x^{3/2}} > 0$

$N > 0: \; 3x^2 - 4x - 4 > 0$

$\Delta/4 = 16 = 4^2$

$x_{\pm} = \frac{2 \pm 4}{3} = \begin{cases} 2 \text{ acc.} \\ -\frac{2}{3} \text{ no} \end{cases}$

f crescente in: $]0,1]$ e in $[2,+\infty[$

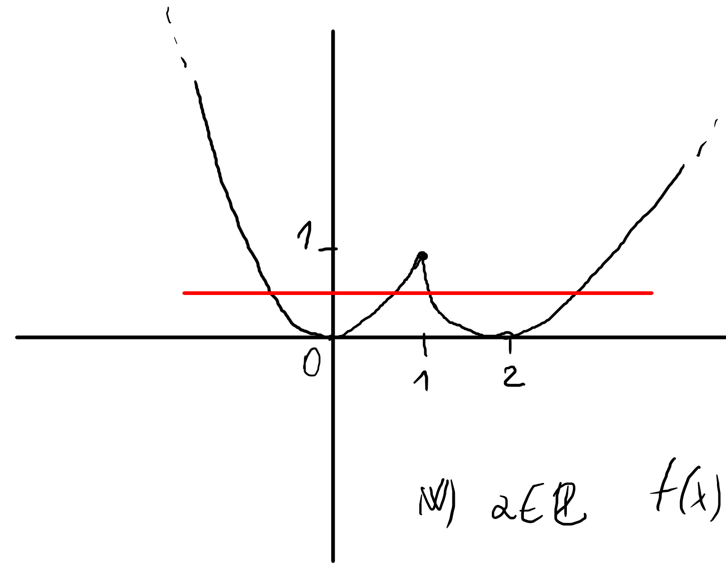
f decrescente: $]-\infty,0[$ e in $[1,2]$

1,2   1 è punto di MASSIMO

2 è punto di MINIMO ASSOLUTO

iii)

$$f''(x) = \begin{cases} \dfrac{3x^2+4x+12}{4x^{5/2}} & x > 1 \\ \dfrac{3}{4}|x|^{-1/2} & x < 1,\ x \neq 0 \end{cases}$$



iv) $a \in \mathbb{R}$ $\quad f(x) = a$

ESERCIZIO

$$f(x) = \int_0^{\log x} e^{-t^2}\,dt$$

i) Dominio =]0,+∞[

$x \in \,]0,1[$ $\quad f(x) < 0$

$x \in \,]1,+\infty[$ $\quad f(x) > 0$

$f(1) = 0$

ii) $f\left(\frac{1}{x}\right) = -f(x)$

$$f\left(\frac{1}{x}\right) = \int_0^{\log\left(\frac{1}{x}\right)} e^{-t^2}\,dt = \int_0^{-\log x} e^{-t^2}\,dt = -\underbrace{\int_0^{\log x} e^{-t^2}\,dt}_{f(x)}$$

$s = -t$

$ds = -dt$

iii) $f'(x)$, $f''(x)$

$f'(x) =$

$$f'(x) = e^{-\log^2 x} \cdot \frac{1}{x}$$

$$f''(x) = e^{-\log^2 x}\left(-2(\log x)\frac{1}{x^2}\right) - e^{-\log^2 x}\left(\frac{1}{x^2}\right)$$

$$= -\frac{e^{-\log^2 x}}{x^2}(2\log x + 1)$$

iv) $f''(x) > 0$: $e^{-\log^2 x}\,\frac{1}{x}(-2\log x - 1) > 0$

$x > 0$

$-2\log x - 1 > 0$

$x < e^{-1/2}$

	0		$e^{-1/2}$
		+	+
		+	−
		+	−
		∪	∩

$x = e^{-1/2}$ FLESSO DISC.

v) f LIMITATA nel DOMINIO $\quad e^{-t^2}$