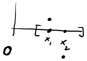
16 Dicembre

Lunedì 20 ultimo giorno.

<u>Esercizio</u> Sia $f:[a,b]\to\mathbb{R}$, $x_1<\dots<x_n$ punti di $[a,b]$ e siano $y_1,\dots,y_n\in\mathbb{R}$

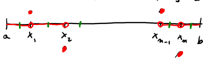
Se $f(x)=\begin{cases}0 & \text{se } x\notin\{x_1,\dots,x_n\}\\ y_j & \text{se } x=x_j\end{cases}$



allora $f\in L[a,b]$ con $\int_a^b f(x)\,dx=0$

Per prima cosa discutiamo del fatto che $f\in L[a,b]$

Spezziamo $[a,x_1]\cup[x_1,x_2]\cup\dots\cup[x_{n-1},x_n]\cup[x_n,b]$



Scompongo ulteriormente in

$*$ $\left[a,\frac{a+x_1}{2}\right]\cup\left[\frac{a+x_1}{2},x_1\right]\cup\left[x_1,\frac{x_1+x_2}{2}\right]\cup\left[\frac{x_1+x_2}{2},x_2\right]\cup\dots$

Per Chasles $f\in L[a,b]\iff$ f è integrabile in ciascuno degli intervalli di quest'ultima scomposizione

$f|_{\left[a,\frac{a+x_1}{2}\right]}\equiv 0\in L\left[a,\frac{a+x_1}{2}\right]$, $f|_{\left[\frac{a+x_1}{2},x_1\right]}$ è monotona

$\Rightarrow f\in L\left[\frac{a+x_1}{2},x_1\right]$

f ristretta a ciascuno degli intervalli in $*$ è monotona e pertanto è integrabile in ciascuno degli intervalli

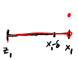
Quindi f è integrabile in $[a,b]$. Per calcolare $\int_a^b f(x)\,dx$

$$\int_a^b f(x)\,dx=\underbrace{\int_a^{\frac{a+x_1}{2}} f(x)\,dx}_{0}+\int_{\frac{a+x_1}{2}}^{x_1} f(x)\,dx+\int_{x_1}^{\frac{x_1+x_2}{2}} f(x)\,dx+\int_{\frac{x_1+x_2}{2}}^{x_2} f(x)\,dx+$$

$$\dots+\int_{\frac{x_{n-1}+x_n}{2}}^{x_n} f(x)\,dx+\int_{x_n}^{\frac{x_n+b}{2}} f(x)\,dx+\underbrace{\int_{\frac{x_n+b}{2}}^{b} f(x)\,dx}_{0}$$

Ora dimostriamo che in ciascuno degli intervalli l'integrale è nullo

Verifichiamo che $\int_{\frac{a+x_1}{2}}^{x_1} f(x)\,dx=0$ $\quad \frac{a+x_1}{2}=z_1$



$$\int_{z_1}^{x_1} f(x)\,dx=\int_{z_1}^{x_1-\delta} f(x)\,dx+\int_{x_1-\delta}^{x_1} f(x)\,dx$$

$$\left|\int_{z_1}^{x_1} f(x)\,dx\right|=\left|\int_{x_1-\delta}^{x_1} f(x)\,dx\right|\le\int_{x_1-\delta}^{x_1}|f(x)|\,dx\le\int_{x_1-\delta}^{x_1}|y_1|\,dx=|y_1|\,\delta$$

In $[x_1-\delta,x_1]$ ho $|f(x)|=\begin{cases}0 & \text{se } x<x_1\\ |y_1| & \text{se } x=x_1\end{cases}$

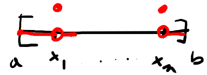
$\Rightarrow\quad |f(x)|\le|y_1|\quad$ in $[x_1-\delta,x_1]$

$$\left|\int_{z_1}^{x_1} f(x)\,dx\right|\le|y_1|\,\delta\qquad\forall\,\delta>0$$

$$\iff\left|\int_{z_1}^{x_1} f(x)\,dx\right|<\delta\qquad\forall\,\delta>0$$

$$\iff\int_{z_1}^{x_1} f(x)\,dx=0.\qquad z_1=\frac{a+x_1}{2}$$

$$f(x)=\begin{cases}0 & \text{se } x\notin\{x_1,\dots,x_n\}\\ y_j & \text{se } x=x_j\end{cases}$$



Un altro modo per dimostrare che $\int_a^b f(x)\,dx=0$ è il seguente

Per prima cosa, come prima, si dimostra $f\in L[a,b]$.

$$F(x)=\int_a^x f(t)\,dt$$

$$F\in C^0([a,b])$$

$F'(x)=f(x)\cancel{=0}$ nei punti dove f è continua

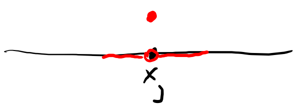
cioè negli $x\notin\{x_1,\dots,x_n\}$

$\Rightarrow F'(x)=0$ se $x\notin\{x_1,\dots,x_n\}$

$F'(x_j)=\,?$

$$F'_d(x_j)=f(x_j^+):=\lim_{x\to x_j^+} f(x)=0$$

$$F'_s(x_j)=f(x_j^-):=\lim_{x\to x_j^-} f(x)=0$$



$$F'(x_j)=0$$

Conclusione $F'\equiv 0$ in $[a,b] \iff F(x)\equiv c$

e siccome $F(x)=\int_a^x f(t)\,dt$ è $F(a)=0 \Rightarrow c=0$

$$F(b)=\int_a^b f(t)\,dt=0$$

$f(x) = \int_x^{2x} e^{t^3}\, dt$ Approssimare $f(1)$ con numero razionale

$y > 0$

$$e^y = \sum_{j=0}^{n} \frac{y^j}{j!} + E_n(y) \qquad E_n(y) = \frac{e^{c_y}}{(n+1)!}\, y^{n+1} \qquad 0 < c_y < y$$

