$$\sum_{n=0}^{n} 4^{-k} = \boxed{\frac{4}{3}\left(1-4^{-(n+1)}\right)}$$

Provare con p. di induzione

Provo con $n=0$

$$4^{-(0)} = \frac{4}{3}\left(1-4^{-(0+1)}\right)$$

$$1 = \frac{4}{3}\left(1-\frac{1}{4}\right)$$

$$1 = \frac{\cancel{4}}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{4}}\right) \qquad 1 = 1$$

Passo dell'induzione

$$\sum_{k=0}^{n+1} 4^{-k} = \boxed{\frac{4}{3}\left(1 - 4^{-(n+2)}\right)} *$$

$$\sum_{k=0}^{n+1} 4^{-k} = \sum_{k=0}^{n} 4^{-k} + 4^{-(n+1)}$$

$$= \frac{4}{3}\left(1 - 4^{-(n+1)}\right) + 4^{-(n+1)}$$

$$= \frac{4}{3} - \frac{4}{3} \cdot 4^{-(n+1)} + 4^{(-n+1)}$$

$$\frac{4}{3} - 4^{-(n+1)}\left(\frac{4}{3} - 1\right)$$

$$\frac{4}{3} - \frac{4}{3}$$

$$\frac{4}{3}\left(1 - \frac{1}{4} \cdot 4^{-(n+1)}\right)$$

$$\frac{4}{3}\left(1 - 4^{-1} \cdot 4^{-(n+1)}\right) = \frac{4}{3}\left(1 - 4^{-(n+2)}\right)$$

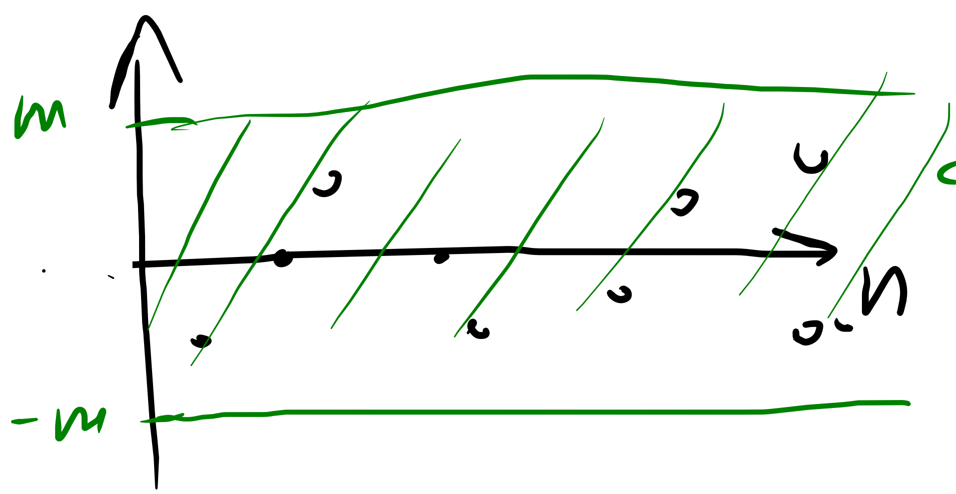
Per il p. di induzione la formula vale $\forall n \geq 0$, $n \in \mathbb{N}$

SUCCESSIONI: definizioni

CONVERGENTE $\quad \lim_{n\to+\infty} a_n = \ell \in \mathbb{R} \qquad a_n = \frac{1}{n}$

DIVERGENTE $\quad \lim_{n\to+\infty} a_n = \pm\infty \qquad a_n = n^2$

LIMITATA: $\exists\, m \in \mathbb{R}$ tale che $|a_n| \leq m \quad \forall n \in \mathbb{N} \qquad a_n = (-1)^n$



$\leftarrow$ la successione sta qui dentro

Successione LIMITATA $\Longleftarrow$ Successione CONVERGENTE

$\not\Longrightarrow$ Non vale per esempio $a_n = (-1)^n$

Successione MONOTONA

non decrescente: $a_n \leq a_{n+1}$

non crescente: $a_n \geq a_{n+1}$

$\forall n \in \mathbb{N}$

Se a_n è una successione monotona, allora ammette sempre limite, finito o no.

