Bell's theorem

Setting

Let us consider two 1/2-spin particles.

Two separated particles, upon which spin measurements can be performed.

a and **b** are the two directions of spin-measurement
A and B are the two outcomes, which can be either Y or N (up and down; + or - …)

Any theory of nature must accommodate this setting, otherwise it is not a theory of nature, since such a situation occurs in nature.

Setting

$$\boxed{p(A, B \,|\, \mathbf{a}, \mathbf{b}, \lambda)}$$

This is the probability that in a measurement of spin of the left particle along direction **a** the outcome is A, and in a measurement of spin of the right particle along direction **b** the outcome is B. (p can also be 0 or 1, if the theory is deterministic)

λ is the state of the two-particle system.

Classical mechanics: λ = positions and momenta of the particles
Quantum mechanics: λ = wave function
Bohmian mechanics: λ = wave function and positions of the particles

We are not committing to any specific theory

Setting

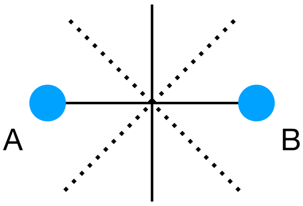
We also define

$p(A \,|\, \mathbf{a}, \lambda)$ = same as before, but with no measurement B
$p(B \,|\, \mathbf{b}, \lambda)$ = same as before, but with no measurement A

The definition of conditional probability implies:

$$p(A, B \,|\, \mathbf{a}, \mathbf{b}, \lambda) \;=\; p(A \,|\, B, \mathbf{a}, \mathbf{b}, \lambda) \cdot p(B \,|\, \mathbf{a}, \mathbf{b}, \lambda)$$

Bell's locality



$$\boxed{\begin{aligned} p(A\,|\,B,\mathbf{a},\mathbf{b},\lambda) &= p(A\,|\,\mathbf{a},\lambda) \\ p(B\,|\,A,\mathbf{a},\mathbf{b},\lambda) &= p(B\,|\,\mathbf{b},\lambda) \end{aligned}}$$

When the two measurements are space-like separated from each other, what happens on one side cannot influence the other side.

Together with the rules of conditional probability, locality implies

$$\boxed{p(A,B\,|\,\mathbf{a},\mathbf{b},\lambda) = p(A\,|\,\mathbf{a},\lambda)\cdot p(B\,|\,\mathbf{b},\lambda)}$$

The theorem

Define:

$$E_\lambda(\mathbf{a},\mathbf{b}) \;=\; p(Y,Y\,|\,\mathbf{a},\mathbf{b},\lambda) + p(N,N\,|\,\mathbf{a},\mathbf{b},\lambda) - p(Y,N\,|\,\mathbf{a},\mathbf{b},\lambda) - p(N,Y\,|\,\mathbf{a},\mathbf{b},\lambda)$$

Sum of agreements minus sun of disagreements.

Then, using Bell's locality condition:

$$E_\lambda(\mathbf{a},\mathbf{b}) \;=\; [p(Y\,|\,\mathbf{a},\lambda) - p(N\,|\,\mathbf{a},\lambda)][p(Y\,|\,\mathbf{b},\lambda) - p(N\,|\,\mathbf{b},\lambda)]$$

And:

$$E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d}) \;=\; [p(Y\,|\,\mathbf{a},\lambda) - p(N\,|\,\mathbf{a},\lambda)][(p(Y\,|\,\mathbf{b},\lambda) - p(N\,|\,\mathbf{b},\lambda)) - (p(Y\,|\,\mathbf{d},\lambda) - p(N\,|\,\mathbf{d},\lambda))]$$

The theorem

$$E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d}) \;=\; \underbrace{[p(Y|\mathbf{a},\lambda) - p(N|\mathbf{a},\lambda)]}_{=\,1-2p(N|\mathbf{a},\lambda)\,\in\,[-1,+1]}[(p(Y|\mathbf{b},\lambda) - p(N|\mathbf{b},\lambda)) - (p(Y|\mathbf{d},\lambda) - p(N|\mathbf{d},\lambda))]$$

Therefore:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| \;\le\; |\underbrace{(p(Y|\mathbf{b},\lambda) - p(N|\mathbf{b},\lambda))}_{\text{r}} - \underbrace{(p(Y|\mathbf{d},\lambda) - p(N|\mathbf{d},\lambda))}_{\text{s}}|$$

$$|E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \;\le\; |\underbrace{(p(Y|\mathbf{b},\lambda) - p(N|\mathbf{b},\lambda))}_{\text{r}} + \underbrace{(p(Y|\mathbf{d},\lambda) - p(N|\mathbf{d},\lambda))}_{\text{s}}|$$

The theorem

So:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \;\le\; |r-s| + |r+s|$$

Taking the square we have:

$$[|r-s| + |r+s|]^2 = 2r^2 + 2s^2 + 2|r^2 - s^2|$$

which is either equal to $4r^2$ or to $4s^2$; in either case, it is less than or equal to 4, since $r, s \in [-1,+1]$. So:

$$|r-s| + |r+s| \;\le\; 2$$

The theorem

So we end up with:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \ \le\ 2$$

This is Bell's inequality, which is the direct consequence of Bell's locality condition alone

$$\boxed{\begin{array}{c} p(A,B\,|\,\mathbf{a},\mathbf{b},\lambda) = p(A\,|\,\mathbf{a},\lambda)\cdot p(B\,|\,\mathbf{b},\lambda) \\ \Downarrow \\ |E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \ \le\ 2 \end{array}}$$

The testability

Problem: not always we have full control of the state of the system (λ). Therefore the above inequalities are not always testable.

