17 Dicembre

Dare un esempio di $f \in L[0,+\infty)$ t.c. $\lim_{x\to+\infty} f(x)$ non esiste.

(Ricordare che se $f \in L[0,+\infty)$ e se $\lim_{x\to+\infty} f(x) =: L \in \overline{\mathbb{R}}$ esiste, allora $L = 0$.)

$\sin\left(\frac{1}{x}\right) \notin L[0,+\infty)$

$\sin\left(\frac{1}{x^2}\right) \in L[0,+\infty)$ $\qquad \lim_{x\to+\infty} \sin\left(\frac{1}{x^2}\right) = 0$

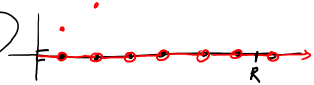
$$f(x) = \begin{cases} 0 & \text{se } x \notin \mathbb{N} \\ x & \text{se } x \in \mathbb{N} \end{cases}$$

$f \overset{?}{\in} L[0,+\infty)$

Per prima cosa $f \in L_{loc}[0,+\infty)$.

Infatti $\forall \; [a,b] \subseteq [0,+\infty)$

f in $[a,b]$ è uguale a 0 eccetto nei punti in $\mathbb{N} \cap [a,b]$. $\int_a^b f(x)\,dx = 0$



* $\lim_{x\to+\infty} f(x)$ non esiste

$$\lim_{R\to+\infty} \underbrace{\int_0^R f(x)\,dx}_{0} = \lim_{R\to+\infty} 0 = 0$$

$f \in L[0,+\infty)$ con $\int_0^{+\infty} f(x)\,dx = 0$

Notare che se esiste $\lim_{x\to+\infty} f(x) = L$ allora per ogni successione $\{x_n\}$ in $[0,+\infty)$ con $\lim_{n\to+\infty} x_n = +\infty$ ho $\lim_{n\to+\infty} f(x_n) = L$.

Se la nostra $f(x)$ ha limite L a $+\infty$, segue che

$L = \lim_{n\to+\infty} f(n) = \lim_{n\to+\infty} n = +\infty$

$L = \lim_{n\to+\infty} f(n+\frac{1}{2}) = \lim_{n\to+\infty} 0$

$L = +\infty$

$L = 0$

assurdo.

Data $f: [0,+\infty) \to \mathbb{R}$, se si ha $\lim_{x\to+\infty} f(x) = L \in \overline{\mathbb{R}}$

allora $\forall \{x_n\}$ in $[0,+\infty)$ con $\lim_{n\to+\infty} x_n = +\infty$, si ha

$$\lim_{n\to+\infty} f(x_n) = L$$

Dim Mi limito al caso $L \in \mathbb{R}$.

Allora $\lim_{x\to+\infty} f(x) = L \iff$

$$\boxed{\forall \varepsilon > 0 \quad \exists K_\varepsilon \text{ t.c. } x > K_\varepsilon \Rightarrow |f(x) - L| < \varepsilon \qquad (1)}$$

Sia ora $\{x_n\}$ in $[0,+\infty)$ t.c. $\lim_{n\to+\infty} x_n = +\infty$.

$$\boxed{\forall K \quad \exists M_K \text{ t.c. } n > M_K \Rightarrow x_n > K. \qquad (2)}$$

Ora dobbiamo dimostrare che $\lim_{n\to+\infty} f(x_n) = L$ cioè che

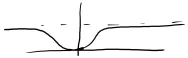
$$\boxed{\forall \varepsilon > 0 \quad \exists N_\varepsilon \text{ t.c. } n > N_\varepsilon \Rightarrow |f(x_n) - L| < \varepsilon}$$

Tentiamo con $N_\varepsilon = M_{K_\varepsilon}$

In effetti $n > N_\varepsilon = M_{K_\varepsilon} \overset{(2)}{\Longrightarrow} x_n > K_\varepsilon \overset{(1)}{\Longrightarrow} |f(x_n) - L| < \varepsilon$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

