20 dicembre

$$\mathrm{Re}\ \frac{1}{z^2+z+1} > 0$$

$z = x+iy$

$\bar z = x - iy$

$$\frac{1}{z^2+z+1} = \frac{\bar z^2+\bar z+1}{|z^2+z+1|^2}$$

$z^2 = x^2-y^2+2xyi$

$\bar z^2 = x^2-y^2-2xyi$

$$\mathrm{Re}\ \frac{\bar z^2+\bar z+1}{|z^2+z+1|^2} = \frac{1}{|z^2+z+1|^2}\,\mathrm{Re}\left(\bar z^2+\bar z+1\right) =$$

$$= \frac{1}{|z^2+z+1|^2}\left(x^2-y^2+x+1\right) > 0$$

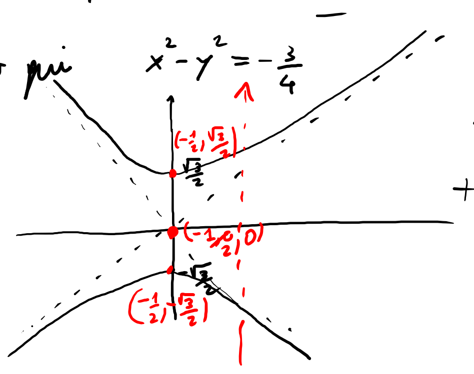
$$x^2-y^2+x+1>0$$

$$x^2+2\frac{x}{2}-y^2+1>0$$

$$\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-y^2+1>0$$

$$\left(x+\frac{1}{2}\right)^2-y^2>-\frac{3}{4}$$

Consideriamo qui $x^2-y^2=-\frac{3}{4}$



$$x^2-y^2+\frac{3}{4}>0$$

$$\lim_{x\to 0^+} \frac{\lg(1+x^a+x^{2a}) - \mathrm{th}(x)}{\underbrace{\int_0^{x^2} \sin\left(\frac{1}{t}\right) dt}_{o(x^2)} + 1 - \cos t}$$

$F(x^2)$

$$F(x) = \int_0^x \sin\left(\frac{1}{t}\right) dt$$

Per un esercizio teorico, che svolgeremo fra poco,

$F'(0) = 0$, $F(0) = 0$. Quindi

$$P_1(x) = F(0) + F'(0)\,x \equiv 0$$ e per Peano

$$F(x) = o(x) \implies F(x^2) = o(x^2)$$

Siano $a, b \in \mathbb{R}$ $a < b$ e sia $f \in L[a, b)$ con f limitata. Allora f è integrabile per Darboux in $[a, b]$.

Verifichiamo che $\sin\left(\frac{1}{x}\right)$ è integrabile per Darboux in $[0, 1]$.

Verifichiamo che $\sin\left(\frac{1}{x}\right) \in L(0, 1]$. Infatti

$\sin\left(\frac{1}{x}\right) \in C^0((0, 1]) \Rightarrow \sin\left(\frac{1}{x}\right) \in L_{loc}(0, 1]$

Succede inoltre $\left|\sin\frac{1}{x}\right| \leq 1$ e $1 \in L(0, 1]$

per il criterio del confronto $\left|\sin\frac{1}{x}\right| \in L(0, 1]$

$\Rightarrow \sin\frac{1}{x} \in L(0, 1]$

Per l'esercizio precedente $\sin\frac{1}{x} \in L[0, 1]$

Consideriamo ora $F(x) = \int_0^x \sin\frac{1}{t}\, dt$ e dimostriamo che $F'(x) = \begin{cases} \sin\frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

$F(0) = 0$ segue dalla definizione.

Noi già sappiamo che $\exists\, G$, derivabile su tutto $\mathbb{R}$, t.c.

$$G'(x) = \begin{cases} \sin\frac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

Consideriamo $F(x) - G(x) \in C^0(\mathbb{R})$.

In $[0, +\infty)$ abbiamo che $F - G \in C^0([0, +\infty))$

con $x > 0$ $F'(x) - G'(x) = \sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) = 0$ $\Rightarrow$ in $(0, +\infty)$ $F' - G' = 0$, $F - G \in C^0([0, +\infty))$

$\Rightarrow F - G$ è costante in $[0, +\infty)$

$F(x) - G(x) = F(0) - G(0) \quad \forall\, x \geq 0$

Analogamente

$F(x) - G(x) = F(0) - G(0) \quad \forall\, x \leq 0$

Quindi $F(x) - G(x) \equiv F(0) - G(0)$

$F(x) = G(x) + F(0) - G(0)$ e siccome, per definizione, G è primitiva di $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$, allora anche F lo è.