Verificare, attraverso la definizione di limite che

(1) $$\lim_{n\to+\infty} \frac{2n+3}{3n-7} = \frac{2}{3}$$

$$\left[(\forall \varepsilon>0)(\exists \nu\in\mathbb{N})(\forall n\in\mathbb{N})\left(n>\nu \Longrightarrow |a_n-\ell|<\varepsilon\right)\right]$$

Applico questo

$$\left|\frac{2n+3}{3n-7}-\frac{2}{3}\right| < \varepsilon$$

$$\left|\frac{2n+3}{3n-7}-\frac{2}{3}\right| = \left|\frac{3(2n+3)-2(3n-7)}{(3n-7)\cdot 3}\right| = \left|\frac{\cancel{6n}+9-\cancel{6n}+14}{3(3n-7)}\right| =$$

$$\left|\frac{23}{3(3n-7)}\right| = \frac{23}{3(3n-7)} < \varepsilon$$

$$\frac{23}{3(3n-7)} < \varepsilon \xRightarrow{\text{isolo } n}$$

$$\frac{23}{3 \cdot \varepsilon} < (3n-7) \qquad \frac{23}{3 \cdot \varepsilon} + 7 < 3n$$

$$n > \frac{23}{9}\frac{1}{\varepsilon} + \frac{7}{3} \qquad |a_n - \ell \to 0$$

Abbiamo provato $\lim_{n \to +\infty} a_n = \frac{2}{3}$

$$\frac{1}{n+3} < \varepsilon$$

$$\frac{1}{n+3} < \frac{1}{n} < \varepsilon \Rightarrow n > \varepsilon$$

(2) $$\lim_{n\to+\infty} \frac{n^4+3}{3n^5+7\cos^3 n+2} = 0$$

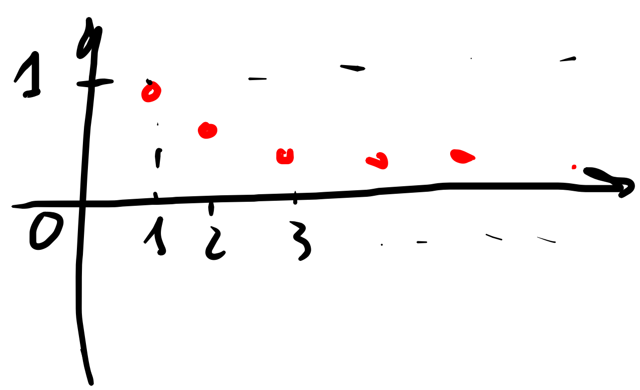
$$\left| \frac{n^4+3}{3n^5+7\cos^3 n+2} - 0 \right| < \varepsilon$$

$$\frac{n^4+3}{3n^5+7\cos^2 n+2} < \frac{n^4+3}{3n^5} = \frac{\cancel{n^4}\left(1+\frac{3}{n^4}\right)}{3n^{\cancel{5}}} =$$

$$= \frac{\left(1 + \frac{3}{n^4}\right)}{3n} < \frac{1+3}{3n} = \frac{4}{3n} < \varepsilon$$

$$n > \frac{4}{3}\frac{1}{\varepsilon} \qquad \forall \varepsilon > 0 \qquad \text{quindi } |a_n - \ell| \to 0$$

$$a_n = \frac{1}{n}$$



$$\frac{1}{n} \leq 1 \qquad \frac{3}{n} \leq 3$$

Le successioni sono limitate

$$a_n = \frac{2n^2-1}{n} = \underbrace{2n}_{\xrightarrow{n\to+\infty} +\infty} - \underbrace{\frac{1}{n}}_{\xrightarrow{n\to+\infty} 0} \xrightarrow{n\to+\infty} +\infty$$

Non è limitata

$$a_n = \frac{\cos(2n)}{n}$$

Sì è limitata

$$|\cos(2n)| \leq 1$$

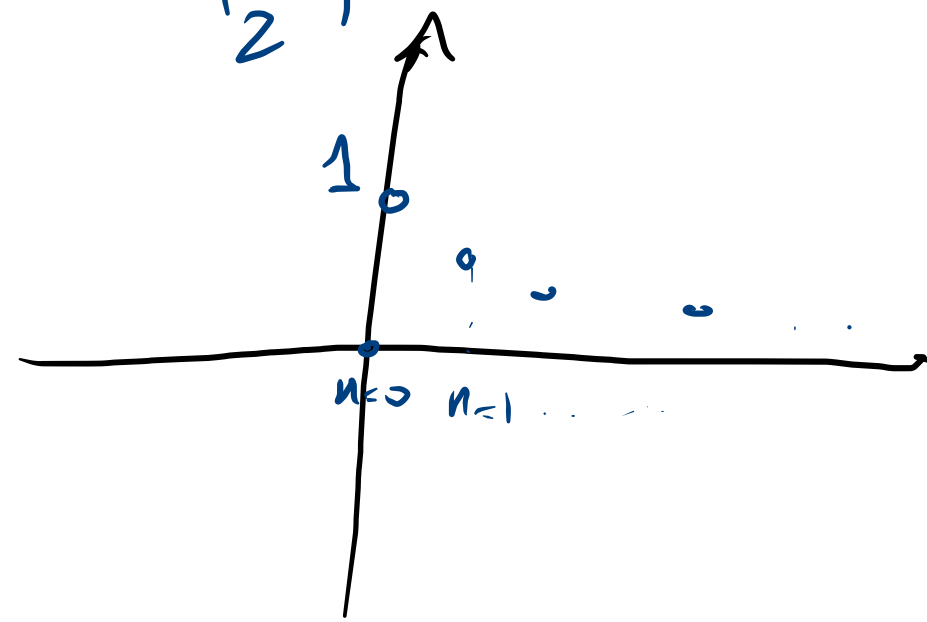
$$\left|\frac{\cos(2n)}{n}\right| \leq \left|\frac{1}{n}\right| \leq 1$$