Calcolare $f^{(n)}(0) = 0$



$$f'(x) = \left(-\frac{1}{x^2}\right)' e^{-\frac{1}{x^2}} = \frac{2}{x^3} e^{-\frac{1}{x^2}} \qquad \text{se } x \neq 0$$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} f'(x) = \lim_{x\to 0} 2\,\frac{e^{-\frac{1}{x^2}}}{x^3}$$

$$\lim_{x\to 0^+} \frac{2\,e^{-\frac{1}{x^2}}}{x^3} \qquad y = \frac{1}{x}$$

$$= \lim_{y\to +\infty} 2\, y^3 e^{-y^2} = \lim_{y\to +\infty} 2\,\frac{y^3}{e^{y^2}} = 0$$

$$\lim_{x\to 0^-} 2\,\frac{e^{-\frac{1}{x^2}}}{x^3} \qquad y = \frac{1}{x}$$

$$= \lim_{y\to -\infty} 2\,\frac{y^3}{e^{y^2}} = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} = \lim_{y\to +\infty} \frac{1}{e^y} = 0 = f(0) \qquad y = \frac{1}{x^2}$$

$$f'(x) = 2\,\frac{e^{-\frac{1}{x^2}}}{x^3}$$

Supponiamo per induzione che $f^{(n)}(x) \overset{\text{per } x\neq 0}{=} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$ e che $f^{(n)}(0) = 0$

dove $P_n(y)$ è un polinomio

Ad es., $P_1(y) = 2y^3$

Allora dimostriamo $f^{(n+1)}(x) \overset{\text{per } x\neq 0}{=} P_{n+1}\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$, $f^{(n+1)}(0) = 0$

$x \neq 0$

$$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = \left(P_n\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}}\right)' =$$

$$= \left(P_n\left(\tfrac{1}{x}\right)\right)' e^{-\frac{1}{x^2}} + P_n\left(\tfrac{1}{x}\right) \frac{2}{x^3} e^{-\frac{1}{x^2}}$$

$$= \Big(\underbrace{-P_n'\left(\tfrac{1}{x}\right) \frac{1}{x^2} + P_n\left(\tfrac{1}{x}\right) \frac{2}{x^3}}_{P_{n+1}\left(\frac{1}{x}\right)}\Big) e^{-\frac{1}{x^2}}$$

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - \overbrace{f^{(n)}(0)}^{0}}{x} = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} f^{(n+1)}(x)$$

$$= \lim_{x\to 0} P_{n+1}\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}} = 0$$

$$\lim_{x\to 0^+} P_{n+1}\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}} \qquad y = \frac{1}{x}$$

$$= \lim_{y\to +\infty} \frac{P_{n+1}(y)}{e^{y^2}} = 0$$

$$\lim_{x\to 0^-} P_{n+1}\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}} \qquad y = \frac{1}{x}$$

$$= \lim_{y\to -\infty} \frac{P_{n+1}(y)}{e^{y^2}} = 0$$

$$f(x) = \int_0^x \frac{1}{1+t+t^6}\,dt \qquad P_6(x)$$

$$|f(1) - P_6(1)|$$

$$\frac{1}{1+y} = \sum_{j=0}^{n} (-1)^j y^j + o(y^n)$$

$$\sum_{j=0}^{n} (-1)^j y^j = \frac{1-(-y)^{n+1}}{1+y}$$

$$o(y^n) = \frac{(-y)^{n+1}}{1+y}$$

$$\frac{1}{1+y} = \sum_{j=0}^{n} (-1)^j y^j + \frac{(-y)^{n+1}}{1+y} \qquad y = t+t^6$$

$$t+t^6 = t(1+o(1))$$

$$\frac{1}{1+t+t^6} = \sum_{j=0}^{5} (-1)^j (t+t^6)^j + o((t+t^6)^5)$$

$$= \sum_{j=0}^{5} (-1)^j (t+t^6)^j + o(t^5)$$

$$= \sum_{j=0}^{5} (-1)^j t^j + o(t^5)$$

$$f(x) = \underbrace{\sum_{j=0}^{5} (-1)^j \frac{x^{j+1}}{j+1}}_{P_6(x)} + o(x^6)$$

$$f(x) = \int_0^x \frac{1}{(1+t)+t^6}\,dt = \int_0^x \frac{1}{1+t} \frac{1}{1+\frac{t^6}{1+t}}\,dt$$

$$\frac{1}{1+y} = 1 - \frac{y}{1+y}$$

$$\frac{1}{1+\frac{t^6}{1+t}} = 1 - \frac{\frac{t^6}{1+t}}{1+\frac{t^6}{1+t}} = 1 - \frac{t^6}{1+t+t^6}$$

$$f(x) = \int_0^x \frac{1}{1+t}\,dt - \int_0^x \frac{1}{1+t} \frac{t^6}{1+t+t^6}\,dt$$