$$e^{t^3} = \sum_{j=0}^{n} \frac{t^{3j}}{j!} + E_n(t^3)$$

$$f(1) = \underbrace{\sum_{j=0}^{n} \frac{1}{j!} \int_1^2 t^{3j}\, dt}_{\overset{n}{Q}} + \int_1^2 E_n(t^3)\, dt$$
errore

$$\int_1^2 t^{3j}\, dt = \frac{t^{3j+1}}{3j+1}\Big]_1^2 = \frac{2^{3j+1}}{3j+1} - \frac{1}{3j+1} \in \mathbb{Q}$$

$$0 < \int_1^2 E_n(t^3)\, dt = \int_1^2 \frac{e^{c_{t^3}}}{(j+1)!}\, t^{3j+3}\, dt \qquad 0 < c_{t^3} < t^3 \le 8$$

$$\le \frac{e^8}{(j+1)!} \int_1^2 t^{3j+3}\, dt = \frac{e^8}{(j+1)!}\, \frac{1}{3j+4} \left(2^{3j+4} - 1\right) < \frac{1}{100}$$

$$\lim_{x\to+\infty} \frac{\lg(1+[x]^{-a}+[x]^{-2a}) - \int_x^{+\infty}\frac{\sin(t)}{t^3}\,dt}{(1+\frac{1}{x})^{\frac{1}{x}}-1}$$

$$\lim_{x\to+\infty}\left(\left(1+\frac{1}{x}\right)^{\frac{1}{x}}-1\right) = 0$$

$$\lim_{y\to0^+}(1+y)^y = 1 = \lim_{y\to0^+} e^{y\lg(1+y)} = e^0$$

$$\text{den} = \left(1+\frac{1}{x}\right)^{\frac{1}{x}}-1 = e^{\frac{1}{x}\lg(1+\frac{1}{x})} - 1 = \cancel{1} + \left(\frac{1}{x}\lg\left(1+\frac{1}{x}\right) + o\left(\frac{1}{x}\lg\left(1+\frac{1}{x}\right)\right)\right) \cancel{-1}$$

$$\text{den} = \frac{1}{x}\underbrace{\lg\left(1+\frac{1}{x}\right)}_{\frac{1}{x}(1+o(1))}(1+o(1)) = \frac{1}{x^2}(1+o(1)) \qquad \lim_{y\to0}\frac{\lg(1+y)}{y} = 1$$

$$\lim_{x\to+\infty} \frac{\lg(1+[x]^{-a}+[x]^{-2a}) - \int_x^{+\infty}\frac{\sin t}{t^3}\,dt}{\frac{1}{x^2}}$$

$$\lim_{x\to+\infty} \frac{\int_x^{+\infty}\frac{\sin t}{t^3}\,dt}{\frac{1}{x^2}} \overset{H}{=} \lim_{x\to+\infty} \frac{-\frac{\sin x}{\cancel{x^3}}}{-\frac{2}{\cancel{x^3}}} = \lim_{x\to+\infty}\frac{\sin x}{2} \quad \text{non esiste}$$

$$= -\int_{+\infty}^{x}\frac{\sin t}{t^3}\,dt$$

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\lg(1+[x]^{-a}+[x]^{-2a}) = [x]^{-a}+[x]^{-2a} - \frac{1}{2}\left([x]^{-a}+[x]^{-2a}\right)^2 + o([x]^{-2a})$$

$$= [x]^{-a} + \frac{1}{2}[x]^{-2a} + o([x]^{-2a}) = [x]^{-a} + o([x]^{-a})$$

$$= [x]^{-a}(1+o(1))$$

$$\frac{\lg(1+[x]^{-a}+[x]^{-2a})}{\frac{1}{x^2}} = \frac{[x]^{-a}}{x^{-2}}(1+o(1)) = \frac{x^{-a}}{x^{-2}}(1+o(1)) = x^{2-a}(1+o(1))$$

$$[x] = x(1+o(1))$$

La nostra $f(x) = x^{2-a}(1+o(1)) - \dfrac{\int_x^{+\infty}\frac{\sin(t)}{t^3}\,dt}{x^{-2}}$

per $a\in(0,2)$ $\quad x^{2-a} \xrightarrow{x\to+\infty} +\infty$

Complessivamente il limite $= +\infty$

per $a \geq 2$ non esiste il limite

$$x^2\int_x^{+\infty}\frac{\sin(t)}{t^3}\,dt = x^2\int_x^{+\infty}\frac{(-\cos t)'}{t^3}\,dt =$$

$$= x^2\left[\frac{\cos x}{x^3} - 3\int_x^{+\infty}\frac{\cos t}{t^4}\,dt\right]$$

$$= \frac{\cos x}{x} - 3x^2\int_x^{+\infty}\frac{\cos t}{t^4}\,dt \qquad \frac{\cos x}{x} \xrightarrow{x\to+\infty} 0$$

$$0 \le x^2\left|\int_x^{+\infty}\frac{\cos t}{t^4}\,dt\right| \le x^2\int_x^{+\infty}\frac{|\cos t|}{t^4}\,dt \le x^2\int_x^{+\infty}\frac{1}{t^4}\,dt$$

$$= x^2\left.\frac{t^{-3}}{-3}\right]_x^{+\infty} = x^2\,\frac{1}{3x^3} \xrightarrow{x\to+\infty} 0$$

$$x^{2-a}(1+o(1)) - o(1) \to \begin{cases} +\infty & a<2 \\ 1 & a=2 \\ 0 & a>2 \end{cases}$$