Solution:
$\lambda = (\mu, \nu)$, where μ are controllable and ν are uncontrollable degrees of freedom

$$E_\mu(\mathbf{a},\mathbf{b}) \ = \ \int E_{(\mu,\nu)}(\mathbf{a},\mathbf{b})\rho(\nu)d\nu$$

This is a physically measurable quantity

Probability distribution; it reflects our ignorance

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| \ \le\ 2$$

The testability

Then:

$$|E_\mu(\mathbf{a},\mathbf{b}) - E_\mu(\mathbf{a},\mathbf{d})| + |E_\mu(\mathbf{c},\mathbf{b}) + E_\mu(\mathbf{c},\mathbf{d})| \leq$$

$$\leq \int d\nu\rho(\nu)\Big[|E_{(\mu,\nu)}(\mathbf{a},\mathbf{b}) - E_{(\mu,\nu)}(\mathbf{a},\mathbf{d})| + |E_{(\mu,\nu)}(\mathbf{c},\mathbf{b}) + E_{(\mu,\nu)}(\mathbf{c},\mathbf{d})|\Big]$$

$$\leq 2\int d\nu\rho(\nu) = 2$$

The inequality still holds.

Application to QM

Let us consider a singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

This state is rotationally invariant, so the spin relation above holds for any direction.

$$p_\lambda^{AB}(\mathbf{a},\mathbf{b}\,|\,Y,Y) = p_\lambda^{AB}(\mathbf{a},\mathbf{b}\,|\,N,N) = \frac{1}{2}\sin^2\frac{\theta_{\mathbf{a},\mathbf{b}}}{2}$$

$$p_\lambda^{AB}(\mathbf{a},\mathbf{b}\,|\,Y,N) = p_\lambda^{AB}(\mathbf{a},\mathbf{b}\,|\,N,Y) = \frac{1}{2}\cos^2\frac{\theta_{\mathbf{a},\mathbf{b}}}{2}$$

Then: $E_\lambda^{AB}(\mathbf{a},\mathbf{b}) = -\cos\theta_{\mathbf{a},\mathbf{b}}$

Application to QM

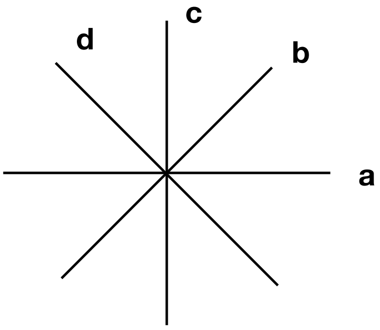
Then:

$$|E_\lambda(\mathbf{a},\mathbf{b}) - E_\lambda(\mathbf{a},\mathbf{d})| + |E_\lambda(\mathbf{c},\mathbf{b}) + E_\lambda(\mathbf{c},\mathbf{d})| =$$

$$= |\cos\theta_{\mathbf{a},\mathbf{b}} - \cos\theta_{\mathbf{a},\mathbf{d}}| + |\cos\theta_{\mathbf{c},\mathbf{b}} + \cos\theta_{\mathbf{c},\mathbf{d}}|$$

Let us choose the four angles as in the picture

$$= \left|\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right| + \left|\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right|$$
$$= 2\sqrt{2} \geq 2$$



The inequality is violated. **QM is nonlocal**

Nonlocality in QM

Where is the source of the nonlocality in QM?
Let is go back to the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

A makes a measurement along the direction **a**. With probability 1/2 the outcome is Y and the state changes to

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \longrightarrow |\uparrow_{\mathbf{a}}\downarrow_{\mathbf{a}}\rangle$$

Also the state of the other particles has changed, no matter how far it is.
This is the source on nonlocality.

Nonlocality in Physics

- Freedman and Clauser (1972)
- **Aspect et al. (1982)**
- Tittel et al. (1998)
- Weihs et al. (1998): experiment under "strict Einstein locality" conditions
- Pan et al. (2000) experiment on the GHZ state
- Rowe et al. (2001): the first to close the detection loophole
- Gröblacher et al. (2007) test of Leggett-type non-local realist theories
- Salart et al. (2008): separation in a Bell Test
- Ansmann et al. (2009): overcoming the detection loophole in solid state
- Giustina et al. (2013), Larsson et al (2014): overcoming the detection loophole for photons
- Christensen et al. (2013): overcoming the detection loophole for photons
- Hensen et al., Giustina et al., Shalm et al. (2015): "loophole-free" Bell tests
- Schmied et al. (2016): Detection of Bell correlations in a many-body system
- Handsteiner et al. (2017): "Cosmic Bell Test" - Measurement Settings from Milky Way Stars
- Rosenfeld et al. (2017): "Event-Ready" Bell test with entangled atoms and closed detection and locality loopholes
- The BIG Bell Test Collaboration (2018): "Challenging local realism with human choices"
- Rauch et al (2018): measurement settings from distant quasars