$$\int_0^x \frac{1}{1+t}\,dt = \underbrace{\int_0^x dt \sum_{j=0}^{5} (-1)^j t^j}_{P_6(x)} - \int_0^x \frac{t^6}{1+t}\,dt$$

$$f(x) = P_6(x) + E(x) \quad \text{dove} \quad E(x) = -\int_0^x \frac{t^6}{1+t} - \int_0^x \frac{1}{1+t} \frac{t^6}{1+t+t^6}\,dt$$

$$|E(1)| = \int_0^1 t^6 \underbrace{\left(1 + \frac{1}{1+t+t^6}\right)}_{\leq 2} \underbrace{\frac{1}{1+t}}_{\leq 1}\,dt$$

$$< 2\int_0^1 t^6\,dt = \frac{2}{7}$$

$$R(x) = \frac{1}{(x+1)^2 (x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{\overset{=1}{C}}{x+2}$$

$$C = R(x)\,(x+2)\Big|_{x=-2} = \frac{1}{(x+1)^2}\Big|_{x=-2} = \frac{1}{(-1)^2} = 1$$

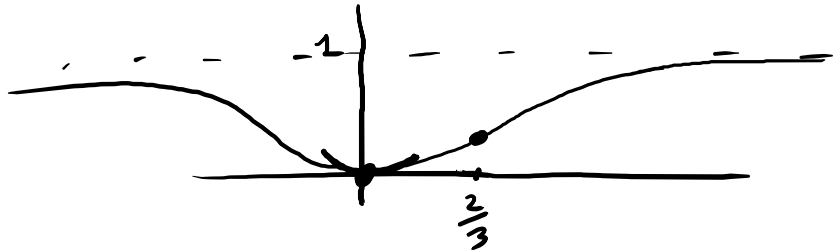
$$B = R(x)\,(x+1)^2\Big|_{x=-1} = \frac{1}{x+2}\Big|_{x=-1} = 1$$

$A+C = 0$ in questo caso $\Rightarrow$ $A = -1$

$$x R(x) = \underbrace{\frac{x}{(x+1)^2\,(x+2)}}_{\xrightarrow{x \to +\infty}\ 0} = \underbrace{A\,\frac{x}{x+1} + C\,\frac{x}{x+2}}_{\downarrow\ A+C} + \underbrace{B\,\frac{x}{(x+2)^2}}_{\xrightarrow{x \to +\infty}\ 0}$$

$A+C = 0$ $\qquad$ $A = -C = -1$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$



$f(0)$

$$\lim_{x \to +\infty} f(x) = e^0 = 1$$

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}} > 0 \qquad \text{per} \quad x > 0$$

$$f''(x) = -\frac{6}{x^4} e^{-\frac{1}{x^2}} + \frac{4}{x^6} e^{-\frac{1}{x^2}} = \frac{2e^{-\frac{1}{x^2}}}{x^4} \left(-3 + \frac{2}{x^2}\right) = 0$$

$$\frac{2}{x^2} = 3 \qquad x^2 = \frac{2}{3} \qquad x = \pm\sqrt{\frac{2}{3}}$$

$$\int_1^{+\infty} \sin(x^p)\,dx \qquad \text{per quali } p \text{ è convergente}$$

sappiamo che è convergente per $p < -1$

$-1 \leq p \leq 0$ non è convergente. In tutti questi casi

$$\sin(x^p) = x^p\,(1 + o(1)) \qquad p < 0$$

$p > 0$

$$y = x^p \qquad x = y^{\frac{1}{p}}$$

$$dy = p\,x^{p-1}\,dx = p\,y^{\frac{p-1}{p}}\,dx$$

$$dx = \frac{1}{p}\,y^{-\frac{p-1}{p}}\,dy$$

$$\lim_{R\to+\infty} \int_1^R \sin(x^p)\,dx$$

$$\int_1^R \sin(x^p)\,dx = \frac{1}{p}\int_1^{R^p} \frac{\sin(y)}{y^{\frac{p-1}{p}}}\,dy = \frac{1}{p}\int_1^{R^p} \frac{\sin y}{y^q}\,dy$$

dove $q > 0$ se $p > 1$

Per $p > 1$, $\frac{p-1}{p} =: q > 0$

Per $p > 1$ $\displaystyle\lim_{R\to+\infty} \frac{1}{p}\int_1^{R^p} \frac{\sin y}{y^q}\,dy$ esiste finito

Invece se $0 < p \leq 1$ non è integrabile

Esercizio Verificare che $x^a \sin(x) \notin L([1,+\infty))$

$\forall\; a \geq 0$