c) $(-1)^n \left(\frac{3}{\pi}\right)^n$ non è limitata

$$\lim_{n \to +\infty} a^n = \begin{cases} 0 & 0 \leq a < 1 \\ 1 & a = 1 \\ +\infty & a > 1 \end{cases}$$

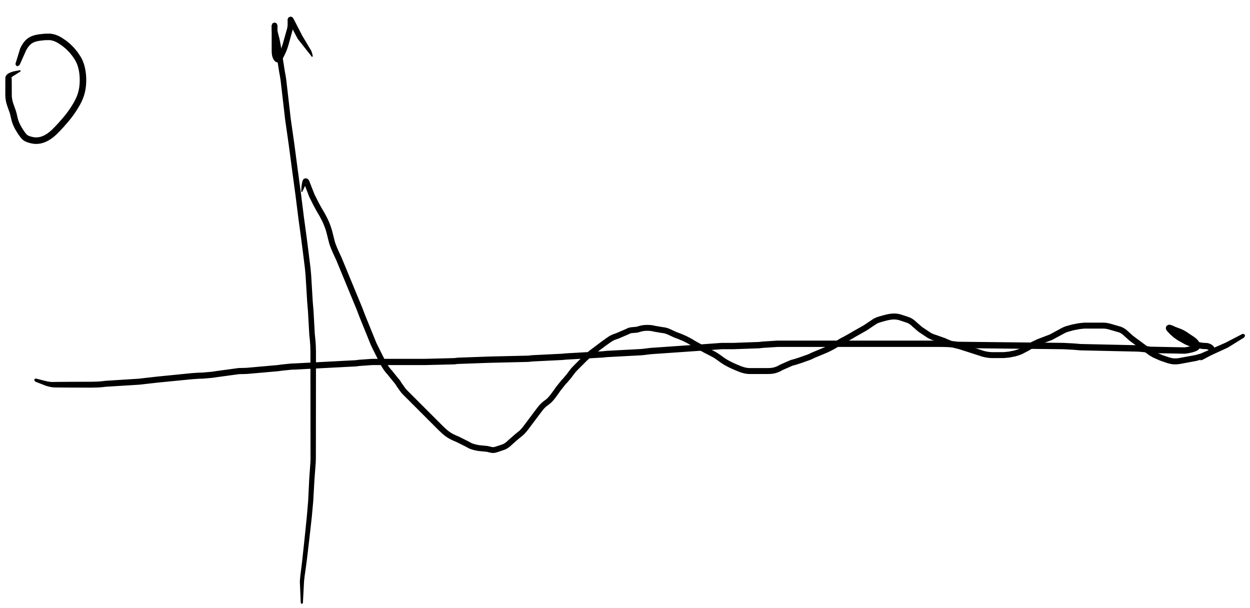
Provare per $a = 2$ $\quad a = \frac{1}{2}$

$a_n = \left(\frac{1}{2}\right)^n$



$$\underbrace{\underbrace{(-1)^n}_{|(-1)^n| \le 1} \quad \underbrace{\left(\frac{3}{\pi}\right)^n}_{\left|\left(\frac{3}{\pi}\right)^n\right| \le 1}}_{\text{la successione è limitata}}$$

$$\lim_{n \to +\infty} (-1)^n \left(\frac{3}{\pi}\right)^n = 0$$



$$\lim_{n\to+\infty} = \frac{2n-1}{n+5} = \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{\cancel{n}\left(1+\frac{5}{n}\right)} = \frac{2}{1} = 2$$

$\frac{1}{n} \xrightarrow{n\to+\infty} 0$

$\frac{5}{n} \xrightarrow{+\infty} 0$

Se converge allora è limitata