$\Rightarrow F'(0) = f(0) = 0$

$$f(x) = \int_0^x dt\; t \int_0^t e^{1+s^2}\, ds = \qquad\qquad P_n(x)$$

$$= e \int_0^x dt\; t \int_0^t e^{s^2}\, ds$$

$$e^y = \sum_{j=0}^{n} \frac{y^j}{j!} + o(y^n)$$

$$e^{s^2} = \sum_{j=0}^{n} \frac{s^{2j}}{j!} + o(s^{2n})$$

$$f(x) = e \int_0^x dt\; t \int_0^t \left(\sum_{j=0}^{n} \frac{s^{2j}}{j!} + o(s^{2n}) \right) ds$$

$$= e \int_0^x dt\; t \sum_{j=0}^{n} \int_0^t \frac{s^{2j}}{j!}\, ds + \int_0^x dt\; t \int_0^t o(s^{2n})\, ds$$

$$= e \int_0^x dt \sum_{j=0}^{n} \frac{t^{2j+2}}{j!\,(2j+1)} + \int_0^x dt\; o(t^{2n+2})$$

$$= e \sum_{j=0}^{n} \int_0^x t^{2j+2}\, dt \,\frac{1}{j!\,(2j+1)} + o(x^{2n+3})$$

$$= \sum_{j=0}^{n} e \frac{x^{2j+3}}{j!\,(2j+1)(2j+3)} + o(x^{2n+3})$$

$$P_{2n+3}(x)$$

$$R(z) = \frac{z^5 + z^2 + 2}{1 + z^5} = \frac{z^5 + 1 + z^2 + 1}{1 + z^5} = 1 + \frac{z^2 + 1}{1 + z^5} \sim S(z)$$

$$z^5 = -1 = \cos\pi + i\sin\pi \qquad z = r(\cos\vartheta + i\sin\vartheta)$$

$$r^5 = 1 \Rightarrow r = 1$$

$$\vartheta_k = \frac{\pi}{5} + \frac{2\pi k}{5} \qquad k = 0, 1, 2, 3, 4$$

$$z_k \qquad z_2 = -1$$

$$z^5 + 1 = (z - z_0)(z - z_1)\underbrace{(z - z_2)}_{(z+1)}(z - z_3)(z - z_4)$$

$$S(z) = \frac{A_0}{z - z_0} + \frac{A_1}{z - z_1} + \frac{A_2}{z + 1} + \frac{A_3}{z - z_3} + \frac{A_4}{z - z_4}$$

$$A_j = S(z)\,(z - z_j)\Big|_{z = z_j}$$

$\text{th}(x) = 1 - 2e^{-2x}(1+o(1))$

$\text{th}(x^a) = 1 - 2e^{-2x^a}(1+o(1)) = 1 + o(x^{-n}) \quad \forall n \in \mathbb{N}$

$$\lim_{x\to+\infty} \frac{\int_x^{2x} \frac{1+t}{(1+[t])^3}\,dt - \lg(\text{th}(x^a) + x^{-a})}{\cancel{\arctan(2x) - \arctan(x)} \quad \frac{1}{2x}} \qquad a > 0$$

$\arctan x = \frac{\pi}{2} - \frac{1}{x} + o\left(\frac{1}{x}\right) \qquad \frac{\arctan x - \frac{\pi}{2}}{-\frac{1}{x}} = (1+o(1))$

$$\lim_{x\to+\infty} \frac{\arctan(x) - \frac{\pi}{2}}{-\frac{1}{x}} \overset{H}{=} \lim_{x\to+\infty} \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}} = \lim_{x\to+\infty} \frac{x^2}{\cancel{1+}x^2} = 1$$

denom $= \cancel{\frac{\pi}{2}} - \frac{1}{2x} - \left(\cancel{\frac{\pi}{2}} - \frac{1}{x}\right) + o\left(\frac{1}{x}\right) = \frac{1}{2x} + o\left(\frac{1}{x}\right) = \frac{1}{2x}(1+o(1))$

$\text{Num} = \int_x^{2x} \frac{1+t}{(1+[t])^3}\,dt - \lg(1 + x^{-a} + o(x^{-n})) \qquad \forall n$

$\int_x^{2x} \frac{1+t}{[t]^3(1+[t]^{-1})^3}\,dt \qquad [t] = t(1+o(1))$

$= \int_x^{2x} \frac{1+t}{t^3}(1+o(1))\,dt = \int_x^{2x} \left(\frac{1}{t^2} + \frac{1}{t^3}\right)(1+o(1))\,dt$

$= \int_x^{2x} \frac{1}{t^2}\,dt + \int_x^{2x} \frac{1}{t^2}\,o(1)\,dt = -t^{-1}\Big]_x^{2x} + \int_x^{2x} \frac{1}{t^2}\,o(1)\,dt$

$= \frac{1}{2x} + \underbrace{\int_x^{2x} \frac{1}{t^2}\,o(1)\,dt}_{o\left(\frac{1}{x}\right)}$

$\text{num} = \frac{1}{2x} + o\left(\frac{1}{x}\right) - \lg(1 + \underbrace{x^{-a} + o(x^{-a})}_{y}) =$

$= \frac{1}{2x} + o\left(\frac{1}{x}\right) - x^{-a} + o(x^{-a})$

$\lg(1+y) = y + o(y)$

$$\text{num} = \begin{cases} -x^{-a} + o(x^{-a}) & \text{se } 0 < a < 1 \\ -\frac{1}{2}x^{-1} + o(x^{-1}) & a = 1 \\ \frac{1}{2x} + o(x^{-1}) & a > 1 \end{cases}$$